

Physics 6010, Fall 2010

Simultaneous Diagonalization of Quadratic Forms. Forced Oscillations.

Relevant Sections in Text: §6.1–6.3, 6.5

A Nice Mathematical Interpretation of Normal Modes: Simultaneous diagonalization of quadratic forms

It is worthwhile giving an alternative description of the derivation of normal modes and characteristic frequencies. This alternative description involves the notion of *simultaneous diagonalization of quadratic forms*. The idea is that the construction of the normal modes amounts to diagonalizing the matrices K and M , and the characteristic frequencies are coming from the diagonal entries of these matrices. As we shall see, strictly speaking, K and M are better thought of as quadratic forms—equivalently, symmetric tensors. One payoff of this slightly more sophisticated point of view is that it explains why we always get a basis of eigenvectors for $M^{-1}K$ and hence why any motion of the dynamical system near equilibrium can be viewed as a superposition of the normal modes..

Recall that M can be viewed as assigning a number $M(\vec{v})$ (twice the kinetic energy) to a vector \vec{v} (a velocity) via

$$M(\vec{v}) = M_{ij}v^i v^j.$$

The vectors (displacements, velocities) *etc.* are elements of a vector space. Denote the basis for the vector space as \vec{e}_i , $i = 1, 2, \dots, n$. Consider a change of basis $\vec{e}_i \rightarrow \vec{e}'_i$ in which the components of the vector \vec{v} change via

$$v^i \rightarrow v'^i = \Lambda_j^i v^j.$$

In a matrix notation:

$$v' = \Lambda v.$$

The kinetic energy — a physically observable quantity — cannot change under a change of basis; the components of M must therefore change as (exercise)

$$M'_{ij} = S_i^k S_j^l M_{kl},$$

or, in matrix notation,

$$M' = S^T M S,$$

where $S = \Lambda^{-1}$. This type of transformation of a matrix is called a *congruence transformation*.

Note that this is *not* the same transformation rule as that of a matrix representing a linear operator. A matrix L representing a linear operator transforms under a change of

basis as a *similarity transformation*: $L \rightarrow S^{-1}LS$. Thus a quadratic form (equivalently, a symmetric rank 2 tensor) is not the same as a linear operator. If the change of basis is an orthogonal transformation, for which $S^T = S^{-1}$, then the distinction between transformation properties disappears.

Because the quadratic form M is positive definite, we can use it to define a scalar product:

$$(\vec{v}, \vec{w}) \equiv M(\vec{v}, \vec{w}) = M_{ij}v^i w^j.$$

Exercise: Show that this does define a scalar product.

As you know, we can always find an orthonormal basis for a vector space with scalar product. Thus we can always find a basis \vec{e}_i for which

$$M(\vec{e}_i, \vec{e}_j) = \delta_{ij}.$$

In this basis the kinetic energy is a sum of squares (exercise):

$$M(\vec{v}) = (v^1)^2 + (v^2)^2 + \dots + (v^n)^2.$$

Thus we have “diagonalized” the quadratic form defined by M .

Under the change to an orthonormal basis with respect to the scalar product defined by M , the matrix K representing the potential energy quadratic form will change by a congruence transformation to some other matrix, not necessarily diagonal. We now consider making a further change of basis characterized by a matrix S that keeps M unchanged from its diagonal form but which diagonalizes K . This change of basis that preserves M must be an orthogonal transformation since in the current basis

$$S^T M S = M \implies S^T S = I.$$

Under an orthogonal transformation the symmetric matrix K transforms via similarity transformation. But it is a standard result from linear algebra that every symmetric matrix can be diagonalized by a similarity transformation with an orthogonal matrix. Note, however, that this result does not allow us to turn K into the identity matrix, only a diagonal matrix. Thus we can always simultaneously diagonalize any two quadratic forms, provided one of them is positive definite so that it can be used to define a scalar product.

Here is a slightly different point of view of the last step in which K is diagonalized. In the basis in which M is the identity matrix but K is not yet diagonalized, our original computation of the characteristic frequencies and normal modes comes from solving

$$(\omega^2 I - K)\vec{a} = 0,$$

where K is the matrix computed arising from the congruence transformation that made M the identity. Evidently, in this basis we are just solving

$$K\vec{a} = \omega^2\vec{a},$$

i.e., the eigenvalue problem for K in this basis. Finding the rotation which diagonalizes K is then essentially equivalent to solving this eigenvalue problem since the eigenvectors define an orthonormal basis in which K is diagonal.

A nice – if somewhat abstract – way to summarize this result is that M defines an inner product relative to which $M^{-1}K$ is symmetric. Every symmetric operator on an inner product space admits a basis of eigenvectors. Finding these eigenvectors amounts to finding the normal modes.

This way of doing the analysis also has the virtue of making it clear the solutions one gets (vectors and frequencies) define a basis for the vector space. This guarantees that all solutions of the EL equations can be obtained from the normal modes by superposition.

When we work in the basis in which the two quadratic forms M and K are diagonal, we are using the directions defined by the normal modes to define new generalized coordinates. More precisely, if we make a change of generalized coordinates by making a change of basis that diagonalizes M and K as described above, then in this system of generalized coordinates (denoted Q^α , $\alpha = 1, \dots, n$) the Lagrangian will have the form (exercise):

$$L = \frac{1}{2} \sum_{\alpha=1}^n \left(\dot{Q}^{\alpha 2} - \omega_\alpha^2 Q^{\alpha 2} \right).$$

The relation between the Q^α and the normal modes Θ_α defined earlier is that the former are normalized so that the kinetic energy is just a sum of squares.

The relation between the normal modes Q^α and the original coordinates x^i is as follows. Let the original vector space basis be denoted by \vec{b}_i , $i = 1, 2, \dots, n$, and let the basis which diagonalizes M and K be denoted by \vec{e}_α (these are the normalized eigenvectors). We have the change of basis matrices

$$\vec{b}_i = S_i^\alpha \vec{e}_\alpha, \quad \vec{e}_\alpha = S_\alpha^i \vec{b}_i.$$

We have

$$\vec{x} = x^i \vec{b}_i = Q^\alpha \vec{e}_\alpha,$$

so that

$$x^i = S_\alpha^i Q^\alpha.$$

Forced Oscillations

A typical scenario in which small oscillations is relevant is where one has a system in stable equilibrium which is subjected to an external force \vec{F} which moves the system

from equilibrium. We allow this force to be time dependent. This introduces an additional potential energy term V_1 to the quadratically approximated Lagrangian given by

$$V_1 = -\vec{F}(t) \cdot \vec{x}.$$

Using the normal modes of vibration, the motion of the system can be reduced to a collection of uncoupled – but now forced – oscillators. Indeed, let us relate the original vector space basis, say \vec{b}_i , to that provided by the normal modes, \vec{e}_α by the change of basis matrix S :

$$\vec{b}_i = S_i^\alpha \vec{e}_\alpha, \quad \vec{e}_\alpha = S_\alpha^i \vec{b}_i.$$

We then have

$$\vec{x} = x^i \vec{b}_i = Q^\alpha \vec{e}_\alpha,$$

where

$$Q^\alpha = S_i^\alpha x^i,$$

and

$$V_1 = -F_i(t)x^i = -F_\alpha(t)Q^\alpha,$$

where

$$F_\alpha(t) = S_\alpha^i F_i(t).$$

So the Lagrangian takes the form

$$L(Q, \dot{Q}, t) = \frac{1}{2} \sum_{\alpha=1}^n \left(\dot{Q}^{\alpha 2} - \omega_\alpha^2 Q^{\alpha 2} + F_\alpha(t) Q^\alpha \right).$$

We have n uncoupled, forced oscillators. Evidently we need only understand how to handle a single forced oscillator to handle the general case.

We thus consider the Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + x F(t),$$

where $k > 0$. The equations of motion are that of a harmonic oscillator subjected to an external, time varying force $F(t)$ (exercise):

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}.$$

Of course, we must assume that F remains suitably small so that the solutions do not violate the approximation needed for their validity.

This inhomogeneous differential equation can be directly integrated as follows. Define

$$\xi(t) = \dot{x}(t) + i\omega x(t).$$

In terms of this complex variable the equation of motion takes the form (exercise)

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}.$$

You can easily see that, when $F = 0$, the solution is of the form $Ae^{i\omega t}$, where A is a constant. So try a solution of the form

$$\xi(t) = A(t)e^{i\omega t}.$$

Plugging into the ODE, we find that A satisfies

$$\dot{A} = e^{-i\omega t} \frac{F(t)}{m},$$

which has solution (exercise)

$$A(t) = \int dt \frac{1}{m} F(t) e^{-i\omega t} + \text{constant}.$$

Putting this all together, we see that the general solution of the forced oscillator equation is (exercise)

$$\begin{aligned} x(t) &= \frac{1}{\omega} \text{Im}(\xi(t)) \\ &= \frac{1}{\omega} \text{Im} \left[e^{i\omega t} \left(B + \int \frac{1}{m} F(t) e^{-i\omega t} dt \right) \right], \end{aligned}$$

where B is an arbitrary complex constant.

As an example, suppose that

$$F(t) = f \cos(\gamma t).$$

Then, provided $\gamma^2 \neq \omega^2$, the general solution is of the form (exercise)

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t),$$

where a and α are real constants. We see that in this case the motion is a superposition of two oscillations at the two frequencies ω and γ inherent in the problem. The relative importance of the forced oscillation component depends, of course, on the size of f , but also on the relative magnitudes of ω and γ .

When $\omega \rightarrow \gamma$ the forced oscillation amplitude diverges and our form of the solution given above becomes invalid; this situation is called *resonance*. To get the correct solution in this case we set $\gamma = \omega$ in our integral expression or the general solution. We then get a solution of the form (exercise)

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{2m\omega} t \sin \omega t,$$

where, again, a and α are constants. Note the linear growth in t , which eventually destroys the harmonic approximation.

We can get a useful picture of the behavior of the system near resonance by an approximation scheme. Let

$$\gamma = \omega + \epsilon,$$

where $\epsilon \ll \omega$. Write the general solution for $x(t)$ (off resonance) in the complex form (exercise):

$$\xi(t) \approx (A + Be^{i\epsilon t})e^{i\omega t}.$$

Over one period, $\frac{2\pi}{\omega}$, the amplitude $C = |A + Be^{i\epsilon t}|$ changes very little. Thus the motion is approximately that of free oscillation with a slowly varying amplitude. In particular, the amplitude is of the form (exercise)

$$C = \sqrt{a^2 + b^2 + 2ab \cos(\epsilon t + \phi)},$$

where $A = ae^{i\alpha}$, $B = be^{i\beta}$, and $\phi = \beta - \alpha$. Thus the amplitude varies (slowly) between the values $|a + b|$ and $|a - b|$. The oscillatory behavior is said to exhibit “beats”.

Typically, a general force $F(t)$ can be Fourier analyzed into sinusoidal components. Likewise, we can Fourier analyze the solution $x(t)$. We can view the above example as illustrating the behavior of a typical Fourier component. The general motion of the system is then a superposition of motions such as given above (exercise).

Finally, let us note that since the Lagrangian for a system executing forced oscillations is explicitly time dependent (provided $\frac{dF}{dt} \neq 0$), there will be no conservation of energy for the oscillator. This should not surprise you, since the oscillator is clearly exchanging energy with its environment. We can compute the energy transferred during a time interval (t_1, t_2) by noting that the oscillator energy can be written as

$$E = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m|\xi|^2,$$

and then using our explicit formula for $\xi(t)$,

$$\xi = e^{i\omega t} \left(B + \int_0^t \frac{1}{m} F(t) e^{-i\omega t} dt \right)$$

to compute the energy at time t . For example, let us suppose that the system is at equilibrium before $t = 0$, a force acts for a period of time after $t = 0$ after which the force is zero again. Then $B = 0$ and the change in the oscillator energy can be written as (exercise)

$$\Delta E = \frac{1}{2m} \left| \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt \right|^2.$$

Thus, the energy transfer is controlled by the absolute value of the Fourier component of the force with frequency ω . If the time during which the force acts is small compared to $\frac{1}{\omega}$, then $e^{i\omega t}$ is approximately constant in the integral, and hence

$$\Delta E \approx \frac{1}{2m} \left| \int_{-\infty}^{\infty} F(t) dt \right|^2.$$

Here the change in energy is controlled solely by the impulse imparted by the force since the time scale is so short that no appreciable change in potential energy occurs while the force acts. In the limit where

$$F(t) = f\delta(t - t_0),$$

this approximation becomes exact (exercise).

Homework Problem

Consider an oscillator with mass m and natural frequency ω , initially at rest, which undergoes a constant force F_0 for a finite period of time T . Show that after the force ceases ($t > T$) the system is oscillating harmonically. Determine the amplitude of this oscillation. Your answer should (only) depend upon F_0, m, ω, T .