Physics 6010, Fall 2010

Infinitesimal Canonical Transformations. Symmetries and Conservation Laws.

Relevant Sections in Text: §8.2, 8.3, 9.1, 9.2, 9.5

Infinitesimal Canonical Transformations

With the notion of canonical transformations in hand, we are almost ready to explore the true power of the Hamiltonian formalism. One final ingredient needs to be developed: continuous canonical transformations and their infinitesimal form.

Consider a family of canonical transformations that depend continuously upon a parameter λ such that for some value of that parameter the transformation is the identity. We write

$$q^i(\lambda) = q^i(q, p, \lambda), \quad p_i(\lambda) = p_i(q, p, \lambda).$$

Let us adjust our parameter such that $\lambda = 0$ is the identity:

$$q^i(q, p, 0) = q^i, \quad p_i(q, p, 0) = p_i.$$

As examples, think about the point transformations corresponding to rotations about an axis, or translations along a direction (exercise).

Let us consider the transformations of the variables arising when the parameter is nearly zero. We have, with $\lambda << 1$,

$$q^i(\lambda) \approx q^i + \lambda\delta q^i, \quad p_i(\lambda) \approx p_i + \lambda\delta p_i,$$

where we have denoted the first-order changes in the canonical variables by

$$\delta q^i = \left.\frac{\partial q^i(q, p, \lambda)}{\partial\lambda}\right|_{\lambda=0}, \quad \delta p_i = \left.\frac{\partial p_i(q, p, \lambda)}{\partial\lambda}\right|_{\lambda=0}.$$

We call δq and δp the *infinitesimal change in q and p induced by the infinitesimal canonical transformation.*

For example, let q be the position of a particle in one dimension. Consider a translational point canonical transformation (exercise):

$$q(\lambda) = q + \lambda, \quad p(\lambda) = p.$$

Here we have the rather trivial infinitesimal transformations:

$$\delta q = 1, \quad \delta p = 0.$$

In our discussion of infinitesimal canonical transformations we have not yet taken account of the fact that the transformation *is* canonical. To first order in the variations of coordinates and momenta we have (for a time-independent transformation)

$$p_i(\lambda)\dot{q}^i(\lambda) = (p_i + \lambda\delta p_i)\dot{q}^i + \lambda p_i(\frac{\partial \delta q^i}{\partial q^j}\dot{q}^j + \frac{\partial \delta q^i}{\partial p_j}\dot{p}_j).$$

For any function $F = F(q,p)$ we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i.$$

Writing out the canonical transformation condition to first order in the variations, and matching up the coefficients of $\dot{q}$ and $\dot{p}$, the infinitesimal canonical transformation requirement yields, respectively, (exercise)

$$\begin{aligned} \delta p_j + p_i\frac{\partial \delta q^i}{\partial q^j} + \frac{1}{\lambda}\frac{\partial F}{\partial q^j} &= 0 \\ p_i\frac{\partial \delta q^i}{\partial p_k} + \frac{1}{\lambda}\frac{\partial F}{\partial p_k} &= 0 \end{aligned} \tag{1}$$

for some function F.

Here we have $2n$ equations for $2n+1$ unknowns $(\delta q^i, \delta p_i, F)$, so we might expect the solution to depend upon a single arbitrary function of the canonical variables. This is in fact the case. To see this, differentiate the first equation with respect to p_k, differentiate the second equation with respect to q^j and take the difference between the two resulting equations. This gives (exercise)

$$\frac{\partial \delta p_j}{\partial p_k} + \frac{\partial \delta q^k}{\partial q^j} = 0. \tag{2}$$

The equations (2) are of the same type as the system of equations one gets when setting the curl of a vector field to zero. In that case you will recall this forces the vector field to be the gradient of a function. Here the same thing works; the solution to (2) is

$$\delta q^k = \frac{\partial G}{\partial p_k}, \quad \delta p_j = -\frac{\partial G}{\partial q^j}, \tag{3}$$

where $G = G(q,p)$ is *any* function.* We have shown that δq and δp must have this form as a necessary condition for the infinitesimal transformation to be canonical. We now substitute this form for the infinitesimal transformation back into our original equations.

* A precise mathematical interpretation of the equations (2) is that they are the *integrability conditions* for the infinitesimal canonical transformation equations (1). The integrability conditions (2) are necessary and sufficient for (1) to have a solution.

We find that the infinitesimal canonical transformation conditions are satisfied for any G if F is given by

$$F = \lambda(G - p_i \frac{\partial G}{\partial p_i} + const.)$$

(The constant can be set to zero without loss of generality (exercise).) So, given any $G = G(q, p)$, the infinitesimal transformation (3) will be canonical. Since a finite transformation can be obtained as a limit of many infinitesimal transformations, it follows that each choice of G determines a 1-parameter family of canonical transformations,

$$q^i \to q^i(\lambda), \quad p_i \to p_i(\lambda), \qquad q^i(0) = q^i, \quad p_i(0) = p_i,$$

whose infinitesimal form is given by (3).

This result is important enough to call it a theorem:

Theorem. *To every function G on phase space there is a 1-parameter family of canonical transformations (which includes the identity), and to every 1-parameter family of canonical transformations (including the identity) there is a function G, such that the infinitesimal canonical transformation takes the form (3).*

The function G is usually called the *infinitesimal generating function* or *infinitesimal generator* of the transformation. Among friends you can call G the generator of the transformation.

Orbits

A one parameter family of canonical transformations moves any given point along a curve. Denoting the curve through the point (q^i, p_i) by $(q^i(\lambda), p_i(\lambda)$ we can compute the tangent vector at the point (q^i, p_i) on the curve by choosing this point to occur, say, when $\lambda = 0$; the tangent vector is

$$(\frac{dq^i(0)}{d\lambda}, \frac{dp_i(0)}{d\lambda}) = (\frac{\partial G(q,p)}{\partial p_i}, -\frac{\partial G(q,p)}{\partial q^i}).$$

Of course we can do this anywhere in phase space since (q^i, p_i) can be any point. We see that the vector field on Γ,

$$(\delta q^i, \delta p_i) = (\frac{\partial G(q,p)}{\partial p_i}, -\frac{\partial G(q,p)}{\partial q^i}),$$

determine the motion of any given point as the canonical transformation unfolds – any given point moves along the "field lines" of the vector field. More precisely, through any given point of phase space there passes a unique curve called the *orbit* of the point. The orbit $(q^i(\lambda), p_i(\lambda))$ that passes through a chosen point (q, p) at $\lambda = 0$ is obtained by solving the following initial value problem*

$$\frac{dq^i(\lambda)}{d\lambda} = \frac{\partial G}{\partial p_i}$$

$$\frac{dp_i(\lambda)}{d\lambda} = -\frac{\partial G}{\partial q^i}$$

with initial conditions

$$q^i(0) = q^i,$$

$$p_i(0) = p_i.$$

The orbit through the point (q^i, p_i) is simply the set of points traced out by (q^i, p_i) as the 1-parameter family of canonical transformations unfolds.

As you can see, these equations are mathematically the same as Hamilton's equations with the role of time being played by λ and the Hamiltonian being given by G. We will return to this similarity shortly. For now, you can use this similarity to help you visualize the orbits of a continuous family of canonical transformations: they are obtained in the same way as the curves swept out by dynamical evolution with Hamiltonian $H = G$.

As an example, suppose

$$G = \frac{1}{2}(q^2 + p^2).$$

The orbits of the canonical tranformation generated by G are determined by

$$\frac{dq}{d\lambda} = p, \quad \frac{dp}{d\lambda} = -q.$$

The orbit through the point (q, p) is given by (exercise)

$$q(\lambda) = q\cos\lambda + p\sin\lambda, \quad p(\lambda) = p\cos\lambda - q\sin\lambda.$$

The orbits are circles except at the origin, where the orbit is a single (fixed) point. Evidently, the 1-parameter family of transformations are just (clockwise) rotations in the q-p plane. Of course this family of orbits describes the time evolution of a harmonic oscillator (with unit mass and frequency). The orbit through a given point (q_0, p_0) is mathematically the same as the time evolution in phase space of the oscillator with initial conditions (q_0, p_0).

* Mathematicians call this construction finding the *flow* of the vector field on phase space whose (q, p) components are given by $(\frac{\partial G}{\partial p_i}, -\frac{\partial G}{\partial q^i})$.

Important examples of infinitesimal generators

After all this formalism, it should be an illuminating relief to study a few simple examples.

Translations

As we have seen, an infinitesimal translation of a degree of freedom q is given by

$$\delta q = 1, \quad \delta p = 0.$$

The "orbit" traced by a point (q, p) in phase space as we vary λ is just a line parallel to the q-axis:

$$q(\lambda) = q + \lambda, \quad p(\lambda) = p.$$

You can easily check that the infinitesimal generator of this transformation is given by

$$G(q, p) = p.$$

More generally, for a system with canonical variables (q^i, p_i) the function

$$G = a^i p_i,$$

where a^i are constants, generates the transformation

$$q^i(\lambda) = q^i + a^i, \quad p_i(\lambda) = p_i.$$

As you have no doubt heard: "momentum is the generator of translations".

Rotations

Let us consider a particle moving in two dimensions. The phase space variables are (x, p_x) and (y, p_y). A rotation by an angle θ is a one parameter family of point transformations (exercise):

$$x(\theta) = x\cos\theta + y\sin\theta$$

$$y(\theta) = y\cos\theta - x\sin\theta.$$

$$p_x(\theta) = p_x\cos\theta + p_y\sin\theta$$

$$p_y(\theta) = p_y\cos\theta - p_x\sin\theta.$$

Can you visualize the orbits? Infinitesimally we get (exercise)

$$\delta x = y, \quad \delta p_x = p_y,$$

$$\delta y = -x, \quad \delta p_y = -p_x.$$

The infinitesimal generator is (exercise)

$$G = yp_x - xp_y.$$

This is the angular momentum! More generally, for a particle moving in 3 dimensions it is easy to extend this result to show that the function

$$G_i = (\mathbf{r} \times \mathbf{p})_i = \epsilon_{ij}{}^k x^i p_k$$

generates rotations about the x^i coordinate axis. Better yet, the function

$$G_{\hat{n}} = \hat{n} \cdot \mathbf{M},$$

where $\hat{n}$ is a constant unit vector and $\mathbf{M}$ is the canonical angular momentum,

$$\mathbf{M} = \mathbf{r} \times \mathbf{p},$$

generates rotations about an axis along $\hat{n}$. One way to see this is to choose your z-axis along $\hat{n}$ whence our previous 2-d result comes into play. Alternatively, as a good exercise you can check that (using vector notation for derivatives)

$$\delta\mathbf{r} = \frac{\partial G_{\hat{n}}}{\partial \mathbf{p}} = \hat{n} \times \mathbf{r}$$

and

$$\delta\mathbf{p} = -\frac{\partial G_{\hat{n}}}{\partial \mathbf{r}} = \hat{n} \times \mathbf{p},$$

which gives the infinitesimal changes of the vectors $\mathbf{r}$ and $\mathbf{p}$ under rotations about $\hat{n}$ which we discussed during our study of symmetries and conservation laws. Thus we find the oft-stated result: "angular momentum generates rotations."

Time evolution as a canonical transformation

We have seen that any function G on phase space is the infinitesimal generator of a 1-parameter family of canonical transformations

$$q^i \to q^i(\lambda), \quad p_i \to p_i(\lambda),$$

where

$$\frac{dq^i(\lambda)}{d\lambda} = \frac{\partial G}{\partial p_i}$$
$$\frac{dp_i(\lambda)}{d\lambda} = -\frac{\partial G}{\partial q^i}$$
$$.$$

We have already noted that these equations are mathematically identical to the Hamilton equations of motion. Indeed, the Hamiltonian is just a function on phase space – though it has distinguished physical significance since it is to define the motion of the system in time. With notational changes $G \to H$, $\lambda \to t$ we see that time evolution is simply a

1-parameter family of canonical tranformations. Thus we say: "the Hamiltonian is the generator of time translations".

We have the now arrived at the following amusing state of affairs. All continuous canonical transformations are mathematically equivalent to dynamical systems in which the parameter λ plays the role of time, the infinitesimal generating function plays the role of Hamiltonian, and the orbits are the solutions of the Hamilton equations. Of course, for a given dynamical system not all continuous families of canonical transformations are physically representing time evolution. For a given dynamical system, there is a distinguished function on phase space, the Hamiltonian, and a distinguished 1-parameter family of canonical transformations that we interpret as motion in time. All other functions on phase space define transformations to equivalent representations of the dynamical system. A key difference between the transformations generated by the Hamiltonian and those generated by other functions is that the time evolution canonical transformation is explicitly time dependent. Let us explore this a bit.

For a particle with 1 degree of freedom and Hamiltonian $H(q, p, t)$, consider the canonical transformations generated by

$$G = \frac{p^2}{2m}.$$

Of course, this is the Hamiltonian for free particle motion, but we are not viewing G as the Hamiltonian at the moment, *e.g.,* the system of interest could be an oscillator. The canonical transformations generated by G are, however, still *mathematically* identical to the motion of a free particle in one dimension. The orbit that passes through the point (q, p) at $\lambda = 0$ is given by (exercise)

$$q(\lambda) = q + \frac{p}{m}\lambda,$$
$$p(\lambda) = p.$$

Using our original canonical transformation notation this means the transformation is

$$q \to Q = q + \frac{p}{m}\lambda, \quad p \to P = p,$$

with inverse

$$q = Q - \frac{P}{m}\lambda, \quad p = P.$$

It is straightforward to check that, for each value of λ, the transformation

$$(q, p) \longleftrightarrow (Q, P)$$

is canonical. Indeed, we have

$$P\dot{Q} - K(Q, P, t) = p\dot{q} - H(q, p, t) + \frac{d}{dt}(\frac{p^2\lambda}{2m}),$$

where

$$K(Q, P, t) = H(Q - \frac{P}{m}\lambda, P).$$

In our previous study of the generating function $G = \frac{p^2}{2m}$ we have viewed the canonical transformations generated by G as just some continuous transformation on phase space depending upon the parameter λ. As mentioned before, we can also view time evolution itself as a canonical transformation generated by the Hamiltonian. For example, free particle dynamics would use the infinitesimal generator G as above, but now we view the transformation as time-dependent. In detail, if we identify λ with t, the time-dependent canonical transformation $(q, p) \longleftrightarrow (Q, P)$ is

$$Q = q + \frac{p}{m}t, \quad P = p$$

with inverse

$$q = Q - \frac{P}{m}t, \quad p = P.$$

Note the physical meaning of the old and new variables: (Q, P) represent the values of the coordinates and momenta at time t given that they had values (q, p) at time $t = 0$. Thus $Q(t)$ and $P(t)$ can be viewed as solutions of the Hamilton equations with Hamiltonian $H = \frac{P^2}{2m}$ and initial conditions $Q(0) = q$, $P(0) = p$. With $H(Q, P) = \frac{P^2}{2m}$, the canonical nature of the transformation can be checked by computing the change in the phase space Lagrangian (exercise). We find (exercise)

$$P\dot{Q} - \frac{P^2}{2m} = p\dot{q} + \frac{d}{dt}\left(\frac{p^2 t}{2m}\right).$$

Note that, in this way of viewing the transformation, the Hamiltonian describing motion of the (q, p) variables in time *vanishes*! This is because (q, p) are playing the role of initial data; initial conditions don't change in time. More on this shortly.

The Poisson bracket and canonical transformations

Canonical transformations define equivalent representations of a Hamiltonian system. The Poisson bracket respects this equivalence: it is *invariant* under canonical transformations. If we take the PB of two functions f and g and get a function h,

$$h = [f, g],$$

then we get the same relation using *any* set of canonical variables. To get a better idea of what this means let us consider a simple example. You can easily check that the following is a canonical transformation (good exercise) for a system with one degree of freedom $q > 0$:

$$Q = e^q, \quad P = e^{-q}p,$$

$$q = \log Q, \quad p = QP.$$

Let $f = q^2$, $g = p$. We have

$$[f, g] = 2q.$$

Now we consider the same computation in the new canonical variables. Corresponding to f and g we have

$$F(Q) = f(q(Q)) = \log^2 Q, \quad G(Q, P) = g(q(Q), p(Q, P)) = QP.$$

Their PB is

$$[F, G] = 2 \log Q = 2q(Q)$$

We see that the two functions $[f, g]$ and $[F, G]$ are related by the coordinate transfomation, *i.e.,* they assign the same number to corresponding points in the (q, p), (Q, P) coordinate systems. As another example, keep the same canonical transformation but now let $f = q$ and $g = p$. We have

$$[f, g] = 1 = [F, G]$$

as we should since the constant function is the same in all coordinate systems.

Let me emphasize that this result is a non-trivial statement about the PB. For example, if the PB were defined by

$$[f, g] = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}$$

it would not be invariant under (all) canonical transformations. Indeed, consider the canonical transformation

$$Q = p, \quad P = -q,$$

and set

$$f = q, \quad g = p$$

so that

$$F = -P, \quad G = Q.$$

Using the new (wrong) PB above we have

$$[f, g] = 1,$$

but we have

$$[F, G] = 0.$$

We have demonstrated the invariance of the PB under canonical transformations in some simple examples. See your text for a proof that the PB is invariant under all canonical transformations; we don't have time for it here. However, we can easily see invariance with respect to infinitesimal transformations as follows. We set $h = [f, g]$. Under an infinitesimal transformation generated by G, we have (using the Jacobi identity and anti-symmetry of the bracket)

$$\begin{aligned}\delta h &= [h, G] \\ &= -[G, h]] \\ &= -[G, [f, g]] \\ &= [f, [g, G]] + [g, [G, f]] \\ &= [f, \delta g] + [\delta f, g]\end{aligned} .$$

The left hand side is the infinitesimal change in the result of the PB. The right hand side is just the effect of changing f and g by the infinitesimal transformation. This means that the change in the PB is only that induced by the change in coordinates.

Given the invariance of the PB under canonical transformations, it follows that the canonical PB relations between coordinates and momenta must take exactly the same form for all choices of canonical coordinates and momenta since the PB's are constant functions (as in our example above). In fact, it can be shown that *a transformation* $(q^i, p_i) \longleftrightarrow (Q^i, P_i)$ *is canonical if and only if*

$$\begin{aligned}&[q^i, q^j] = [Q^i, Q^j] = 0 \\ &[p_i, p_j] = [P_i, P_j] = 0 \\ &[q^i, p_j] = [Q^i, P_j] = \delta_{ij}.\end{aligned}$$

Here we view $Q = Q(q, p, t)$ and $P = P(q, p, t)$ as functions of the variables (q, p) – as we did in our example. Otherwise, the above result is trivial (exercise).

As an example, let us return to the canonical transformation

$$Q = q + \frac{p}{m}\lambda, \quad P = p$$

Because $[f, f] = 0$ (exercise), it is immediate that

$$[Q, Q] = 0 = [P, P],$$

and it is easy to see that

$$[Q, P] = [q + \frac{p}{m}\lambda, p] = [q, p] = 1.$$

Often times the easiest way to create/verify a canonical transformation is to verify the canonical PB relations. For example, consider a point transformation for a system with one degree of freedom,

$$q \to Q = Q(q).$$

Here $Q(q)$ is some given invertible function. How to complete this to a canonical transformation? Define $P = P(q,p)$ and compute

$$[Q,P] = \frac{dQ}{dq}\frac{\partial P}{\partial p}.$$

So we must have

$$\frac{dQ}{dq}\frac{\partial P}{\partial p} = 1.$$

Now, $\frac{dQ}{dq}$ must be non-vanishing for the transformation to be invertible. So we have

$$P = \frac{dq}{dQ}p + f(q),$$

where $f(q)$ is any function. We can verify from the phase space Lagrangian that this is indeed a canonical transformation:

$$P\dot{Q} = (\frac{dq}{dQ}p + f(q))\frac{dQ}{dq}\dot{q} = p\dot{q} + \frac{dh}{dt},$$

where

$$h(q) = \int dq\, f(q)\frac{dQ}{dq} + \text{const}.$$

Symmetries and Conservation Laws

We will now explore a very beautiful formulation of the link between symmetries and conservation laws. In fact, by investing our time in all this canonical transformation business we are now in a position to show how this link is virtually an obvious identity!

Symmetries

We have seen that canonical transformations relate equivalent representations of a given dynamical system. The change from one representation to another involves a redefinition of the Hamiltonian if the transformation is time dependent. If the transformation is time independent, the functional form of the Hamiltonian changes merely by virtue of the change of variables alone. For simplicity, let us restrict attention to time independent canonical transformations. While, in general, such a transformation will change the functional form of the Hamiltonian (just as the functional form of a function $f(x,y)$ will change when passing to polar coordinates), particular Hamiltonians may admit special transformations which happen to leave their functional form unchanged (just as a function $f(x,y)$ depending only upon the distance from the origin will be unchanged by a rotation about the origin). Let us suppose we have a Hamiltonian which admits such a canonical transformation, *i.e.,*

$$K(Q,P,t) \equiv H(q(Q,P,t), p(Q,P,t), t) = H(Q,P,t).$$

Such a canonical transformation is called a *symmetry* of the dynamical system since the Hamiltonian, which governs the behavior of the system, is insensitive to the transformation.

A simple example of a symmetry is the translation of a free particle in one dimension (Hamiltonian $H(q,p,t) = \frac{p^2}{2m}$):

$$Q = q + \lambda, \quad P = p.$$

We have (exercise)

$$K(Q,P,t) = \frac{[p(Q,P,t)]^2}{2m} = \frac{P^2}{2m} = H(Q,P,t).$$

Another way to display the symmetry mathematically is that under the replacement $q \to q + \lambda$, $p \to p$, the Hamiltonian doesn't change:

$$H(q+\lambda,p,t) = \frac{p^2}{2m} = H(q,p,t).$$

To see what it means for a Hamiltonian *not* to be invariant, consider a particle moving in the plane governed by a Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \alpha x^2 + \beta y.$$

Evidently, this particle sees a harmonic oscillator type of interaction in the x direction and a constant force interaction along the y-direction. Physically, the different interactions distinguish the x and y directions – for this reason there is no rotational symmetry. Indeed, you can easily check that a rotation in the plane changes the form of the Hamiltonian. The kinetic energy is rotationally invariant, but under a rotation by an angle α,

$$x \to x\cos\alpha + y\sin\alpha, \quad y \to y\cos\alpha - x\sin\alpha,$$

the potential energy

$$V(x,y) = \alpha x^2 + \beta y \longrightarrow \alpha(x^2\cos^2\alpha + 2xy\sin\alpha\cos\alpha + y^2\sin^2\alpha) + \beta(y\cos\alpha - x\sin\alpha) \neq V(x,y).$$

The transformed potential – and the correpsondingly transformed Hamiltonian — describes the same system in the rotated reference frame. The value of the Hamiltonian at corresponding points in the two reference frames is the same, but that does *not* mean the Hamiltonian has rotational symmetry. For the system to admit a symmetry the Hamiltonian should be completely insensitive to the transformation - it should "look the same" after the transformation.

On the other hand, consider a particle moving in a central potential in the plane. The Hamiltonian is of the form

$$H(\vec{r},\vec{p}) = \frac{p^2}{2m} + V(r), \quad r = \sqrt{x^2+y^2},$$

e.g.,

$$V = \alpha(x^2 + y^2).$$

Under a point canonical transformation the position and momentum rotate in the usual vectorial way:

$$\vec{r} \to \vec{r}\,' = R(\alpha)\vec{r}, \quad \vec{p} \to \vec{p}\,' = R(\alpha)\vec{p}.$$

Here we have denoted the 2×2 matrix representing a rotation by an angle α by $R(\alpha)$. The Hamiltonian only depends upon the lengths of these vectors, (r, p), which are rotationally invariant:

$$r' = r, \quad p' = p.$$

Thus, for a particle moving in a central potential

$$H(\vec{r}\,', \vec{p}\,') = H(\vec{r}, \vec{p}),$$

and the rotational point canonical transformation is a symmetry of Hamiltonians of this form.

Now, given a Hamiltonian $H(q, p, t)$, suppose that there is a 1-parameter family of canonical transformations

$$(q^i, p_i) \to (q^i(\lambda), p_i(\lambda))$$

which is a symmetry of the Hamiltonian:

$$H(q(\lambda), p(\lambda)) = H(q, p).$$

This implies that the infinitesimal change in H due to the transformation vanishes:

$$\frac{d}{d\lambda} H(q(\lambda), p(\lambda)) = 0.$$

Another way to write this expression is

$$\delta H = \frac{\partial H}{\partial q^i} \delta q^i + \frac{\partial H}{\partial p_i} \delta p_i = 0,$$

where δq^i and δp_i are the infinitesimal changes in the canonical variables due to the canonical transformations. Denoting the generating function by G, we have

$$\delta H = [H, G] = 0.$$

For example, in our translational example above, we have

$$H(q, p, t) = \frac{p^2}{2m}, \quad G = p,$$

and

$$\delta H = [H, G] = \frac{\partial H}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial G}{\partial q} = 0.$$

As a good exercise you should check that in our rotational example, with

$$G = xp_y - yp_x,$$

we get

$$\delta H = [\frac{p^2}{2m} + U(r), G] = 0.$$

Conservation Laws

A conserved quantity is any function on the phase space which does not change in time as one evolves the system according to Hamilton's equations. Let me belabor this. The time variation of some physical quantity $F(q,p)$ arises by (i) solving Hamilton's equations to get a curve $(q^i(t), p_i(t))$ in phase space, and (ii) evaluating $F(q,p)$ on the curve to get $F(t) \equiv F(q(t), p(t))$. The time rate of change of F for the given curve in phase space is

$$\frac{dF(t)}{dt} = \frac{\partial F(q,p)}{\partial q^i}\bigg|_{q(t),p(t)} \dot{q}^i(t) + \partial F(q,p)\bigg|_{q(t),p(t)} \dot{p}_i = [F(q,p), H]\bigg|_{q(t),p(t)}.$$

Thus, a quantity $F = F(q,p)$ is conserved if

$$\frac{dF}{dt} = [F, H] = 0.$$

But we have just seen that a continuous symmetry implies the Hamiltonian has a vanishing PB with the generator G of the symmetry. Thus the symmetry condition immediately implies conservation of the generator:

$$\dot{G} = [G, H] = -[H, G] = 0.$$

As you can see, in the Hamiltonian formulation of mechanics the existence of a symmetry:

$$[H, G] = 0$$

is the same as the existence of a conservation law

$$[G, H] = 0$$

because of the identity $[G, H] = -[H, G]$. In this sense we can say that, in the Hamiltonian formulation of mechanics, symmetries and conservation laws are identified—they are "two sides of the same coin".

Theorem. *Given a Hamiltonian system* (Γ, H)*, the infinitesimal generating function* $G: \Gamma \to R$ *of a continuous symmetry of* H *is conserved by the time evolution generated by* H*. Every conserved quantity for a Hamiltonian system generates a continuous symmetry of that system.*

Simple examples of this result are as follows.

Translational symmetry and conservation of momentum

Suppose that the Hamiltonian does not depend upon a given canonical variable, say, q^1:

$$\frac{\partial H}{\partial q^1} = 0.$$

Then, clearly, under the canonical transformation

$$q^1 = q^1 + \lambda,$$

(all other variables unchanged) we have H being unchanged:

$$H(q^1 + \lambda, q^2, \dots, p_1, p_2, \dots, t) = H(q^1, q^2, \dots, p_1, p_2, \dots, t).$$

Put differently, we have (exercise)

$$[H, p_1] = 0.$$

But this means that, using Hamilton's equations (exercise),

$$\dot{p}_1 = 0.$$

Thus translational symmetry in a canonical variable implies that its associated momentum is conserved.

Note: the "coordinate" being translated may have a very different physical meaning than the familiar rectilinear coordinate of a particle. The conserved quantity may therefore have a different physical interpretation than what we usually call "momentum".

Rotational Symmetry and Conservation of Angular Momentum

Suppose a particle, with position $\mathbf{r}$ and canonical momentum $\mathbf{p}$ moves in a potential that is rotationally invariant, *i.e.,*

$$H = \frac{p^2}{2m} + U(r).$$

Then it is easy to see that the Hamiltonian is rotationally invariant under the rotational canonical transformation (good exercise). This means that the PB of H with any component of the angular momentum must vanish, *i.e.,* (canonical) angular momentum is conserved. It is of course frequently stated in the literature that rotational symmetry implies conservation of angular momentum. In the Hamiltonian formulation, rotational symmetry *is* conservation of angular momentum!

Time translation symmetry and conservation of energy

Suppose the Hamiltonian of a system doesn't depend upon time:

$$H = H(q, p).$$

Obviously, as a function on phase space H does not vary in time. This implies that the value of the Hamiltonian will not change as one moves along any curve in phase space satisfying Hamilton's equations, *i.e.,* H is conserved. One often writes

$$\frac{\partial H}{\partial t} = 0 \Longleftrightarrow \frac{dH}{dt} = 0$$

In formulas, the change in the Hamiltonian under time evolution is:

$$\frac{dH}{dt} = [H, H] = 0.$$

Thus if the Hamiltonian has time translation as a symmetry this is the same as saying that it is a constant of the motion.

We finish this discussion with an important fact: If two functions $f(q,p)$ and $g(q,p)$ are conserved, then so is their Poisson bracket $[f, g]$. To see this we use the Jacobi identity. Let H be the Hamiltonian, we have

$$\frac{d[f,g]}{dt} = [[f, g], H] = -[[g, H], f] - [[H, f], g].$$

By assumption f and g are conserved, so that their PBs with H must vanish, and we get our result. This result can be useful for finding new conserved quantities. But it is possible that the PB of two known conservation laws results in a constant function, *e.g.,* 0, which is certainly conserved but trivially so (e.g., consider a system with a conserved momentum and a conserved Hamiltonian – what does the PB give you?). Sometimes the PB of two known conservation laws results in a (constant multiple of another) known conservation law. But, if you have two (or more) conservation laws you should always take all their PBs to see if there are additional conserved quantities.

Symmetry groups

Let's summarize our discussion thus far. We have seen that a continuous canonical transformation is generated by a function $G(q, p)$ – the infinitesimal generator – via

$$\delta q^i = [q^i, G], \quad \delta p_i = [p_i, G].$$

If the Hamiltonian H is unchanged by this transformation – G generates a symmetry of the dynamical system – then

$$[H, G] = 0,$$

The Hamilton equations then imply

$$\dot{G} = [G, H] = -[H, G] = 0.$$

Thus the infinitesimal generator of a symmetry is a conserved quantity. Conversely, if a function $G(q, p)$ is conserved then it generates a symmetry of the Hamiltonian.

We have also seen that (because of the Jacobi identity) given two conserved quantities $f(q, p)$ and $g(q, p)$ (and hence two symmetries) their Poisson bracket $[f, g]$ is also conserved (and is also generator of a symmetry). Now imagine we have (somehow) found *all* the conserved quantities, or at least we have found a set of conserved quantities whose PBs yield no new conserved quantities. In this case an important mathematical structure emerges which has proved to be of no small significance in physics. This structure is that of a *Lie Group*. Lie groups arise in a number of contexts. For us, their significance is that they characterize the symmetry structure of a given system. I would like to explore this a little bit.

Recall that a group is a set G, with elements $(g_1, g_2, \ldots) \in G$, equipped with a binary operation which takes any pair of elements g_1 and g_2 and defines a third element $g_1 \cdot g_2 \in G$. This binary operation is called (perhaps misleadingly) "group multiplication" or "group product". This product must be associative:

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \equiv g_1 \cdot g_2 \cdot g_3.$$

It is also required that there is a distinguished element $e \in G$, called the identity, which satisfies for any $g \in G$

$$e \cdot g = g \cdot e = g.$$

Finally, for G to be a group every element $g \in G$ must have an "inverse" $g^{-1} \in G$ which satisfies

$$g^{-1} g = g g^{-1} = e.$$

In physics, groups usually arise as sets of transformations. As an example, consider the group G of rotations of the x-y plane about the origin. Each rotation is specified by an angle θ. Thus G is a 1-d space (in fact it is a circle). The group multiplication takes two rotations, specified by θ_1 and θ_2 and makes a third rotation by $\theta_1 + \theta_2$. Clearly this "product" is associative (and commutative in this example). The identity transformation is the rotation by the angle $\theta = 0$. The inverse of a rotation by θ is a rotation by $-\theta$. This example has the group elements labeled by a continuously variable parameter (the angle). Groups whose elements are labeled by continuous parameters are called *Lie Groups* (named after Sophus Lie who pioneered the concept in the late 1800s).

As another example, let us consider the (Lie) group of all rotations in 3-d space about some chosen origin. As you know, a rotation is characterized by an axis (2 parameters) and an angle (1 parameters). Thus the set of rotations is a 3-d space, a 3-d Lie group. A group element can be characterized by a 3×3 matrix R, which transforms a (column) vector v to the rotated vector Rv. For example, a rotation by θ about the z axis is given by

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

These 3 matrices are examples of *orthogonal* transformations, which are linear transformations satisfying $R^T = R^{-1}$. Rotations are represented by orthogonal transformations because rotations preserve the dot product. In matrix notation, the dot product of two vectors v and w is $v^T w = w^T v$. Under a rotation R this dot product must not change

$$v^T w = (Rv)^T(Rw) = v^T R^T R w \Longrightarrow RR^T = I.$$

This implies that $\det(R) = \pm 1$. Since, by a choice of axes any single rotation can be put into the form of, say, R_z, and since R_z has unit determinant, we see that rotations are orthogonal transformations of unit determinant. It can be shown, conversely, that every orthogonal transformation of unit determinant is a rotation (about some axis by some angle). Therefore the group of rotations can be viewed as the set of such matrices with the group product being the matrix product, which is associative. It is easy to see that the identity is such a matrix. And, of course, given R, R^T is also such a matrix and is the inverse to R. Thus the set of orthogonal matrices with unit determinant is a group, called $SO(3)$; it is the rotation group.*

The rotation group in the plane can be obtained as a *subgroup* of $SO(3)$; its elements are simply $R_z(\theta)$. This group is called $SO(2)$. Note that the order of rotations in 2-d does not matter:

$$R_z(\theta_1)R_z(\theta_2) = R_z(\theta_1 + \theta_2) = R_z(\theta_2)R_z(\theta_1).$$

* In general, the set of orthogonal transformations of an n dimensional vector space is a group called $O(n)$. The set of orthogonal transformation with unit determinant is also a group and is called $SO(n)$.

We say that $SO(2)$ is a commutative or *Abelian* group. In 3-d the order of rotations *does* matter. For example, you can check that $R_z(\theta)R_x(\phi) \neq R_x(\phi)R_z(\theta)$.

What does all this have to do with symmetries and conservation laws? The set of canonical transformations which leaves the Hamiltonian unchanged – the symmetries – forms a group (usually labeled by continuous parameters, hence a Lie group). This is the *symmetry group* of the dynamical system. Let us write the elements of this group as $\phi_1, \phi_2, \ldots$. The element ϕ is a canonical transformation taking a point $Z^\alpha = (q^i, p_i)$ in phase space to the point

$$\tilde{Z}^\alpha = \phi^\alpha(Z),$$

and such that

$$H(\phi(Z)) = H(Z).$$

Thus ϕ is a coordinate transformation on phase space which (i) only changes the form of the phase space Lagrangian by addition of a time derivative, (ii) does not change H. The group product is *composition* of the transformations:

$$(\phi_1 \cdot \phi_2)(Z) = \phi_1(\phi_2(Z)).$$

Evidently the composition of two symmetries is another symmetry (exercise). The identity element e is the identity transformation:

$$e(Z) = Z,$$

which is clearly a symmetry. Since canonical transformations are invertible, the inverse transformation is used for the group inverse:

$$\phi^{-1}(\phi(Z)) = Z.$$

As a nice exercise you can check that if ϕ is a symmetry, then so is ϕ^{-1}. Thus the set of symmetries of a dynamical system form a group. Often times one can find a subset of all possible symmetries which forms a group by itself, this is a subgroup of the symmetry group.

Example: Rotational symmetry of central force problems

Let us consider a particle moving in a central force field. The Hamiltonian is, as usual,

$$H = \frac{p^2}{2m} + V(r).$$

Here we can view position and momentum as column vectors $\vec{r}$, $\vec{p}$ and

$$p^2 = (\vec{p})^T \vec{p}, \quad r^2 = (\vec{r})^T \vec{r}.$$

We already know that rotations act on this system as a point canonical transformation. Let $R \in SO(3)$ be the rotation of interest, the transformation is

$$\vec{r} \to R\vec{r}, \quad \vec{p} \to R\vec{p}.$$

We know that this transformation will not change p^2 and r^2. Thus the Hamiltonian is invariant under rotations. We say that $SO(3)$ is a symmetry (sub)group for central force problems.

Note however that here $SO(3)$ is being used in a slightly different way than in our previous example of 3-d rotations. Here we are using elements of $SO(3)$ to perform transformations on a 6-d space. Nonetheless, the transformations on the 6-d space completely "mimic" the transformations we used to define $SO(3)$. In particular, to every element of $SO(3)$ there is defined a corresponding transformation on the phase space. The products of two transformations on phase space corresponds to the product of the two transformations of $SO(3)$. We say that we are using a different *representation* of $SO(3)$. In general, any time you have a group G and you find a set of transformations on some space which corresponds to the elements of G and whose composition corresponds to the group product we say we have a representations of G. In mechanics the set of symmetries always is a representation of some group, which we call the symmetry group of the dynamical system.

Let us note that the complete symmetry group for central force problems is larger than SO(3). First of all, we know that

$$\frac{dH}{dt} = 0.$$

This means that if we replace $\vec{r}$ and $\vec{p}$ with any of the 1-parameter family $\vec{r}(\lambda)$, $\vec{p}(\lambda)$, which are solutions of the Hamilton equations with $\vec{r}(0) = \vec{r}$ and $\vec{p}(0) = \vec{p}$, the λ dependence will drop out – we then get a 1-parameter family of additional symmetries. There may be even more symmetries, depending upon the form of $V(r)$; more on this later. As an example of this time translation symmetry. Consider a harmonic oscillator-type Hamiltonian:

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}q^2.$$

H generates the canonical transformations

$$q(\lambda) = q\cos\lambda + p\sin\lambda, \quad p(\lambda) = p\cos\lambda - q\sin\lambda.$$

You can easily check that

$$H(q(\lambda), p(\lambda)) = \frac{1}{2}(p\cos\lambda - q\sin\lambda)^2 + \frac{1}{2}(q\cos\lambda + p\sin\lambda)^2 = H(q,p).$$

The symmetry group generated by H is the group of rotations of the phase plane, so it is $SO(2)$, although, of course, the physical meaning of the transformations is different than the rotations in the x-y plane.

Symmetry algebras

If the symmetry (sub)group of a dynamical system is a Lie group, that is, if the symmetries are continuous transformations, then we can consider infinitesimal transformations. The infinitesimal transformations lead to an infinitesimal analog of the Lie Group: the *Lie algebra*. Rather trying to explain this concept in all generality, let us focus on a key example and then have a look at symmetries.

Let us return to rotations in 3-d. We have displayed 3 (of the infinity of) rotation matrices, corresponding to rotations about the x, y, and z axes. Let us expand these to first order to get the infinitesimal forms:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{pmatrix} + \dots,$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{pmatrix} + \dots,$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \dots.$$

Evidently for each of $i = (1,2,3) = (x,y,z)$ we have

$$R_i(\theta) = I + \theta G_i + \dots,$$

where

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The G_i are called the infinitesimal generators of the rotation (about x, y and z) because they define the infinitesimal rotation and, by iteration, any finite rotation as well. Note that the generators are anti-symmetric matrices. This is easy to understand. Any rotation R is an orthogonal matrix:

$$I = RR^T.$$

Infinitesimally we have

$$I = (I + \theta G + \dots)^T (I + \theta G + \dots) = I + \theta(G^T + G) + \mathcal{O}(\theta^2) \quad \Longrightarrow \quad G^T = -G.$$

It is easy to see that any anti-symmetric matrix can be expressed as a linear combination of the G_i, so the three infinitesimal infinitesimal generators exhibited above can be used to define any rotation.

There is an important relation among the generators:

$$[G_i, G_j] = \epsilon_{ijk} G_k,$$

which you can easily check by explicit computation. These commutation relations characterize the product of two rotations at the infinitesimal level. Indeed, consider the commutator of two infinitesimal rotations:

$$\begin{aligned} R_x(\alpha)R_y(\beta) - R_y(\beta)R_x(\alpha) &= (I + \alpha G_x + \ldots)(I + \beta G_y + \ldots) - (I + \beta G_y + \ldots)(I + \alpha G_x + \ldots) \\ &= \alpha\beta[G_x, G_y] + \ldots . \end{aligned}$$

Another way to look at this is to consider the infinitesimal change in the position vector under two successive rotations. Writing

$$\delta_i \vec{r} = G_i \vec{r},$$

we have

$$\vec{r} \longrightarrow R_i(\theta)\vec{r} = \vec{r} + \theta\delta_i\vec{r} + \ldots,$$

and we have (exercise)

$$\delta_i\delta_j\vec{r} - \delta j\delta_i\vec{r} = [G_i, G_j]\vec{r} = \epsilon_{ijk} G_k \vec{r}.$$

Although we shall not prove it here, the commutator of the generators provides the infinitesimal form of the group product rule. In particular, the fact that the commutator of two generators gives another generator is the group "closure" property. As you may know, the commutator of linear operators satisfies the Jacobi identity. This is the infinitesimal form of the associativity of the group product. It is a fundamental result of Lie group theory that the infinitesimal generators and their commutation relations completely characterize the Lie group – one can (with a few technical subtleties) reconstruct the group from its infinitesimal generators. This infinitesimal form of the group is known as a *Lie algebra*. In general, a Lie algebra is a vector space of generators (here the space of anti-symmetric matrices) along with an anti-symmetric "product" or "bracket", which takes any two elements of the vector space and produces another such element, and which satisfies the Jacobi identity* (here the commutator). Here we have obtained the Lie algebra of SO(3), usually denoted by $so(3)$. In general, if we pick a basis G_i for the vector space of generators, then the Lie bracket must be of the form

$$[G_i, G_j] = C_{ij}^k G_k.$$

* Note that the Jacobi identity implies that, in general, Lie algebras are non-associative algebras.

The constants C_{ij}^k are called the *structure constants* of the Lie algebra. Roughly speaking, the structure constants completely characterize the algebra (and hence the group). We see that for rotations the structure constants of $so(3)$ are $C_{ij}^k = \epsilon_{ij}{}^k$.

Let us now return to the group of symmetries of a Hamiltonian dynamical system. We have briefly sketched the structure of this Lie group. Let us now uncover the Lie algebra. Denoting again the points in phase space by $Z^\alpha = (q^i, p_i)$, the infinitesimal transformation corresponding to some continuous symmetry $Z^\alpha \to Z^\alpha(\lambda)$ is given by

$$Z^\alpha \to Z^\alpha + \lambda \delta Z^\alpha + \dots,$$

where

$$\delta Z^\alpha = [Z^\alpha, G],$$

and G is the infinitesimal generating function for the symmetry. We have seen that the PB of two symmetry generators is again a symmetry generator. Assuming we have found all symmetry generators (or at least a subset which closes under the PB) we have again constructed a Lie algebra – the Lie algebra of symmetries. Indeed, the vector space is the vector space of symmetry generating functions (clearly one can add symmetry generating functions and multiply with constants to get again a symmetry generating function). The Lie algebra bracket is the PB, which does satisfy the Jacobi identity.

Let us return to our example of a particle moving in a central potential. We have seen that the rotational invariance of the Hamiltonian means that the infinitesimal generators of the symmetry – the angular momenta:

$$\vec{L} = \vec{r} \times \vec{p},$$

are conserved. Since these generate a symmetry (Lie) group, we can ask what is the Lie algebra? It is easy to check that the components of the angular momentum satisfy the following PB relations (exericse)

$$[L_i, L_j] = \epsilon_{ijk} L_k.$$

You might have guessed this result. We have already seen that the rotational symmetry transformations of the Hamiltonian system provides a representation of $SO(3)$. Likewise, the infinitesimal symmetry transformations provide a representation of Lie algebra $so(3)$.*

The set of all conserved quantities for a Hamiltonian dynamical system form a (representation of a) Lie algebra, as outlined above. It is almost always easier to characterize

* As mentioned previously, the complete symmetry group of the system is bigger than $SO(3)$, likewise the infinitesimal symmetry algebra is bigger than $so(3)$. The rotations provide a subgroup/subalgebra.

the symmetries of a Hamiltonian system in terms of Lie algebras instead of Lie groups. A basic job for any serious classical (or quantum) mechanic is to find the Lie algebra of symmetries of the dynamical system of interest. This amounts to finding a basis for the vector space of conserved quantities, G_i, and computing their PBs to find the structure constants:

$$[G_i, G_j] = C_{ij}^k G_k.$$

The symmetry algebra of the Kepler problem

Let us consider the relative motion Kepler problem. The conservation laws for the (relative motion) Kepler problem include the energy H, angular momentum $\vec{L}$ and Laplace-Runge-Lenz (LRL) vector $\vec{A}$. What symmetries do they generate? Well, we have of course time translation and rotations, but what of the LRL vector?

To get a manageable answer to this question let us get rid of the time translation symmetry as follows. First, we note that since both $\vec{L}$ and $\vec{A}$ have vanishing PBs with H, the transformations they generate will not change the value of the energy, that is, they will generate transformations in phase space which move among phase space points all having the same value of H. Focus attention on the subset of phase space which has a given fixed value E for the Hamiltonian. For the moment, let us focus on the bound states where $E < 0$. To identify the Lie algebra of symmetries it is convenient to work with slightly modified conserved quantities. Define

$$\vec{D} = \frac{\vec{A}}{\sqrt{2m|E|}}.$$

A straightforward (but tedious) computation of PBs then yields:

$$[D_i, L_j] = \epsilon_{ijk} D_k,$$

and

$$[D_i, D_j] = \epsilon_{ijk} L_k.$$

The first of these results is easy enough to understand. Recall that L_i generates infinitesimal rotations about the x^i axis. The PB $[D_i, L_j]$ is then the infinitesimal change of the i^{th} component of $\vec{D}$ under a rotation about x^j. Recall that for any vector, say, $\vec{D}$, its infinitesimal change under a rotation about an axis $\hat{n}$ is given by

$$\delta_n \vec{D} = \hat{n} \times \vec{D}.$$

This means (exercise)

$$\delta_n D_x = n_y D_z - n_z D_y$$

and so forth (cyclic permutations). You can then easily see that, choosing the rotation axis to be x, y or z:

$$(\delta_j \vec{D})_i = \epsilon_{ijk} D_k,$$

in accord with the $[D, L]$ PB. You can also understand the PB

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

in the same fashion (exercise). Indeed, for any vector with Cartesian components V_i constructed solely from $\vec{r}$ and $\vec{p}$ using dot and cross products (no other "fixed vectors", *e.g.,* a prescribed electric field) one will have

$$[V_i, L_j] = \epsilon_{ijk} V_k.$$

To summarize, the symmetry algebra of bound state motion with a given energy is given by the PBs of 6 generators $\vec{L}$ and $\vec{D}$:

$$[L_i, L_j] = \epsilon_{ijk} L_k, \quad [D_i, L_j] = \epsilon_{ijk} D_k, \quad [D_i, D_j] = \epsilon_{ijk} L_k.$$

It is not too hard to verify that this algebra is providing a representation of $so(4)$, the Lie algebra of $SO(4)$ (orthogonal 4×4 matrices with unit determinant). Indeed, just as we did with $so(3)$ you can see that $so(4)$ can be identified with the algebra of antisymmetric 4×4 matrices. Every such matrix is a linear combination of 6 basic matrices which correspond to infinitesimal rotations in a single $x^i - x^j$ plane, $i, j = 1, 2, 3, 4$. Here then we have a representation of $SO(4)$ on the 6-d phase space of the (relative) Kepler problem. Similar computations uncover the symmetry algebra in the cases $E \geq 0$ (see the text).

The $SO(4)$ symmetry group of the (reduced) Kepler problem has been much studied. In the quantum theory of a hydrogenic atom the same algebra appears (now represented via Hermitian operators and commutators). Just using the fact that the quantum versions of $\vec{L}$ and $\vec{D}$ satisfy the $so(4)$ algebra one can completely deduce the spectrum of such atoms without solving any differential equations!