Physics 6010, Fall 2016

Some examples.

Relevant Sections in Text: §1.3–1.6

Examples

After all this formalism it is a good idea to spend some time developing a number of illustrative examples. These examples represent some of the situations you need to get good at, mechanically speaking.

Example: Newtonian particle in different coordinate systems.

We have already noted that the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}, t),$$

will give the equations of motion corresponding to Newton's second law for a particle moving in 3-d under the influence of a potential V. To see this, you can use Cartesian coordinates:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z, t).$$

Using,

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x},$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x},$$

the EL equation for the x coordinate is easily seen to be (exercise)

$$-\frac{\partial V}{\partial x} - m\ddot{x} = 0.$$

Of course, the y and z coordinates get a similar treatment. We then get (exercise)

$$m\ddot{\vec{r}} + \nabla V = 0.$$

Let us consider cylindrical coordinates,

$$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

Can you write down the equations of motion following from $\vec{F} = m\vec{a}$ in cylindrical (ρ, θ, z) coordinates? As discussed earlier, one can change coordinates in the Lagrangian without disturbing the validity of the Euler-Lagrange quations. Thus it is relatively straightforward to write down the equations of motion in cylindrical coordinates using the EL equations – a distinct practical advantage of this formalism. We just need to express the kinetic energy (T) and the potential energy (V) in terms of cylindrical coordinates and take the difference to make the Lagrangian, from which the EL equations are easily computed.

To compute the kinetic energy we take the kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and substitute

$$\dot{x} = \dot{\rho}\cos\theta - \dot{\theta}\rho\sin\theta, \quad \dot{y} = \dot{\rho}\sin\theta + \dot{\theta}\rho\cos\theta,$$

to get

$$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2).$$

To get the potential energy we just substitute into $V(x, y, z, t)$ for x and y in terms of ρ and θ. The Lagrangian is of the form

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2) - V(\rho, \theta, z, t).$$

Caution:

The notation for the potential energy function is the usual one used by physicists, but it can be misleading because, strictly speaking, it violates standard mathematical notational rules. $V(\rho, \theta, z, t)$ means *the potential energy at the location defined by* (ρ, θ, z) *and at the time* t. $V(\rho, \theta, z, t)$ is *not* the function obtained by substituting $x \to \rho, y \to \theta$ in $V(x, y, z, t)$, which a strict interpretation of the function notation would require. What we are calling $V(\rho, \theta, z, t)$ is in fact the function obtained from $V(x, y, z, t)$ by the substitution $x \to \rho\cos\theta$, *etc.* For example, $V(x, y, z, t) = x^2 t$ corresponds to $V(\rho, \theta, z, t) = \rho^2 t\cos^2\theta$, which is a violation of the rules of function notation.

The EL equations are (exercise)

$$m\ddot{\rho} - \rho\dot{\theta}^2 + \frac{\partial V}{\partial\rho} = 0,$$

$$m\frac{d}{dt}(\rho^2\dot{\theta}) + \frac{\partial V}{\partial\theta} = 0,$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0.$$

Exercise: Does the $\rho\dot{\theta}^2$ *term in the radial equation represent an attractive or repulsive effect in the radial motion?*

Exercise: Repeat this computation for spherical polar coordinates.

Example: Double Pendulum

Consider a system consisting of two plane pendulums (pendula?) connected in series. It would be good for your brain if you figured out how to write the equations of motion using Newton's second law, but I forgive you if you don't try it. The Lagrangian analysis is relatively straightforward.

To begin with, we have two particles moving in a plane. We denote their x and y positions via (x_1, y_1) and (x_2, y_2), where the origin of coordinates is placed at the fixed point of the double pendulum. The masses are m_1 and m_2. The motion of the particles is constrained: the lengths are l_1 and l_2; pendulum 1 is attached to a fixed point in space and pendulum 2 is attached to the end of pendulum 1. Mathematically we have

$$x_1^2 + y_1^2 = l_1^2, \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2.$$

These two constraints on the 4 cartesian coordinates leaves 2 degrees of freedom for this system. As we have already mentioned, the configuration of the system is uniquely specified once the angular displacement of each pendulum is specified. Let us choose these angles relative to a vertical line oriented downward. These angles – generalized coordinates for this system – are denoted by θ_1 and θ_2.

The kinetic energy for mass 1 is easily seen to be (exercise)

$$T_1 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2.$$

To find T_2 we note that, defining l_2 and θ_2 in the same way as for mass 1, we have

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2,$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2.$$

Along any dynamical trajectory $(x_1(t), y_1(t), x_2(t), y_2(t))$ we have (exercise)

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2,$$

$$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2.$$

The kinetic energy of mass 2 is then (exercise)

$$\begin{aligned} T_2 &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right]. \end{aligned}$$

The potential energy for particle 1 is (exercise)

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos\theta_1$$

and for particle 2 we get (exercise)

$$V_2 = m_2 g y_2 = -m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right).$$

The total potential energy is then

$$V = V_1 + V_2.$$

All together, the Lagrangian for this system is (exercise)

$$L = \frac{1}{2}(m_1+m_2)l_1\dot\theta_1^2 + \frac{1}{2}m_2 l_2^2\dot\theta_2^2 + m_2 l_1 l_2 \cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + (m_1+m_2)g l_1\cos\theta_1 + m_2 g l_2 \cos\theta_2.$$

It is now straightforward, albeit a little boring, to compute the EL equations. Note that the third term in the Lagrangian represents a coupling of the velocities of the two masses through the kinetic energy. This is a consequence of interaction of the pendulums induced by the constraints discussed above.

Example: Plane pendulum with moving support.

Consider a plane pendulum (mass m_2, length l) whose point of support is a mass m_1 which can slide horizontally. Let us compute the Lagrangian and EL equations. To begin with, we have two particles moving in two dimensions with coordinates (x_1, y_1) and (x_2, y_2). The motion is constrained; we have

$$y_1 = 0, \quad (x_2 - x_1)^2 + y_2^2 = l^2.$$

Here we have set the x axis along the line upon which m_1 moves. We then have (exercise)

$$(x_1, y_1) = (x, 0)$$

and

$$(x_2, y_2) = (x + l\sin\phi, -l\cos\phi).$$

Thus the generalized coordinates are the horizontal displacement $x_1 \equiv x$ for m_1 and the angle ϕ made with the vertical for the pendulum with mass m_2. The kinetic energy is therefore (exercise)

$$T = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2\left(\dot x^2 + 2l\dot x\dot\phi\cos\phi + l^2\dot\phi^2\right).$$

The potential energy is (exercise)

$$V = -m_2 g l \cos\phi.$$

The Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right) + m_2 g l \cos\phi.$$

Once again note how the constraints have coupled the motion via the kinetic energy.

The EL equations for x are (exercise)

$$(m_1 + m_2)\ddot{x} + \frac{d}{dt}(m_2 l\dot{\phi}\cos\phi) = 0.$$

The EL equations for ϕ are (exercise)

$$m_2 l^2\ddot{\phi} + \frac{d}{dt}(m_2 l\dot{x}\cos\phi) + m_2 l(\dot{x}\dot{\phi} - g)\sin\phi = 0.$$

Note that the x equation of motion implies

$$(m_1 + m_2)\dot{x} + (m_2 l\dot{\phi}\cos\phi) = \text{constant}.$$

We say that this quantity is a *constant of motion* or an *integral of motion* or that this quantity is *conserved* by the motion.

Exercise: What is the physical meaning of this conserved quantity?

In both of our pendulum examples the Lagrangians have no explicit t dependence. An example of a system with a t-dependent Lagrangian can be obtained from the previous example as follows.

Suppose the point of support of the pendulum is forced to oscillate with amplitude A and frequency ω:

$$x = A\cos\omega t.$$

Then the Lagrangian is (exercise)

$$L = \frac{1}{2}(m_1 + m_2)A^2\omega^2\sin^2\omega t + \frac{1}{2}m_2\left(-2lA\omega\sin\omega t\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right) + m_2 g l\cos\phi.$$

Note the explicit t dependence of the Lagrangian. There is now only a single degree of freedom, ϕ. The first term in the Lagrangian is purely a function of time and will not contribute to the EL equations (exercise); it can be dropped from L. The EL equations for ϕ are now

$$m_2[l^2\ddot{\phi} - \frac{d}{dt}(Al\omega\sin\omega t\cos\phi) - l(A\omega\sin\omega t\dot{\phi} + g)\sin\phi] = 0.$$

As an exercise you can check that this equation correctly describes the reduction of the original ϕ EL equations by the substitution $x = A\cos\omega t$.

Example: Charged particle in a Prescribed Electromagnetic Field

Two of the fundamental interactions allow themselves to be treated fruitfully using classical mechanics: gravity and electromagnetism. Of course, we have in mind macroscopic systems here. (The other fundamental interactions – strong and weak – only operate microscopically and require a quantum treatment.) Here we will give a Lagrangian formulation of the dynamics of a charged (test) particle in a given electromagnetic field. While a fully relativistic treatment is certainly feasible, we will stick to a non-relativistic (slow motion) treatment for simplicity.

We consider a particle with mass m and electric charge q moving in a given electromagnetic field $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$. The equations of motion come from the *Lorentz force law*,* which asserts that the force $\vec{F}$ at time t on a charge q located at position $\vec{r}$ and moving with velocity $\vec{v}$ is given by

$$\vec{F}(\vec{r}, \vec{v}, t) = q\vec{E}(\vec{r}, t) + \frac{q}{c}\vec{v} \times \vec{B}(\vec{r}, t).$$

Here, of course,

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}.$$

Thus the equations of motion for the curve $\vec{r} = \vec{r}(t)$ are

$$m\ddot{\vec{r}}(t) - q\vec{E}(\vec{r}(t), t) + \frac{q}{c}\dot{\vec{r}}(t) \times \vec{B}(\vec{r}(t), t) = 0.$$

It is not possible to find a Lagrangian whose EL equations correspond to the Lorentz force law without introducing the *electromagnetic potentials*. Recall that 4 of the eight Maxwell equations, the "homogeneous equations",

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0,$$

$$\nabla \cdot \vec{B} = 0.$$

are equivalent to the existence of a scalar potential ϕ and a vector potential $\vec{A}$ so that

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

and

$$\vec{B} = \nabla \times \vec{A}.$$

* I will use Gaussian units. See the text for formulas which use SI units.

You can easily check that these relations lead to electromagnetic fields satisfying the homogeneous Maxwell equations (exercise). Conversely, given any electromagnetic field satisfying the homogeneous Maxwell equations, one can find a function $\phi(\vec{r}, t)$ and a vector field $\vec{A}(\vec{r}, t)$ such that the above relations are satisfied.

The principal issue that arises here is that of *gauge transformations*: For each configuration of the electromagnetic field there are infinitely many potentials that can describe it. You can easily check that if ϕ and $\vec{A}$ correspond to a given $\vec{E}$ and $\vec{B}$, then so do

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t},$$

$$\vec{A}' = \vec{A} + \nabla \Lambda,$$

where $\Lambda(\vec{r}, t)$ is *any* function. This change in potentials is called a gauge transformation. Because the potentials are not uniquely defined by the electromagnetic fields, and because the effect of the electromagnetic field on matter is via the Lorentz force law – involving only $\vec{E}$ and $\vec{B}$, the potentials have no direct physical significance, *e.g.,* one cannot measure ϕ by studying the motion of test particles.

In terms of the potentials, the equations of motion are (in an inertial reference frame)

$$m\ddot{\vec{r}} + q\left(\nabla\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{1}{c}\dot{\vec{r}} \times \nabla \times \vec{A}\right) = 0.$$

We now show that the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2}m(\dot{\vec{r}})^2 - q\phi(\vec{r}, t) + \frac{q}{c}\vec{A}(\vec{r}, t) \cdot \dot{\vec{r}}$$

yields these equations of motion as EL equations. To do this, we consider the EL equation for $x(t)$ and compare with the x component of the Lorentz force law. The y and z EL equations are handled in an identical manner. We have (exercise)

$$\frac{\partial L}{\partial x} = -q\frac{\partial \phi}{\partial x} + \frac{q}{c}\frac{\partial \vec{A}}{\partial x} \cdot \dot{\vec{r}},$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x,$$

so that, on a curve $\vec{r} = \vec{r}(t)$, (exercise)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + \frac{q}{c}\dot{\vec{r}} \cdot \nabla A_x + \frac{q}{c}\frac{\partial A_x}{\partial t}.$$

Here ∇A_x means to take the gradient of the function A_x. The EL equations are therefore

$$m\ddot{x} + \frac{q}{c}\dot{\vec{r}} \cdot \nabla A_x + \frac{q}{c}\frac{\partial A_x}{\partial t} + q\frac{\partial \phi}{\partial x} - \frac{q}{c}\frac{\partial \vec{A}}{\partial x} \cdot \dot{\vec{r}} = 0.$$

Using (exercises)

$$(\nabla\phi)_x = \frac{\partial \phi}{\partial x},$$

$$\left(\frac{\partial \vec{A}}{\partial t}\right)_x = \frac{\partial A_x}{\partial t},$$

$$\left(\dot{\vec{r}} \times \nabla \times \vec{A}\right)_x = -\dot{\vec{r}} \cdot \nabla A_x + \frac{\partial \vec{A}}{\partial x} \cdot \dot{\vec{r}},$$

it is easy to see that the EL equation for x is the same as the x component of the Lorentz force law (exercise).

There is a technical issue of interest here. The Lagrangian is built from potentials $(\phi, \vec{A})$. As we have already pointed out, there are infinitely many potentials for any given electromagnetic field. Thus there are infinitely many Lagrangians describing motion in a single electromagnetic field. Each of these Lagrangians will yield the same Lorentz force law, which is built from $(\vec{E}, \vec{B})$. Each of the Lagrangians can be related via a gauge transformation of the potentials. Apparently, under a gauge transformation of the potentials the Lagrangian changes in just the right way so that the EL equations do not change. How does this happen? You will explore this in a homework problem.

Example: Lagrangians for Systems

Often times it is useful to view a dynamical system as consisting of several subsystems. For example, the solar system can be modeled as consisting of ten particles interacting gravitationally. *In the absence of interaction,* the Lagrangian L_0 for the total (non-interacting) system can be viewed as the sum of Lagrangians for each of its parts:

$$L_0 = L_1 + L_2 + L_3 + \ldots.$$

Here L_1, L_2, *etc.* are the Lagrangians for the subsystems. For example, if we have a system of (non-interacting) Newtonian subsystems each Lagrangian is of the form (for the i^{th} subsystem)

$$L_i = T_i - V_i.$$

Here V_i is the potential energy of the i^{th} system due to *external* forces — not due to inter-system interactions, which we are ignoring for a moment. It is easy to see that L_0 correctly describes the motion of the system of non-interacting systems through its EL equations. This is because the EL equations for the k^{th} system involve derivatives of L_0 with respect to coordinates and velocities of the k^{th} system and this just picks out L_k from the sum $L_0 = \sum_j L_j$.

As a simple example, let us consider a system consisting of the planets, viewed as non-interacting point particles. They all move in a central force field due to the sun which is non-dynamical in this model. The Lagrangian for the k^{th} planet, with position $\vec{r}_k$ is of the form

$$L_k = \frac{1}{2} m_k \dot{\vec{r}}_k^2 + \frac{GMm_k}{r_k},$$

where m_k is the mass of the planet, M is the mass of the sun, and G is Newton's constant.

Of course, non-interacting systems are an idealization and, ultimately, are of little physical interest. Interactions are what makes the world what it is. One of the main ways to introduce interactions between the subsystems is to introduce a potential energy function, $V = V(q_1, q_2, \ldots, t)$ which couples various degrees of freedom. We then have the Lagrangian for the interacting system given by

$$L = L_0 - V.$$

The effect of this potential energy function in the Lagrangian is to *couple* the motion of the subsystems. You can see this by noting the EL equations for a degree of freedom can now, in general, depend upon the other degrees of freedom (exercise).

As an example, let us consider the Lagrangian for a pair of electons in a Helium atom. We view the nucleus as fixed, with charge Q; it is part of the "environment". The "system" consists of the two electrons, each with mass m and charge q. The configuration space is $R^3 \times R^3 = R^6$, and we label points in the configuration space with position vectors $\vec{r}_1$ and $\vec{r}_2$. The Lagrangian is

$$L = \frac{1}{2} m \dot{\vec{r}}_1^2 - \frac{qQ}{|\vec{r}_1|} + \frac{1}{2} m \dot{\vec{r}}_2^2 - \frac{qQ}{|\vec{r}_2|} - \frac{q^2}{|\vec{r}_1 - \vec{r}_2|}.$$

The first two terms represent electron 1 moving in the Coulomb field of the nucleus. Likewise for the next two terms regarding electron two. With these first 4 terms alone the electrons orbit the nucleus independently and do not interact among themselves. The last term represents the interaction between the electrons, which is Coulomb repulsion. It is this term which couples the motion of the two electrons and makes the EL equations somewhat complex, lacking an explicit solution.

The other principal way to mathematically represent interactions is via constraints. In this more phenomenological approach, the coupling between subsystems will typically occur via the *kinetic* energy function. We will give a couple of examples in what follows.