Physics 6010, Fall 2016

Introduction. Configuration space. Equations of Motion. Velocity Phase Space.

Relevant Sections in Text: §1.1–1.4

Introduction

This course principally deals with the variational principles of mechanics, particularly the Lagrangian and Hamiltonian descriptions of dynamical systems. So, this won't be a course in which we spend a lot of time doing things like solving for the motion of a triple pendulum coupled by springs while sliding on an inclined plane with friction in a viscous fluid as viewed in a rotating reference frame. To be sure, we will analyze the motion of some important dynamical systems. But the emphasis will be more on formalism – the Lagrangian and Hamiltonian approaches, in particular – and less on solving equations. There are many reasons for this; here are a few.

First of all, when analyzing a dynamical system the first task is to figure out what is the correct mathematical description, what are the equations of motion, etc. The Lagrangian and Hamiltonian formalisms are the most powerful ways to do this. Next, as I shall repeatedly emphasize, most interesting dynamical systems are simply too complex in their behavior for one to find closed-form, analytical expressions for their motion. In other words, the solutions to the equations of motion exist, but you will never be able to write them down in practice. How then to extract physical information about the system? There are a variety of sophisticated methods for getting at aspects of the physical behavior. Many of these methods stem from the powerful vantage point provided by the Lagrangian and/or Hamiltonian formulations of mechanics. Finally, one can view "classical mechanics" as an approximation to the more fundamental "quantum mechanics", this approximation being valid, *e.g.,* for macroscopic systems. A key link between these two descriptions of dynamics is made via the Lagrangian and Hamiltonian formalisms. Indeed, a lot of the structural features of quantum mechanics have clear classical analogs, and these are uncovered via the Lagrangian and Hamiltonian formulations. One might say the Lagrangian and Hamiltonian forms of mechanics are a classical imprint of the quantum world. So, by learning these techniques you are better prepared to study quantum mechanics and you are acquiring tools which can handle all kinds of dynamics – classical and/or quantum.

In mechanics we have four principal tasks: (1) determine the *configuration space* or the *phase space* for the system of interest; (2) find the underlying *dynamical law* – the equations of motion – governing the motion of a system; (3) use the dynamical law to find the allowed motions of a system; (4) out of all allowed motions find those which describe the physical situations of interest. Let us now briefly discuss each of these tasks.

Configuration space

Mechanics concerns itself with *dynamical systems* – systems whose configuration evolves in time according to some deterministic law. To define a dynamical system, then, one first needs to specify what one means by "configuration". We need to define a *configuration space* for the system. This is a set of variables which characterizes what the system is doing at a given time — it is the set of quantities which define the system we are studying. Normally, the configuration space is a continuous space (a "manifold"). The dimension of the configuration space is then called the number of *degrees of freedom* of the system.

We shall see that a better characterization of what a system is doing at any given time is given by the *phase space* of the system. There are various versions of the phase space. For now you can think of it as the space of possible *initial conditions* for the system, *i.e.,* the set of conditions you need to uniquely specify a solution to the equations of motion.

Example: Point particles

In many ways the simplest dynamical system is that of a "point particle", which can represent any number of physical systems whenever the internal structure of the system can be neglected for the purposes at hand. Thus, when calculating the forces on a car going around a curve, one can often idealize the car as a mathematical point located at the center of mass. A good first pass at the motion of the earth-sun system consists of assuming that these bodies are represented by point masses located at their respective centers of mass. A point particle can be viewed as the discrete building block in a continuum of matter, *etc.*

Mathematically, a "particle" is a dynamical system whose configuration space is the set of allowed positions of the particle.* For a particle moving in three dimensional space the configuration space is simply Euclidean space (E^3). Because E^3 is three dimensional, we say that the particle has "3 degrees of freedom". By fixing an origin, the configuration space can be viewed as R^3. We can label the configuration (or position) of the particle by any system of coordinates on R^3, *e.g.,* Cartesian coordinates (x, y, z), spherical polar coordinates (r, θ, ϕ), *etc.* Evidently, the number of coordinates needed to characterize the configuration of the system (here a particle) is the same as the dimension of the configuration space and is the same as the number of degrees of freedom. It is often convenient to specify the configuration of the particle by its *position vector* $\vec{r} \in R^3$,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Dynamical evolution is a continuous change in position of the particle, *i.e.,* a curve $\vec{r} = \vec{r}(t)$ in R^3. This curve can be specified by giving the coordinate location of the particle

* As we shall see, in some sense every dynamical system can be viewed as a "particle" provided we work on a manifold with a sufficient number of dimensions.

as a function of time, *e.g.,*

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

Unless otherwise notified, we assume the curve is smooth (infinitely continuously differentiable). Note that we are employing the traditional abuse of notation in which we use the same label for the function (*e.g.,* $x(t)$) as for the coordinate (*e.g.,* x).

As an elementary example, here is a curve in configuration space:

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

The motion is helical, moving up the z axis.

Exercise: What physical system would give such a motion for a particle?

Given the parametric form of the curve (or "trajectory") in configuration space, $\vec{r}(t)$, a *tangent vector* $\vec{v}(t)$ to the curve at the point $\vec{r}(t)$ can be computed by simple differentiation:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}.$$

Exercise: Show that this **is** *a tangent vector.*

This vector along the curve represents the *velocity* of the particle when it is at position $\vec{r}(t)$. Note that we have a velocity vector at each point of the curve $\vec{r}(t)$. We can take yet another derivative and obtain the *acceleration*:

$$\mathbf{a}(t) = \frac{d^2\vec{r}(t)}{dt^2},$$

and so on.

Exercise: Compute the velocity and acceleration of the helical curve given above. Check that they point in the direction you expect.

Constraints. The Spherical Pendulum.

It often occurs that for one reason or another the motion of a particle is restricted to some subset of E^3. For example, a *spherical pendulum* is a mass fixed by a rod to a point in space. We can model the spherical pendulum as a point mass moving in E^3 subject to a *constraint* that its distance from a fixed point is constant in time (reflecting the rigidity of the rod). Choosing the fixed point as the origin of Cartesian coordinates we have

$$x^2 + y^2 + x^2 = l^2,$$

where l is the length of the rod. Mathematically, the true configuration space is not E^3 but instead is a surface: a 2-dimensional sphere, $S^2 \subset E^3$. Motion in configuration space is in this case a curve on S^2. We can still view the motion as occurring in E^3 provided we take account of the constraint. For example, the velocity of the particle now must obey

$$\frac{d}{dt}(x^2 + y^2 + x^2) = 0 \quad \Longrightarrow \quad x\dot{x} + y\dot{y} + z\dot{z} = \vec{r} \cdot \vec{v} = 0,$$

which means that the velocity is everywhere tangent to the sphere.* Of course, for such a system it is most convenient to express everything in terms of spherical polar coordinates (r, θ, ϕ). Then we can fix $r = l$ and use θ and ϕ to specify the configuration of the system. A trajectory of the pendulum is then specified by specifying the functions $\theta(t)$ and $\phi(t)$ so that

$$\theta = \theta(t)$$

$$\phi = \phi(t).$$

The velocity vector at time t has θ and ϕ components given by $(\dot{\theta}, \dot{\phi})$, where

$$\dot{\theta} = \frac{d\theta(t)}{dt}, \quad \dot{\phi} = \frac{d\phi(t)}{dt}.$$

This example is a good illustration of the two principal ways to handle dynamical systems involving *holonomic constraints*, which are constraints which can be expressed as the vanishing of functions on configuration space: (1) One can work with the original configuration space and adjoin the constraints (and their differential consequences) to the equations of motion. Or, (2) one can solve the constraints and work with coordinates on the true configuration space. Such coordinates are usually referred to as *generalized coordinates.*

Generalized Coordinates

More complicated systems can often be modeled as collections of N point particles. In this case the configuration space is $3N$-dimensional, E^{3N} or R^{3N} (exercise). There also may be constraints which reduce the size of the configuration space, *i.e.,* reduce the number of degrees of freedom. For example, a *rigid body* has but 6 degrees of freedom no

* Note that this means the constraint does no work. This is pretty typical.

matter how many particles we imagine it to contain. This is because we can completely specify the configuration of a given rigid body by (i) locating its center of mass (or any other given point on the body), (ii) specifying its orientation relative to a fixed set of axes. The center of mass location requires 3 coordinates. The orientation requires 3 angles (*e.g.,* Euler angles) So, (i) and (ii) together require 6 variables.

Whether or not we view a system as made of particles, we will generally have a configuration space of some sort or other. This configuration space may be determined via constraints, and we may not choose to explicitly solve the constraints, but in principle there is always some N dimensional space which serves as the configuration space. Normally the configuration space is specified by displaying the *generalized coordinates* that are needed to uniquely determine the configuration of the system. A spherical pendulum has spherical coordinates θ and ϕ as its generalized coordinates — the configuration is space is S^2. A system consisting of a pair of coupled plane pendulums has a pair of angles θ_1 and θ_2, both running around a circle, as its generalized coordinates; the configuration space is a two-dimensional torus, T^2. And so forth.

Exercise: What is the configuration space for a triple plane pendulum? What happens for N such pendulums?

In general, a dynamical system with N degrees of freedom will have its configuration described by N generalized coordinates, which we shall denote by q^i, $i = 1, 2, \ldots, N$. (Of course, in specific examples more descriptive notation — *e.g.,* θ, ϕ, *etc.* — would be used.) These are coordinates on some space, technically known as a *manifold*, *e.g.,* R^3, S^2, T^2, *etc.* Motion of the system is specified by a curve on this manifold

$$q^i = q^i(t).$$

Of course, one of the principal goals of mechanics is to characterize these curves in a useful fashion. So, now you see a sense in which any dynamical system is a "particle" moving in some (higher-dimensional) configuration space.

Velocity phase space

For the most part we shall restrict our attention to dynamical systems described by second order differential equations of motion. Each solution to the equations of motion — each physically allowed trajectory — is then uniquely determined by its initial position and velocity. Thus one sometimes says that the *state* of the system is determined by specifying the position and velocity at some given time. The positions are points in some manifold, labeled by coordinates q^i. The velocities are possible tangent vectors along curves through the points. The set of all positions and possible tangent vectors is called the *velocity phase space.* A good way to think of the velocity phase space is that it *is is the space of possible initial conditions for the equations of motion.* Points in the phase space are determined by the pair $(q^i, \dot{q}^i)$, $i = 1, 2, \ldots, N$, where N is the number of degrees of freedom. Thus the velocity phase space has $2N$ dimensions. Mathematicians call the velocity phase space the *tangent bundle* to the manifold parametrized by q^i.

Let me emphasize an important notational point which always seems to cause confusion. The variable $\dot{q}^i$ used as part of the velocity phase space is not the derivative of anything. It's just a vector. We have here another convenient abuse of notation. Recall how q^i does double duty: it represents a point in configuration space and it can denote a function of t when we have a curve in configuration space, $q^i = q^i(t)$. Similarly, $\dot{q}^i$ has two completely different uses. It can denote a possible tangent vector or it can represent a family of tangent vectors to a curve $q^i(t)$, in which case

$$\dot{q}^i(t) = \frac{dq^i(t)}{dt}.$$

Put differently, a point in velocity phase space is a set of numbers $(q^i, \dot{q}^i)$. In this context, these variables are not functions of t. Think of them as $2N$ numbers specifying initial conditions. Each of these ordered pairs represents a point in configuration space a curve could go through and a possible tangent vector to that curve. Infinitely many curves in configuration space can correspond to the same $(q^i, \dot{q}^i)$. A curve in the velocity phase space looks like:

$$q^i = q^i(t), \quad \dot{q}^i = \dot{q}^i(t).$$

Such a curve may not actually correspond to any motion of the system – we need $\dot{q}^i(t)$ to actually represent the tangent to $q^i(t)$, that is, we need to choose $\dot{q}^i(t) = \frac{dq^i(t)}{dt}$. A curve in which $\dot{q}^i(t)$ is in fact the tangent to the curve $q^i(t)$ is then

$$q^i = q^i(t), \quad \dot{q}^i = \frac{dq^i(t)}{dt}.$$

Thus a curve in configuration space "lifts" to define a curve in velocity phase space, but not every curve in velocity phase space is the lift of a curve in configuration space.

So, to summarize, sometimes $\dot{q}^i$ is just a set of variables, sometimes it represents a collection of functions of t (usually) obtained by differentiation of $q^i(t)$. The trick is to be able to know which way the notation is being used based upon context. It rarely happens that both notations could make sense in a given situation, but it takes a little experience to see this.

Exercise: Characterize the typical curves in the velocity phase space corresponding to the motion of (i) a free particle, (ii) a particle moving in a uniform gravitational field, (iii) a harmonic oscillator.

The Extended Velocity Phase Space

It will be useful to introduce one more playground for our dynamical systems: the *extended velocity phase space* (EVPS). This space is obtained by taking the $2N$ dimensional phase space and adjoining a time axis. So, points in the EVPS are a collection of $2N+1$ numbers $(t, q^i, \dot{q}^i)$; the EVPS is a manifold of dimension $2N+1$. A curve in velocity phase space is then represented as a *graph*: $(q^i = q^i(t), \dot{q}^i = \dot{q}^i(t))$. The idea here is that many physical quantities of interest are functions on the EVPS.

Exercise: Show that the energy and angular momentum of a Newtonian particle moving in a (possibly time-dependent) potential are functions on the EVPS.

Equations of motion, Newton's second law

Let us now discuss the idea of a "dynamical law". For a system with N degrees of freedom, denoted by q^i, the dynamical law normally is a system of N ordinary differential equations which can be defined by an expression such as*

$$\Delta_i(t, q, \dot{q}, \ddot{q}, \ldots) = 0, \quad i = 1, 2, \ldots, N.$$

These are the *equations of motion* in the sense that they define differential equations for a curve $q^i = q^i(t)$ with $\dot{q}^i = \frac{dq^i}{dt}$, $\ddot{q}^i = \frac{d^2q^i}{dt^2}$, *etc.* Of course, one of the goals of this course is to learn methods to find the equations of motion and extract information from them.

Generally speaking, we would be very happy if we could find all possible solutions to the equations of motion. The solutions of the equations of motion which are of interest will usually be characterized as the solutions satisfying appropriate initial conditions. It is an empirical fact that, by and large, the motion of a system is uniquely determined once one knows the initial position and velocity. Thus the equations of motion for a dynamical system are typically second-order.

It is worth emphasizing that almost every dynamical law you can write down does *not* allow for an explicit, analytical solution to the equations of motion. It's not that the solutions don't exist, it is just that the motion is too complicated for some relatively simple collection of elementary functions to describe it. Thus one typically has to use a combination of numerical methods and high powered "tricks" to extract information about the physical behavior of the system from the equations of motion. Numerical methods are "beyond the scope of this course" (*i.e.,* "Somebody else can teach *that.*"). Many of

* Even if we do not work on the true configuration space, that is, we have constraints hanging around, the dynamical law will still be a system of differential equations. Some of them will just be purely algebraic.

the "tricks", on the other hand, stem from results of the formalism of Lagrangian and Hamiltonian mechanics, which is the central theme of the course, of course.

Probably the most familiar equations of motion to you are those following from Newton's second law. For example, for a particle labeled by position vector $\vec{r}$, the *force* on the particle at position $\vec{r}$ and time t is denoted by $\vec{F}(\vec{r}, t)$. At each t, $\vec{F}$ defines a *vector field* on R^3. Given a force, the equations of motion take the form

$$m\frac{d^2\vec{r}(t)}{dt^2} - \vec{F}(\vec{r}(t), t) = 0,$$

where m is the *inertial mass* of the particle. Here I have been a little pedantic with notation in that I have explicitly included the function argument — the time t — of the curve. Usually this is suppressed in order for the equation to be more pleasing to the eye. I include it here to emphasize the fact that the dynamical law is an equation for a vector-function of time, $\vec{r}(t)$, and, in particular, that one must substitute this vector function into the "$\vec{r}$ slot" of $\vec{F}(\vec{r}, t)$ to get the equation of motion. Thus the force term in the equation of motion gets its time dependence from two places: (i) the **explicit** *time dependence* coming from the "t slot" in $\vec{F}(\vec{r}, t)$ and (ii) the **implicit** *time dependence* coming from the substitution of the curve into the "$\vec{r}$ slot" of $\vec{F}(\vec{r}, t)$.

This "explicit" and "implicit" business happens a lot in mechanics, and it sometimes confuses the beginner, so let me belabor the point here. These two sources of time dependence in the force term in the equations of motion arise because (i) the force vector field depends upon what time it is, *i.e.,* it is changing in time (*e.g.,* think of a forced oscillator), and (ii) the force term in the equation of motion represents the force at time t *on the particle*. Because the force (in general) varies with position, and because the particle (in general) is moving, this causes additional time dependence in the force term. Here's a simple example. Consider a forced pendulum moving in the small angle approximation so that it is just a forced harmonic oscillator. Maybe the pendulum is a kid on a swing whom you are pushing. Let the displacement from equilibrium be denoted by x and let the force you exert (in the direction of motion) at time t be denoted by $f(t)$. The net force on the particle/kid in this one-dimensional system has component along x given by $F(x, t) = -kx + f(t)$, were k is a constant determining the restoring force (built from the acceleration due to gravity). You can see the x dependence, and you can see the explicit t dependence. For a given motion of the system, $x = \cos(t)$, say, the force varies in time due to both of these dependencies: $F(t) = F(x(t), t) = -\cos(t) + f(t)$.

You have no doubt had experience solving Newton's second law for a particle with some simple choices of the force vector field. The general solution for a given force law will involve two integration constants for each degree of freedom since the equations are second order. By letting these constants vary over all possible values you obtain all possible allowed motions of the system consistent with the chosen force. Specific initial conditions select specific values for the two integration constants thus uniquely specifying the motion.

Velocity dependent forces can also be accommodated, although they are less common. Sticking with the example of a single particle, the force vector field is now (in general) a function of seven variables — 3 position coordinates, 3 velocity components, and time:

$$\vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t).$$

The equations of motion for the curve $\vec{r}(t)$ take the form

$$m\frac{d^2\vec{r}(t)}{dt^2} - \vec{F}(\vec{r}(t), \frac{d\vec{r}(t)}{dt}, t) = 0.$$

The most significant example here is the magnetic force on a charged particle in which*

$$\vec{F}(\vec{r}, \dot{\vec{r}}, t) = \frac{q}{c}\dot{\vec{r}} \times \mathbf{B}(\vec{r}, t).$$

Here q is the particle's charge, c is the speed of light, and $\mathbf{B}(\vec{r}, t)$ is the magnetic field at the location $\vec{r}$ and time t.

In Newton's theory of mechanics, all dynamical systems can be handled via equations of motion of the type "$F = ma$". I will call dynamical systems of this type "Newtonian systems". While the use of Newtonian systems to model macroscopic phenomena is extremely useful, strictly speaking nature does not consist of Newtonian systems, if only because of relativity and quantum mechanics. Thus it is not always feasible to analyze things using "$F = ma$". But even Newtonian systems are not always best treated using Newton's second law explicitly. For example, consider a rigid body with, say, Avagadro's number of particles. You now have to analyze 10^{26} equations of $F = ma$ type! The presence of constraints means you really only need 6 equations of motion – how should we obtain them? The Lagrangian and Hamiltonian formulations of mechanics we shall study provide a significant generalization of Newtonian mechanics – with a much wider domain of applicability – and provide a powerful means to handle such complex Newtonian systems as well.

The Big Picture: Observables, States, and Laws

Virtually all successful physical descriptions of nature have similar structures (if disparate details). In particular, there are 3 key ingredients in a physical theory: Observables, States, and Laws. Different theories can, of course, take widely different approaches to mapping these ingredients to the physical world and giving a mathematical representation of them. It is worth spelling this out briefly for classical mechanics.

* Here we use Gaussian units.

Observables

Observables are, naturally, quantities which can be measured (at least in principle). They are things like, the energy of a system, the position of a particle, *etc.* In classical mechanics, observables mathematically correspond to functions on phase space. (Think about it: position, momentum, energy, angular momentum, *etc.* — all functions on phase space.)

States

Depending upon what a system is doing, one can get various outcomes if you measure an observable. The *state* of a system characterizes the outcome of all possible measurements. So, if you know the state of a system at a given time, you know everything there is to know about the system at that time. In classical mechanics, states mathematically correspond to a point in phase space. Think about it: if observables are functions on phase space, then if you know what point in phase space the system occupies you know the value of all the observables. Normally, observables are quantities which may change if the state of the system changes. Indeed, the state of the system changes if and only if at least one observable changes – that's how we know something changed! Depending upon one's model of a physical system, some quantities which we commonly view as something we measure may not be classified as "observables" in our scheme of terminology. For example, if the system of interest is a particle of mass m we will assume the mass is the same for *all* states of the system. A better way to think of the mass is then as a "parameter" which characterizes the definition of the system rather than the state of the system.

Laws

Laws come in various forms, but for dynamical theories (*e.g.,* classical mechanics, quantum mechanics, electrodynamics,. . .) they will normally include a statement defining the change of the state of the system in time. In classical mechanics, the equations of motion play this role. We have already discussed how the laws of motion are usually second order differential equations. Thus the space of states can be taken to be the velocity phase space. Solutions to the equations of motion define a family of curves in the state space along which observables change. Assuming the usual existence and uniqueness theorems for ordinary differential equations, the dynamical law defines a unique curve through each point in the space of states.

Exercise: How are observables, states, laws mathematically defined in quantum mechanics? How about in electrodynamics?

As I mentioned earlier, the law you are probably most familiar with is "$F = ma$". One of the principal goals of this course is to introduce you to a new, more general law, based on the *Euler-Lagrange equations.* We shall see where this law comes from soon. For now, I simply state it. If a system is characterized by an extended velocity phase space $(q^i, \dot{q}^i, t)$, the motion of the system can be characterized by a function $L = L(q, \dot{q}, t)$ – called the *Lagrangian* – according to the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0.$$

You should not think of this (at least initially) as an equation for L. Rather, given L, this is a recipe for writing down the dynamical law for the system. The art of the physicist is to determine what Lagrangian to use for a given dynamical system. For Newtonian systems one chooses $L = T - V$, where T and V are the kinetic and potential energies, respectively. To see why, let us compute the Euler-Lagrange equations for a particle of mass m moving a a force field $\vec{F}$ with potential energy function $V(\vec{r}, t)$ (so that $\vec{F} = -\nabla V$). Using Cartesian coordinates to parametrize the velocity phase space, the Lagrangian is:

$$L = T - V = \frac{1}{2}mv^2 - V(\vec{r}, t) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z, t).$$

It is easy to compute the required derivatives:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}, \quad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}, \quad \frac{\partial L}{\partial y} = -\frac{\partial V}{\partial z}, \quad \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z},$$

so that the Euler-Lagrange equations of motion become $\vec{F} = m\vec{a}$:

$$-\frac{\partial V}{\partial x} - m\ddot{x} = 0, \quad -\frac{\partial V}{\partial y} - m\ddot{y} = 0, \quad -\frac{\partial V}{\partial z} - m\ddot{z} = 0.$$

We shall now spend quite a bit of time finding out where the Lagrangian version of the dynamical law comes from and explore the salient properties of the Lagrangian formalism.