Physics 6010, Fall 2010

Symmetries and Conservation Laws: Noether's Theorem

Relevant Sections in Text: §2.6, 2.7

Noether's Theorem

We have seen that the familiar conservation laws for energy, momentum, and angular momentum all follow from symmetry properties of the Lagrangian. In your homework you had to exhibit additional conservation laws in nature. It is natural to ask whether other–indeed, if all–conservation laws arise via symmetries of a Lagrangian. The answer is yes. This result, which is a key reason for the power of the Lagrangian formalism, is known as *Noether's theorem.*

While we cannot prove Noether's theorem in detail here, we can give a good picture of how it works. Here it is.

Consider an infinitesimal transformation

$$\delta q^i = F^i(q, \dot{q}, t), \quad \delta\dot{q}^i = \frac{d}{dt}F^i.$$

The idea here is that, given a curve $q^i = q^i(t)$ the infinitesimal transformation defines a varied curve

$$q^i(t) + \delta q^i(t) = q^i(t) + F^i(q(t), \frac{dq(t)}{dt}, t).$$

Note however that the variation here does *not* need to vanish at any endpoints. The change in the Lagrangian (on a curve) due to this infinitesimal transformation can be computed via

$$\delta L = \frac{\partial L}{\partial q^i}F^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}F^i.$$

We say the Lagrangian is *invariant* under this transformation — equivalently: $\delta q = F$ defines an *infinitesimal symmetry* of the Lagrangian — if there exists a function G on the velocity phase space such that

$$\delta L = \frac{dG}{dt}.$$

Because two Lagrangians differing by a time derivative give the same equations of motion, this a suitable notion of symmetry. (Note that G may vanish in many cases.)

Noether's theorem then guarantees that

$$Q = \frac{\partial L}{\partial \dot{q}^i}F^i - G$$

is conserved when the EL equations hold:

$$\mathcal{E}_i \equiv \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0.$$

To see this is a straightforward computation. On the one hand we have for any variation (exercise)

$$\delta L = \mathcal{E}_i \delta q^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right).$$

For a variation which is an infinitesimal symmetry we have

$$\delta L = \frac{dG}{dt}.$$

Putting these two results together we have for an infinitesimal symmetry $\delta q^i = F^i$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}F^i - G\right) = -\mathcal{E}_i F^i.$$

Thus, for solutions of the EL equations Q is conserved.

Noether's Theorem (informally stated):

There is a one to one correspondence between Lagrangian symmetries and conservation laws. Given a Lagrangian symmetry, δq^i (which may involve a function G in a total derivative), the corresponding conserved quantity is

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i - G.$$

Remark:

Note that discrete symmetries (*e.g.*, reflections) will not give rise to conservation laws since they do not have an infinitesimal form.

As an example of the use of this theorem, let us reconsider conservation of momentum and energy. Consider a Lagrangian L with q^1 cyclic:

$$\frac{\partial L}{\partial q^1} = 0.$$

Consider the following transformation:

$$q^i(\alpha) = q^i + \alpha\delta^{i1}.$$

This is a translation of q^1. The infinitesimal transformation is (exercise)

$$\delta q^i = \delta^{i1}.$$

In this case the total derivative term does not appear, *i.e.*, $G = 0$:

$$\delta L = 0.$$

We get (exercise)

$$\begin{aligned} Q &= \sum_j \frac{\partial L}{\partial \dot{q}^j} \delta^{1j} \\ &= p_1. \end{aligned}$$

Next consider conservation of energy. For conservation of energy to hold we must assume that $\frac{\partial L}{\partial t} = 0$. The infinitesimal symmetry is

$$\delta q^i = \dot{q}^i.$$

This transformation can be viewed as a time translation since on a curve $q^i(t)$, infinitesimally, (exercise)

$$q^i(t + \epsilon) \approx q^i(t) + \epsilon \frac{dq(t)}{dt}.$$

That this *is* a Lagrangian symmetry when $\frac{\partial L}{\partial t} = 0$ is easily seen:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q^i} \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i \\ &= \frac{dL}{dt}. \end{aligned}$$

The last equality follows from the assumption that the Lagrangian has no explicit time dependence. We conclude that $\delta q^i = \dot{q}^i$ is a Lagrangian symmetry in which the total derivative function is given by $G = L$. Applying Noether's theorem we get (exercise)

$$Q = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L = E.$$

As an exercise you can use Noether's theorem to re-derive the relation between rotational symmetry and conservation of angular momentum.

Finally, we remark we do not need to have (explicitly) a cyclic coordinate to get a conservation law. We just need an infinitesimal transformation for which the Lagrangian only changes by a total time derivative. For example, consider a particle moving in a central force field:

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - U(|\vec{r}|).$$

Working in Cartesian coordinates, all 3 coordinates appear in U:

$$U = U(\sqrt{x^2 + y^2 + z^2}).$$

However, the following infinitesimal transformation is a symmetry (exercise)

$$\delta x = -y, \quad \delta y = x.$$

This is an infinitesimal rotation about the z axis. From Noether's theorem, the corresponding conserved quantity is

$$Q = (mv_x)(-y) + (mv_y)(x) = (\vec{r} \times \vec{p})_z,$$

which is the z component of angular momentum. As an exercise you can figure out the symmetries for the other 2 components.

Homework Problem

Consider a closed system of Newtonian particles interacting pairwise by central forces. Show that this system admits a **boost** *symmetry, corresponding to transformation to a new inertial reference frame moving with constant relative velocity $\vec{V}$, i.e., the position of each particle is transformed according to*

$$\vec{r} \rightarrow \vec{r} + \vec{V}t.$$

Use Noether's theorem to compute the corresponding conserved quantities. What is their physical interpretation?