*Physics 6010, Fall 2010*

*Small Oscillations. Normal Modes.*

*Relevant Sections in Text: §6.1–6.3, 6.4*

**Small Oscillations: One degree of freedom.**

We now leave the 2-body problem and consider another, rather important class of systems that can be given a complete analytic treatment. I assume you already know that the motion of a system in the vicinity of a point of stable equilibrium is approximated by the superposition of harmonic oscillations. This approximation is very valuable and we shall spend some time studying it.

Let us begin by considering the motion of a one-dimensional system near a critical point of the potential energy. While only the simplest systems can be reduced to one degree of freedom, this case is still particularly important because – as we shall soon see – multi-dimensional systems (in the vicinity of stable equilibrium) can be reduced to multiple copies of the one dimensional case.

We assume the Lagrangian is of the form:

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q).$$

Note in particular that we assume (for now) that the system is closed, *i.e.,* energy is conserved. Let $q_0$ be a critical point of the potential, so that $V'(q_0) = 0$. Let us approximate the motion by assuming that $x := q - q_0$ is "small". We expand the Lagrangian in a Taylor series in $x$, keeping only the first non-trivial terms (exercise):

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \dots.$$

Here we have set $m = a(q_0)$ and $k = V''(q_0)$. Of course, the kinetic energy must be positive, so that $m > 0$. Note that we also have dropped the irrelevant constant $V(q_0)$. If desired we can adjust the zero point of potential energy so that this constant is zero.

The EL equations are (in the domain of validity of the approximation)

$$m\ddot{x} = -kx.$$

This equation is easily integrated:

$$x = Re(Ae^{i\omega t}), \quad \text{when } k \neq 0.$$

$$x = at + b, \quad \text{if } k = 0,$$

where $\omega = \pm\sqrt{\frac{k}{m}}$ and $A$ is a complex constant encoding the two real integration constants, which can be fixed by initial conditions. If $k > 0$, then $q_0$ is a point of stable equilibrium, and we get harmonic motion. In particular, if $x$ is small initially and the initial velocity is sufficiently small, then $x(t)$ remains small (exercise), so that our approximation is self-consistent. On the other hand, if $k \leq 0$, then the motion of the particle need not maintain our approximation of small $x$; our approximation is not self-consistent and must be abandoned after a very short time. Of course, the case $k < 0$ corresponds to unstable equilibrium, for which a small perturbation leads to a rapid motion away from equilibrium. If $k = 0$, then the critical point $q_0$ is not a maximum or minimum but is a saddle point ("neutral equilibrium"); our approximation again becomes invalid, although the time scale for this is larger than the case of unstable equilibrium.

Thus in a neighborhood of a point of stable equilibrium it is consistent to make the *harmonic approximation* to the potential and kinetic energies. In the harmonic approximation the motion of the system is mathematically the same as that of a simple harmonic oscillator.

A very familiar example of all of this is the planar pendulum of mass $m$ and length $l$ for which $q = \theta$ is the deflection from a vertical position. We have

$$V(\theta) = mgl(1 - \cos\theta),$$

and

$$a(\theta) = \frac{1}{2}ml^2.$$

The Lagrangian in the harmonic approximation near equilibrium at $\theta = 0$ is (exercise)

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{mgl}{2}\theta^2,$$

so that the harmonic motion has "mass" $ml^2$ and angular frequency" $\omega = \sqrt{\frac{g}{l}}$. *(Exercise:* what happens to motion near $\theta = \pi$?)

**Example**

Consider a mass $m$ which is constrained to move on a straight line. The mass is bound to a fixed point by harmonic force with potential energy $V = \frac{1}{2}K(r - R)^2$, where $K$ is a constant, $r$ is the distance of the particle to the fixed point. The distance from the point to the line is $l > R$. A mechanical model of this system is a mass sliding on a straight track; the mass being connected to a fixed point by a spring. Our goal is to find the stable equilibrium position(s) and compute the frequency of small oscillations about the equilibrium. Evidently, $r^2 = x^2 + l^2$, where $x$ is the position of the particle along the

given line, with $x = 0$ the location at distance $l$ from the center of force. Using $x$ as the generalized coordinate, the Lagrangian for this system is (exercise)

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}K(\sqrt{x^2 + l^2} - R)^2.$$

You can check that $x = 0$ is the only point of stable equilibrium. (As an exercise you can check that the point $x = 0$ is a point of unstable equilibrium if we assume $R \geq l$.) You can expand the potential to second order to find that (exercise)

$$k = \frac{K(l - R)}{l},$$

so that

$$\omega = \sqrt{\frac{K(l - R)}{lm}}.$$

Note that $K(l - R)$ is the force needed to move a particle from $r = R$ to the point $r = l$ (exercise) in the presence of the potential $V$.

**Example**

We return to our example of a plane pendulum of length $l$ with horizontally moving point of support. The mass $m_1$ (the point of support) has position $x$, and the angular deflection of the pendulum (mass $m_2$) is denoted by $\phi$. The kinetic energy is

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi),$$

and the potential energy is

$$V = -m_2 gl\cos\phi.$$

We have already seen that we can use conservation of the momentum conjugate to $x$ to effect a Lagrangian reduction which eliminates the $x$ degree of freedom. In detail, the conservation law we need is

$$P = (m_1 + m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{constant}.$$

This conservation law is just the conservation of the $x$ component of the center of mass of the system (exercise).

Let us compute the frequency of small oscillations of the pendulum in the reference frame in which $P = 0$, so that (exercise)

$$(m_1 + m_2)x + m_2 l\sin\phi = \text{constant}.$$

In this reference frame, which is the rest frame of the $x$ component of the center of mass, the reduced kinetic energy is (exercise)

$$T = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \frac{m_2}{m_1 + m_2} \cos^2 \phi\right),$$

and the reduced potential energy is (exercise)

$$V = -m_2 g l \cos \phi.$$

Clearly $\phi = 0$ is an equilibrium point (exercise). Expanding in powers of $\phi$ we get, in the harmonic approximation,

$$T \approx \frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} \dot{\phi}^2,$$

$$V \approx \frac{1}{2} m_2 g l \phi^2.$$

It is now straightforward to see that the frequency of small oscillations is (exercise)

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}.$$

Of course, in a generic inertial reference frame the motion of the system in the harmonic approximation is small oscillations at the above frequency superimposed with a uniform translation of the center of mass along the $x$ direction.

As a sanity check, let us consider the limit in which $\frac{m_2}{m_1} << 1$, *i.e.,* the mass at the point of support is becoming large. Physically, we expect that the point of support moves with a uniform translation along $x$. In the rest frame of the point of support (which is now the approximate center of mass) we have a traditional plane pendulum problem. In this limit we get (exercise)

$$\omega \approx \sqrt{\frac{g}{l}},$$

as expected.

**Oscillations of systems with more than one degree of freedom.**

So far we have studied (small) oscillations of systems with a single degree of freedom. Systems with more degrees of freedom can exhibit much more intricate behavior in the vicinity of stable equilibrium. However, this more intricate behavior can always be viewed as the superposition of harmonic motions of decoupled degrees of freedom — the *normal modes* of vibration. We shall first spend a little time developing the general theory, and then we shall spend some time on examples.

Suppose we have a system with $n$ generalized coordinates $q^i$, $i = 1, 2, \ldots, n$, and a Lagrangian of the form

$$L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - V(q).$$

Recall that we use the summation convention; there is a double sum in the first term of $L$. The "metric" $g_{ij}(q)$ is a *symmetric* array which may depend upon the configuration coordinates. There is no loss of generality by assuming the metric to be symmetric,

$$g_{ij} = g_{ji},$$

since only the symmetric combination appears in the sum over $i$ and $j$ (exercise). Usually, the metric is diagonal (*e.g.,* in spherical polar coordinates), but we have seen examples where the metric is off diagonal (*e.g.,* pendulum with moving point of support).

Critical points of $V$, *i.e.,* points $q_0^i$ such that

$$\left( \frac{\partial V}{\partial q^i} \right) (q_0) = 0,$$

define equilibrium configurations of the system (exercise). We will sometimes suppose that the equilibrium is stable, *i.e.,* $q_0^i$ is a (local) minimum of $V$. But you should always be thinking about the other types of equilibrium as we go.

Let us approximate the motion in the neighborhood of a point of equilibrium by defining

$$x^i = q^i - q_0^i,$$

and expanding the Lagrangian in a Taylor series about $x^i = 0$. To the first non-trivial order we get (exercise)

$$L \approx \frac{1}{2} M_{ij} \dot{x}^i \dot{x}^j - \frac{1}{2} K_{ij} x^i x^j$$

where

$$M_{ij} = g_{ij}(q_0), \qquad K_{ij} = \left( \frac{\partial^2 V}{\partial q^i \partial q^j} \right) (q_0),$$

and we have dropped an irrelevant additive constant $V(q_0)$, *i.e.,* we have adjusted the zero of potential energy to be at $q_0^i$. We have that $M_{ij} = M_{ji}$ and we assume that the potential energy is sufficiently smooth so that the matrix of second partial derivatives is symmetric at the critical point $q_0$:

$$K_{ij} = K_{ji}.$$

The (approximate) EL equations are (exercise)

$$M_{ij} \ddot{x}^j + K_{ij} x^j = 0,$$

which are coupled system of $n$ homogeneous, linear ODEs with constant coefficients. Defining $\vec{x}$ as a column vector with entries $x^i$, and viewing $M_{ij}$ and $K_{ij}$ as (symmetric) matrices $M$ and $K$, we can write the EL equations in the matrix form (exercise):

$$M\ddot{\vec{x}} = -K\vec{x}.$$

Let us note that if $q_0^i$ is a point of stable equilibrium then the symmetric matrix $K$ is positive definite, that is, it can have only positive eigenvalues.* This is because a negative or zero eigenvalue will correspond to displacements $x^i$ which either lower or do not change the potential energy in an arbitrarily small neighborhood of the equilibrium point (exercise), which contradicts our assumption of stable equilibrium. Conversely, since every symmetric matrix can be diagonalized, if the eigenvalues are all positive definite then the point $q_0^i$ is a minimum. Put differently, $q_0^i$ is a point of stable equilibrium if and only if the *quadratic form*

$$K(\vec{x}) := K_{ij}x^i x^j$$

is *positive definite*, which means $K(\vec{x}) > 0$ for all $\vec{x} \neq 0$. Physically this means that any displacement $x^i$ from equilibrium will increase the potential energy. All this discussion is just restating standard results from multivariable calculus.

Likewise, positivity of the kinetic energy implies that in any physical application the symmetric matrix $M$ should be positive definite. This means the quadratic form

$$M(x) := M_{ij}x^i x^j$$

is positive definite, *i.e.,* $M(\vec{x}) > 0$ for all $\vec{x} \neq 0$. We assume this in what follows.

Here's our strategy for analyzing the (approximate) EL equations. We can write the EL equations as (exercise)

$$\ddot{\vec{x}} + (M^{-1}K)\vec{x} = 0,$$

where $M^{-1}$ exists because $M$ is positive definite (exercise). Suppose we can find an eigenvector $\vec{a}$ of $M^{-1}K$ with eigenvalue $\omega^2$. (For now, we allow $\omega$ to be complex.) Then it is easy to see that we will get a solution the form

$$\mathbf{x} = \vec{a}\cos(\omega t + \beta).$$

Thus, provided $\omega^2 > 0$, *i.e.,* $\omega$ is real and non-vanishing, some combination of the displacements of the system (determined by the eigenvector) is behaving as a harmonic oscillator with frequency $\omega$. If we can find enough eigenvectors, we will be able to build the general solution to the EL equations by superposition. Because the general solution should involve

* Note that a symmetric, real matrix always admits complete set of eigenvectors with real eigenvalues.

$2n$ integration constants — corresponding to initial positions and initial velocities, we will need $n$ independent solutions of the above form (exercise), *i.e.,* $n$ linearly independent eigenvectors, to get the general solution in this manner.

Our goal, then, is to determine the eigenvalues $\omega^2$ and the eigenvectors $\vec{a}$. We will write our eigenvalue equation in the equivalent form (exercise):

$$(K - \omega^2 M)\vec{a} = 0.$$

Thus $\vec{a}$ is an eigenvector of the matrix $(K - \omega^2 M)$ with eigenvalue zero. You will recall the basic result from linear algebra that the square matrix $K - \omega^2 M$ has an eigenvector with eigenvalue zero if and only if (exercise)

$$\det(K - \omega^2 M) = 0.$$

This is the *characteristic equation* for $\omega$; it is a polynomial equation for $\omega$ of order $2n$. Thus $\omega$, called the *characteristic frequency*, arises as a root of a polynomial of order $2n$.

In general there are $2n$ roots of the characteristic equation. But notice that if $\omega$ is a solution, then so is $-\omega$ since they both yield the same $\omega^2$. Since it is $\omega^2$ which determines the eigenvector, we see that changing the sign of $\omega$ does not give a new solution (linearly independent eigenvector). Thus, without loss of generality, we can assume that $\omega > 0$. Thus we get $n$ characteristic frequencies. At this point, as far as we know, these frequencies may be non-vanishing and real, or zero, or imaginary. The real solutions correspond to stable directions in configuration space. The imaginary solutions correspond to directions in configuration space relative to which the equilibrium is unstable. The vanishing frequencies correspond to directions in configuration space relative to which we have neutral equilibrium. Corresponding to each of these roots is an eigenvector $\vec{a}$ and hence a solution $\vec{x}$ to the EL equations.

It is not hard to see that if the potential energy quadratic form is positive definite then $\omega^2 > 0$, so that there are $n$ real, *positive* characteristic frequencies. To see this, simply note that for any displacement $\vec{x}$ solving the equations of motion we have

$$\omega^2 M(\vec{x}) = K(\vec{x}).$$

Because both quadratic forms are positive definite, $M(\vec{x}) > 0$ and $K(\vec{x}) > 0$, it follows that

$$\omega^2 > 0,$$

as desired.

Physically speaking, the existence of $n$ positive roots for $\omega$ stems from our assumption that $q_0$ is a point of stable equilibrium, which mathematically means $K(\vec{x})$ is positive

definite. If $K(\vec{x})$ could be negative or zero, allowing for complex or vanishing frequencies, this would lead to exponential or linear (rather than oscillatory) solutions. Henceforth we assume we have stable equilibrium and just focus on the sinusoidal solutions. You should have no problem adapting our discussion to the other cases (and we will have an example of neutral equilibrium in a little while)

Once we have a characteristic frequency $\omega$, we can reconstruct the corresponding $a^i$ by solving the equation $(K - \omega^2 M)\vec{a} = 0$. The solution is guaranteed to exist (because $\omega$ solves the characteristic equation). In this way we have, in fact, found an eigenvector $(\vec{a})$ of $M^{-1}K$ with eigenvalue $\omega^2$ (exercise). It is, in general, possible to find $n$ roots of the characteristic equation along with $n$ *orthogonal* — and hence linearly independent — eigenvectors (more on this shortly). Each of the (real) solutions obtained through this procedure evolves in time harmonically at the characteristic frequency; these solutions are called *normal modes*. Let us denote the characteristic frequencies by $\omega_\alpha$ and the corresponding normalized eigenvectors by $\vec{a}_\alpha = \{a^i_\alpha\}$, $\alpha = 1, 2, \ldots, n$. The general solution to the EL equations is then a superposition of the normal modes:

$$x^k(t) = \sum_\alpha c_\alpha a^k_\alpha \cos(\omega_\alpha t + \beta_\alpha),$$

where $c_\alpha$ and $\beta_\alpha$ are constants (of integration) determined by initial conditions.

**Normal Modes - The Recipe**

Let us summarize the construction of the normal modes of vibration described by the Lagrangian, which approximates the dynamics of a system near equilibrium:

$$L = \frac{1}{2} M_{ij} \dot{x}^i \dot{x}^j - \frac{1}{2} K_{ij} x^i x^j, \quad i, j = 1, \ldots, n.$$

First we solve the characteristic equation

$$\det(\omega^2 M - K) = 0$$

for the characteristic frequencies $\omega_\alpha$, $\alpha = 1, \ldots, n$.

We then solve the linear equations

$$(\omega^2_\alpha M - K)\vec{a}_\alpha = 0$$

for the corresponding vectors $\vec{a}_\alpha$, $\alpha = 1, \ldots, n$. The normal modes of vibration are*

$$\vec{\Theta}_\alpha(t) = \vec{a}_\alpha \cos(\omega_\alpha t + \beta_\alpha).$$

* Here we assume $\omega_\alpha \neq 0$. If a characteristic frequency vanishes the corresponding normal mode is of the form $\vec{\Theta} = (c + dt)\vec{a}$, where $c$ and $d$ are constants.

The general motion of the system in the vicinity of stable equilibrium (in the harmonic approximation) is a superposition of the normal modes. The superposition goes over the amplitudes and phases of each oscillator. These coefficients are determined by initial conditions. We have

$$\vec{x}(t) = \sum_{\alpha=1}^{n} C_\alpha \vec{\Theta}_\alpha(t).$$

**An Elementary Example**

As a very simple example of finding normal modes and characteristic frequencies, let us consider a system described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega_0^2(x^2 + y^2) - \frac{1}{2}m\alpha^2(x-y)^2.$$

This system can be viewed as two identical one-dimensional harmonic oscillators (natural frequency $\omega_0$ ) with a coupling by a harmonic force (natural frequency $\alpha$) (exercise).

The matrices $M$ and $K$ are given by (good exercise!)

$$M_{ij} = m\delta_{ij},$$

and

$$K_{xx} = K_{yy} = m(\omega_0^2 + \alpha^2), \quad K_{xy} = K_{yx} = -m\alpha^2.$$

We now have

$$K - \omega^2 M = m \begin{pmatrix} \alpha^2 + \omega_0^2 - \omega^2 & -\alpha^2 \\ -\alpha^2 & \alpha^2 + \omega_0^2 - \omega^2 \end{pmatrix}.$$

The characteristic equation is (exercise)

$$0 = \det(K - \omega^2 M) = m^2 \left[ (\alpha^2 + \omega_0^2 - \omega^2)^2 - \alpha^4 \right].$$

The characteristic frequencies are then given by (exercise)

$$\omega_1 = \omega_0, \quad \omega_2 = \sqrt{\omega_0^2 + 2\alpha^2}.$$

The corresponding normalized normal modes are determined by

$$\vec{a}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We have

$$\Theta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \beta_1), \quad \Theta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{\omega_0^2 + 2\alpha^2}\, t + \beta_2).$$

Evidently, the oscillation at frequency $\omega_0$ corresponds to the two masses moving exactly in phase, so the coupling does not come into play, while the oscillation at frequency $\sqrt{\omega_0^2 + 2\alpha^2}$ has the two masses moving exactly out of phase.

The general motion of the system is a superposition of these two normal modes. We can write the general motion of the system in vector form as

$$\vec{x}(t) = C_1\vec{a}_1\cos(\omega_1 t + \beta_1) + C_2\vec{a}_2\cos(\omega_2 t + \beta_2),$$

or, more explicitly,

$$x(t) = \frac{1}{\sqrt{2}}\Big(C_1\cos(\omega_1 t + \beta_1) + C_2\cos(\omega_2 t + \beta_2)\Big)$$
$$y(t) = \frac{1}{\sqrt{2}}\Big(C_1\cos(\omega_1 t + \beta_1) - C_2\cos(\omega_2 t + \beta_2)\Big).$$

Here $(C_1, C_2, \beta_1, \beta_2)$ are real constants which are determined by initial conditions. Despite the fact that the motion is very regular, *i.e.,* it is the superposition of harmonic oscillations, the appearance of the motion can be quite complicated.

**Example: Double Pendulum**

Let us return to the coplanar double pendulum. The Lagrangian is

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + (m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2\cos\theta_2.$$

Recall that $m_1$ has a fixed point of support, while $m_2$ is supported at the location of $m_1$. Stable equilibrium occurs at $\theta_1 = 0 = \theta_2$. We expand in Taylor series about equilibrium to get the approximate Lagrangian:

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}(m_1+m_2)gl_1\theta_1^2 - \frac{1}{2}m_2 g l_2\theta_2^2.$$

Note that the coupling between degrees of freedom occurs through the kinetic terms. The characteristic equation is (exercise)

$$0 = \det\begin{pmatrix} (\omega^2 l_1^2 - gl_1)(m_1+m_2) & \omega^2 m_2 l_1 l_2 \\ \omega^2 m_2 l_1 l_2 & (\omega^2 l_2^2 - gl_2)m_2 \end{pmatrix}.$$

The roots are (exercise)

$$\omega_\pm^2 = \frac{g}{2m_1 l_1 l_2}\left\{(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)\left[(m_1+m_2)(l_1+l_2)^2 - 4m_1 l_1 l_2\right]}\right\}.$$

To carry on, let us consider a special case. Suppose the two pendula are identical: $m_1 = m_2 = m$, $l_1 = l_2 = l$. Then the characteristic frequencies become

$$\omega_\pm^2 = \frac{g}{l}(2 \pm \sqrt{2}).$$

Note that all dependence upon the mass drops out. In this case the normal modes are determined by

$$\mathbf{a}_\pm = C \begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}.$$

The normal mode $\Theta_+$ has both masses having velocities out of phase; the normal mode $\Theta_-$ has the velocities in phase.

**Example: Linear triatomic molecule**

As another application of our theory let us consider the motion near equilibrium of a linear triatomic molecule. By this we mean that we have two atoms of mass $m$ located symmetrically on either side of an atom of mass $M$. Let all three atoms lie on a line. For simplicity we only consider longitudinal motion, *i.e.,* motion along the extent of the molecule. We assume that, near equilibrium, the restoring force on the atoms is $-kx$, where $x$ is the displacement of an atom from equilibrium. In detail, let $x_1$ and $x_2$ denote the longitudinal displacements of each $m$ from equilibrium, and let $X$ denote the displacement of $M$ from equilibrium. The potential energy is approximated by (exercise)

$$V = \frac{k}{2}\left[(X - x_1)^2 + (X - x_2)^2\right].$$

The kinetic energy is simply

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}M\dot{X}^2.$$

Ordering the coordinates on the configuration space as $(x_1, x_2, X)$, the matrices of interest are (exercise)

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix},$$

$$K = \begin{pmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{pmatrix}.$$

The characteristic equation is (exercise)

$$\omega^2(k - \omega^2 m)[k(M + 2m) - \omega^2 Mm] = 0.$$

The solutions are (exercise)

$$\omega_1 = 0\,(!!!), \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{k}{m}\left(\frac{M + 2m}{M}\right)}.$$

The appearance of a zero frequency mode is, at first, a little disconcerting, but simply reflects the possible motion in which all three masses move with a uniform translation (see below). The other two modes represent oscillations near equilibrium. To see all this, we need to compute the normal modes. We get (exercise)

$$a_1 = N_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = N_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad a_3 = N_3 \begin{pmatrix} 1 \\ 1 \\ -\frac{2m}{M} \end{pmatrix},$$

where the $N$'s are normalization constants. The motion of mode 1 is rigid motion at constant velocity. To see this we simply note that

$$x_1 = x_2 = X = at + b$$

solves the equations of motion and with zero frequency (exercise). Note that a spatial translation of the system is a symmetry of the Lagrangian, so the total (center of mass) momentum is conserved. If desired, we can work in the rest frame of the center of mass; this reduction eliminates the center of mass single degree of freedom and leaves two oscillatory degrees of freedom. The motion of mode 2 has the central atom at rest and the two endpoint atoms oscillating 180 degrees out of phase with the same amplitude. The third mode has the endpoint atoms in phase, with the same amplitude while the center atom moves 180 degrees out of phase with them and with a different amplitude. These latter two modes keep the center of mass at rest.

Finally, let us briefly and qualitatively consider the general, non-longitudinal motion of the molecule. Of course, if we allow for vibrations perpendicular to the line defined of the molecule we have more degrees of freedom (namely, 9) to consider. While the explicit computations are more lengthy, nothing conceptually new arises. There will be a number of zero frequency modes corresponding to rigid motions (translations and rotations) of the molecule. The remaining modes will be truly vibrational. Of the 9 degrees of freedom and the 9 corresponding modes, 5 will be of the zero-frequency type (exercise). Naively, there are 3 translations and 3 rotations, but the linear nature of the molecule, and its modeling via point masses, means that rotations about the molecular axis are not motions of the system. Thus there are only 5 zero frequency modes: 3 translations + 2 rotations. This leaves 4 vibrational modes. Two of these modes–the longitudinal modes–we have already studied. The other two modes are transverse to the axis of the molecule. If we think of this axis as the $x$-axis, then the transverse vibrations come from displacements in the $y$ and $z$ directions. Clearly there is nothing to physically distinguish $y$ from $z$, so we expect that the frequencies for the two transverse normal modes will be degenerate. Indeed, there is nothing to pick out which orthogonal directions to the molecule should be $y$ and $z$. The system exhibits a symmetry under rotations about the molecule axis. Thus the normal modes for transverse vibrations will be along any two perpendicular directions each of which is perpendicular to the molecule axis.

The rigid motions of the molecule, which give rise to the zero frequency modes, correspond to symmetries and conservation laws. Since the potential energy is changed only by a relative motion of the atoms, it clearly will exhibit a symmetry with respect to any transformation that leaves the relative position of the atoms unchanged, *i.e.,* rotations and translations of the molecule as a whole. The kinetic energy is, of course, invariant under such transformations (exercise). Thus the Lagrangian is invariant under the rigid rotations and translations of the molecule. The corresponding conservation laws are 5 in number (3 translations and 2 rotations–exercise); they are the 3 components of the center of mass momentum (no external forces) and 2 components of angular momentum (no external torques).