Physics 6010, Fall 2010

Some examples. Constraints and Lagrange Multipliers.

Relevant Sections in Text: §1.3–1.6

Example: Newtonian particle in different coordinate systems.

We have already noted that the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}, t),$$

will give the equations of motion corresponding to Newton's second law for a particle moving in 3-d under the influence of a potential V. To see this use Cartesian coordinates:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z, t).$$

Using,

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x},$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x},$$

the EL equation for the x coordinate is easily seen to be (exercise)

$$-\frac{\partial V}{\partial x} - m\ddot{x} = 0.$$

Of course, the y and z coordinates get a similar treatment. We then get (exercise)

$$m\ddot{\vec{r}} + \nabla V = 0.$$

Let us consider cylindrical coordinates,

$$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

Can you write down the equations of motion following from $\vec{F} = m\vec{a}$ in cylindrical (ρ, θ, z) coordinates? It is completely straightforward using the EL equations – a distinct practical advantage of this formalism. We just need to express the kinetic energy (T) and the potential energy (V) in terms of cylindrical coordinates and take the difference to make the Lagrangian, from which the EL equations are easily computed.

To compute the kinetic energy we take the kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and substitute

$$\dot{x} = \dot{\rho}\cos\theta - \dot{\theta}\rho\sin\theta, \quad \dot{y} = \dot{\rho}\sin\theta + \dot{\theta}\rho\cos\theta,$$

to get

$$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2).$$

To get the potential energy we just substitute into $V(x, y, z, t)$ for x and y in terms of ρ and θ. The Lagrangian is of the form

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2) - V(\rho, \theta, z, t).$$

Caution:

The notation for the potential energy function is the usual one used by physicists, but it can be misleading because, strictly speaking, it violates standard mathematical notational rules. $V(\rho, \theta, z, t)$ means *the potential energy at the location defined by* (ρ, θ, z) *and at the time* t. $V(\rho, \theta, z, t)$ is *not* the function obtained by substituting $x \to \rho, y \to \theta$ in $V(x, y, z, t)$, which a strict interpretation of the function notation would require. What we are calling $V(\rho, \theta, z, t)$ is in fact the function obtained from $V(x, y, z, t)$ by the substitution $x \to \rho\cos\theta$, *etc.* For example, $V(x, y, z, t) = x^2 t$ corresponds to $V(\rho, \theta, z, t) = \rho^2 t\cos^2\theta$.

The EL equations are (exercise)

$$m\ddot{\rho} - \rho\dot{\theta}^2 + \frac{\partial V}{\partial \rho} = 0,$$

$$m\frac{d}{dt}(\rho^2\dot{\theta}) + \frac{\partial V}{\partial \theta} = 0,$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0.$$

Exercise: Does the $\rho\dot{\theta}^2$ term in the radial equation represent an attractive or repulsive effect in the radial motion?

Exercise: Repeat this computation for spherical polar coordinates.

Lagrangians for Systems

Often times it is useful to view a dynamical system as consisting of several subsystems. For example, the solar system can be modeled as consisting of ten particles interacting gravitationally. *In the absence of interaction,* the Lagrangian L_0 for the total (non-interacting) system can be viewed as the sum of Lagrangians for each of its parts:

$$L_0 = L_1 + L_2 + L_3 + \ldots.$$

Here L_1, L_2, *etc.* are the Lagrangians for the subsystems. For example, if we have a system of (non-interacting) Newtonian subsystems each Lagrangian is of the form (for the i^{th} subsystem)

$$L_i = T_i - V_i.$$

Here V_i is the potential energy of the i^{th} system due to *external* forces — not due to inter-system interactions, which we are ignoring for a moment. It is easy to see that L_0 correctly describes the motion of the system of non-interacting systems through its EL equations. This is because the EL equations for the k^{th} system involve derivatives of L_0 with respect to coordinates and velocities of the k^{th} system and this just picks out L_k from the sum $L_0 = \sum_j L_j$.

As a simple example, let us consider a system consisting of the planets, viewed as non-interacting point particles. They all move in a central force field due to the sun which is non-dynamical in this model. The Lagrangian for the k^{th} planet, with position $\vec{r}_k$ is of the form

$$L_k = \frac{1}{2} m_k \dot{\vec{r}}_k^2 + \frac{GMm_k}{r_k},$$

where M is the mass of the sun and G is Newton's constant.

Of course, non-interacting systems are an idealization and, ultimately, are of little physical interest. Interactions are what makes the world what it is. One of the main ways to mathematically represent interactions between the subsystems is to introduce a potential energy function, $V = V(q_1, q_2, \dots, t)$ which couples various degrees of freedom. We then have the Lagrangian for the interacting system given by

$$L = L_0 - V.$$

The effect of this potential energy function in the Lagrangian is to *couple* the motion of the subsystems. You can see this by noting the EL equations for a degree of freedom can now, in general, depend upon the other degrees of freedom (exercise).

As an example, let us consider the Lagrangian for a pair of electons in a Helium atom. We view the nucleus as fixed, with charge Q; it is part of the "environment". The "system" consists of the two electrons, each with mass m and charge q. The configuration space is $R^3 \times R^3 = R^6$, and we label points in the configuration space with position vectors $\vec{r}_1$ and $\vec{r}_2$. The Lagrangian is

$$L = \frac{1}{2} m \dot{\vec{r}}_1^2 - \frac{qQ}{|\vec{r}_1|} + \frac{1}{2} m \dot{\vec{r}}_2^2 - \frac{qQ}{|\vec{r}_2|} - \frac{q^2}{|\vec{r}_1 - \vec{r}_2|}.$$

The first two terms represent electron 1 moving in the Coulomb field of the nucleus. Likewise for the next two terms regarding electron two. With these first 4 terms alone the electrons orbit the nucleus independently and do not interact among themselves. The last term represents the interaction between the electrons, which is Coulomb repulsion. It is this term which couples the motion of the two electrons and makes the EL equations somewhat complex, lacking an explicit solution.

The other principal way to mathematically represent interactions is via constraints. In this scenario, the coupling between subsystems will typically occur via the *kinetic* energy function. We will give a couple of examples in what follows.

Example: Double Pendulum

Consider a system consisting of two plane pendulums (pendula?) connected in series. Don't even try to write down the equations of motion using Newton's second law! The Lagrangian analysis is straightforward.

To begin with, we have two particles moving in a plane. We denote their x and y positions via (x_1, y_1) and (x_2, y_2), where the origin of coordinates is placed at the fixed point of the double pendulum. The masses are m_1 and m_2. The motion of the particles is constrained: the lengths are l_1 and l_2; pendulum 1 is attached to a fixed point in space and pendulum 2 is attached to the end of pendulum 1. Mathematically we have

$$x_1^2 + y_1^2 = l_1^2, \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2.$$

These two constraints on the 4 cartesian coordinates leaves 2 degrees of freedom for this system. As we have already mentioned, the configuration of the system is uniquely specified once the angular displacement of each pendulum from (say) the vertical is specified. These angles – generalized coordinates for this system – are denoted by θ_1 and θ_2.

The kinetic energy for mass 1 is easily seen to be (exercise)

$$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2.$$

To find T_2 we note that, defining l_2 and θ_2 in the same way as for mass 1, we have

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2,$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2.$$

Along any curve we have (exercise)

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2,$$

$$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2.$$

The kinetic energy of mass 2 is then (exercise)

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right]. \end{aligned}$$

The potential energy for particle 1 is (exercise)

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos\theta_1$$

and for particle 2 we get (exercise)

$$V_2 = m_2 g y_2 = -m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right).$$

The total potential energy is then

$$V = V_1 + V_2.$$

All together, the Lagrangian for this system is (exercise)

$$L = \frac{1}{2}(m_1+m_2)l_1\dot{\theta}_1^2+\frac{1}{2}m_2 l_2^2\dot{\theta}_2^2+m_2 l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2+(m_1+m_2)gl_1\cos\theta_1+m_2 g l_2\cos\theta_2.$$

It is straightforward, albeit a little tedious, to compute the EL equations. Note that the third term in the Lagrangian represents a coupling of the velocities of the two masses through the kinetic energy. This is a consequence of interaction of the pendulums induced by the constraints discussed above.

Example: Plane pendulum with moving support.

Consider a plane pendulum (mass m_2, length l) whose point of support is a mass m_1 which can slide horizontally. Let us compute the Lagrangian and EL equations. To begin with, we have two particles moving in two dimensions with coordinates (x_1, y_1) and (x_2, y_2). The motion is constrained; we have

$$y_1 = 0, \quad (x_2 - x_1)^2 + y_2^2 = l^2.$$

Here we have set the x axis along the line upon which m_1 moves. We then have (exercise)

$$(x_1, y_1) = (x, 0)$$

and

$$(x_2, y_2) = (x + l\sin\phi, -l\cos\phi).$$

Thus the generalized coordinates are the horizontal displacement $x_1 \equiv x$ for m_1 and the angle ϕ made with the vertical for the pendulum with mass m_2. The kinetic energy is therefore (exercise)

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right).$$

The potential energy is (exercise)

$$V = -m_2 g l \cos\phi.$$

The Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right) + m_2 g l \cos\phi.$$

Once again note how the constraints have coupled the motion via the kinetic energy.

The EL equations for x are (exercise)

$$(m_1 + m_2)\ddot{x} + \frac{d}{dt}(m_2 l\dot{\phi}\cos\phi) = 0.$$

The EL equations for ϕ are (exercise)

$$m_2 l^2\ddot{\phi} + \frac{d}{dt}(m_2 l\dot{x}\cos\phi) + m_2 l(\dot{x}\dot{\phi} - g)\sin\phi = 0.$$

Note that the x equation of motion implies

$$(m_1 + m_2)\dot{x} + (m_2 l\dot{\phi}\cos\phi) = \text{constant}.$$

We say that this quantity is a *constant of motion* or an *integral of motion* or that this quantity is *conserved* by the motion.

Exercise: What is the physical meaning of this conserved quantity?

In both of our pendulum examples the Lagrangians have no explicit t dependence. An example of a system with a t-dependent Lagrangian can be obtained from the previous example as follows.

Suppose the point of support of the pendulum is forced to oscillate with amplitude A and frequency ω:

$$x = A\cos\omega t.$$

Then the Lagrangian is (exercise)

$$L = \frac{1}{2}(m_1 + m_2)A^2\omega^2\sin^2\omega t + \frac{1}{2}m_2\left(-2lA\omega\sin\omega t\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right) + m_2 g l\cos\phi.$$

Note the explicit t dependence of the Lagrangian. There is now only a single degree of freedom, ϕ. The first term in the Lagrangian is purely a function of time and will not contribute to the EL equations (exercise); it can be dropped from L. The EL equations for ϕ are now

$$m_2[l^2\ddot{\phi} - \frac{d}{dt}(Al\omega\sin\omega t\cos\phi) - l(A\omega\sin\omega t\dot{\phi} + g)\sin\phi] = 0.$$

As an exercise you can check that this equation correctly describes the reduction of the original ϕ EL equations by the substitution $x = A\cos\omega t$.

Example: Charged particle in a Prescribed Electromagnetic Field

Two of the fundamental interactions allow themselves to be treated fruitfully using classical mechanics: gravity and electromagnetism. Of course, we have in mind macroscopic systems here. (The other fundamental interactions – strong and weak – only operate microscopically and require a quantum treatment.) Here we will give a Lagrangian formulation of the dynamics of a charged (test) particle in a given electromagnetic field. While a fully relativistic treatment is certainly feasible, we will stick to a non-relativistic (slow motion) treatment for simplicity.

We consider a particle with mass m and electric charge q moving in a given electromagnetic field $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$. The equations of motion come from the *Lorentz force law*,* which asserts that the force $\vec{F}$ at time t on a charge q located at position $\vec{r}$ and moving with velocity $\vec{v}$ is given by

$$\vec{F}(\vec{r}, \vec{v}, t) = q\vec{E}(\vec{r}, t) + \frac{q}{c}\vec{v} \times \vec{B}(\vec{r}, t).$$

Here, of course,

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}.$$

Thus the equations of motion for the curve $\vec{r} = \vec{r}(t)$ are

$$m\ddot{\vec{r}}(t) - q\vec{E}(\vec{r}(t), t) + \frac{q}{c}\dot{\vec{r}}(t) \times \vec{B}(\vec{r}(t), t) = 0.$$

It is not possible to find a Lagrangian whose EL equations correspond to the Lorentz force law without introducing the *electromagnetic potentials*. Recall that 4 of the eight Maxwell equations, the "homogeneous equations",

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0,$$

$$\nabla \cdot \vec{B} = 0.$$

are equivalent to the existence of a scalar potential ϕ and a vector potential $\vec{A}$ so that

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

and

$$\vec{B} = \nabla \times \vec{A}.$$

* We use Gaussian units. The text uses mks units.

You can easily check that these relations lead to electromagnetic fields satisfying the homogeneous Maxwell equations (exercise). Conversely, given any electromagnetic field satisfying the homogeneous Maxwell equations, one can find a function $\phi(\vec{r}, t)$ and a vector field $\vec{A}(\vec{r}, t)$ such that the above relations are satisfied.

Remark: One issue that arises here is that of *gauge transformations*: For each configuration of the electromagnetic field there are infinitely many potentials that can describe it. You can easily check that if ϕ and $\vec{A}$ correspond to a given $\vec{E}$ and $\vec{B}$, then so do

$$\phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t},$$

$$\vec{A}' = \vec{A} + \nabla\Lambda,$$

where $\Lambda(\vec{r}, t)$ is *any* function. This change in potentials is called a gauge transformation. Because the potentials are not uniquely defined by the electromagnetic fields, and because the effect of the electromagnetic field on matter is via the Lorentz force law – involving only $\vec{E}$ and $\vec{B}$, the potentials have no direct physical significance, *e.g.,* one cannot measure ϕ by studying the motion of test particles.

In terms of the potentials, the equations of motion are (in an inertial reference frame)

$$m\ddot{\vec{r}} + q\left(\nabla\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\dot{\vec{r}} \times \nabla \times \vec{A}\right) = 0.$$

We now show that the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2}m(\dot{\vec{r}})^2 - q\phi(\vec{r}, t) + \frac{q}{c}\vec{A}(\vec{r}, t) \cdot \dot{\vec{r}}$$

yields these equations of motion as EL equations. To do this, we consider the EL equation for $x(t)$ and compare with the x component of the Lorentz force law. The y and z EL equations are handled in an identical manner. We have (exercise)

$$\frac{\partial L}{\partial x} = -q\frac{\partial\phi}{\partial x} + \frac{q}{c}\frac{\partial\vec{A}}{\partial x} \cdot \dot{\vec{r}},$$

and

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x} + \frac{q}{c}A_x,$$

so that, on a curve $\vec{r} = \vec{r}(t)$, (exercise)

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = m\ddot{x} + \frac{q}{c}\dot{\vec{r}} \cdot \nabla A_x + \frac{q}{c}\frac{\partial A_x}{\partial t}.$$

Here ∇A_x means to take the gradient of the function A_x. The EL equations are therefore

$$m\ddot{x} + \frac{q}{c}\dot{\vec{r}} \cdot \nabla A_x + \frac{q}{c}\frac{\partial A_x}{\partial t} + q\frac{\partial\phi}{\partial x} - \frac{q}{c}\frac{\partial\vec{A}}{\partial x} \cdot \dot{\vec{r}} = 0.$$

Using (exercises)

$$(\nabla\phi)_x = \frac{\partial\phi}{\partial x},$$

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_x = \frac{\partial A_x}{\partial t},$$

$$\left(\dot{\vec{r}}\times\nabla\times\vec{A}\right)_x = -\dot{\vec{r}}\cdot\nabla A_x + \frac{\partial\vec{A}}{\partial x}\cdot\dot{\vec{r}},$$

it is easy to see that the EL equation for x is the same as the x component of the Lorentz force law (exercise).

There is a technical issue of interest here. The Lagrangian is built from potentials $(\phi,\vec{A})$. As we have already pointed out, there are infinitely many potentials for any given electromagnetic field. Thus there are infinitely many Lagrangians describing motion in a single electromagnetic field. Each of these Lagrangians will yield the same Lorentz force law, which is built from $(\vec{E},\vec{B})$. Each of the Lagrangians can be related via a gauge transformation of the potentials. Apparently, under a gauge transformation of the potentials the Lagrangian changes in just the right way so that the EL equations do not change. How does this happen? You will explore this in a homework problem.

Constraints

Often times we consider dynamical systems which are defined using some kind of restrictions on the motion. For example, the spherical pendulum can be defined as a particle moving in 3-d such that its distance from a given point is fixed. Thus the *true* configuration space is defined by giving a simpler (usually bigger) configuration space along with some *constraints* which restrict the motion to some subspace. We now give a systematic treatment of this idea and show how to handle it using the Lagrangian formalism.

For simplicity we will only consider *holonomic* constraints, which are restrictions which can be expressed in the form of the vanishing of some set of functions – the constraints – on the configuration space and time:

$$C_\alpha(q,t) = 0, \quad \alpha = 1,2,\dots m.$$

We assume these functions are smooth and independent so that if there are n coordinates q^i, then at each time t the constraints restrict the motion to a nice $n-m$ dimensional space. For example, the spherical pendulum has a single constraint on the three Cartesian configuration variables (x,y,z):

$$C(x,y,z) = x^2+y^2+z^2-l^2 = 0.$$

This constraint restricts the configuration to a two dimensional sphere of radius l centered at the origin. To see another example of such constraints, see our previous discussion of the double pendulum and pendulum with moving point of support.

We note that the constraints will restrict the velocities:

$$\frac{d}{dt}C_\alpha = \frac{\partial C_\alpha}{\partial q^i}\dot{q}^i + \frac{\partial C_\alpha}{\partial t} = 0.$$

For example in the spherical pendulum we have

$$\vec{r}\cdot\dot{\vec{r}} = 0.$$

There are two ways to deal with such constraints. Firstly, one can simply solve the constraints, *i.e.,* find an independent set of generalized coordinates. We have been doing this all along in our examples with constraints. For the spherical pendulum, we solve the constraint by

$$x = l\sin\theta\cos\phi, \quad y = l\sin\theta\sin\phi, \quad z = l\cos\theta,$$

and express everything in terms of θ and ϕ, in particular the Lagrangian and EL equations. In principle this can always be done, but in practice this might be difficult. There is another method in which one can find the equations of motion without having to explicitly solve the constraints. This is known as the method of *Lagrange multipliers.* This method is not just popular in mechanics, but also features in "constrained optimization" problems, *e.g.,* in economics. As we shall see, the Lagrange multiplier method is more than just an alternative approach to constraints – it provides additional physical information about the forces which maintain the constraints.

Lagrange Multipliers

The method of Lagrange multipliers in the calculus of variations has an analog in ordinary calculus. Suppose we are trying to find the critical points of a function $f(x,y)$ subject to a constraint $C(x,y) = 0$. That is to say, we want to find where *on the curve defined by the constraint* the function has a maximum, minimum, saddle point. Again, we could try to solve the constraint, getting a solution of the form $y = g(x)$. Then we could substitute this into the function f to get a (new) function $h(x) = f(x, g(x))$. Then we find the critical points by solving $h'(x) = 0$ for $x = x_0$ whence the critical point is $(x_0, g(x_0))$ This is analogous to our treatment of constraints in the variational calculus thus far (where we solved the constraints via generalized coordinates before constructing the Lagrangian and EL equations). There is another method, due to Lagrange, which does not require explicit solution of the constraints and which gives useful physical information about the constraints.

To begin with, when finding a critical point (x_0, y_0) subject to the constraint $C(x, y) = 0$ we are looking for a point on the curve $C(x, y) = 0$ such that a displacement tangent to the curve does not change the value of f to first order. Let the tangent vector to $C(x, y) = 0$ at the point (x_0, y_0) on the curve be denoted by $\vec{t}$. We want

$$\vec{t} \cdot \nabla f(x_0, y_0) = 0 \quad \text{where } C(x_0, y_0) = 0.$$

Evidently, at the critical point the gradient of f is orthogonal to the curve $C(x, y) = 0$. Now, any vector orthogonal to the curve – orthogonal to $\vec{t}$ at (x_0, y_0) – will be proportional to the gradient of C at that point.* Thus the condition for a critical point (x_0, y_0) of f (where $C(x_0, y_0) = 0$) is that the gradient of f and the gradient of C are proportional at (x_0, y_0). We write

$$\nabla f + \lambda \nabla C = 0,$$

for some λ. This requirement is meant to hold only on the curve $C = 0$, so without loss of generality we can take as the critical point condition

$$\nabla(f + \lambda C) = 0, \quad C = 0.$$

This constitutes *three* conditions on 3 unknowns; the unknowns being (x, y) and λ. The function λ is known as a *Lagrange multiplier*. In fact, if we artificially enlarge our x-y plane to a 3-d space parametrized by (x, y, λ) we can replace the above critical point condition with

$$\tilde{\nabla}(f + \lambda C) = 0,$$

where $\tilde{\nabla}$ is the gradient in (x, y, λ) space. You should prove this as an exercise.

To summarize: the critical points (x_0, y_0) of a function $f(x, y)$ constrained to a curve $C(x, y) = 0$ can be obtained by finding unconstrained critical points (x_0, y_0, λ_0) of a function

$$\tilde{f}(x, y, \lambda) = f(x, y) + \lambda C(x, y).$$

We can do the same thing with our variational principle. Suppose we have an action for n degrees of freedom q^i, $i = 1, 2, \ldots, n$:

$$S[q] = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t), t)$$

where the configuation space is subject to m constraints

$$C_\alpha(q, t) = 0, \quad \alpha = 1, 2, \ldots, m.$$

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* This follows from the basic calculus result that the gradient of a function is orthogonal to the locus of points where the function takes a constant value.

Let the solutions to the constraints be given in terms of generalized coordinates s^A, $A = 1, 2, \ldots, n-m$,

$$q^i = F^i(s,t),$$

i.e.,

$$C_\alpha(F^i(s,t),t) = 0.$$

The functions F^i determine the graph of the solution set of $C_\alpha = 0$ in the configuration space. The correct equations of motion can be obtained by substituting the solutions $q^i = F^i(s,t)$ into the Lagrangian for q^i, thus defining a Lagrangian for s^A, and computing the resulting EL equations for s^A. Using the same technology you used in your homework to study the effect of a point transformation on the EL equations (principally the chain rule), it is not hard to see that the correct equations of motion are then

$$\frac{\partial F^i}{\partial s^A}\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)_{q=F(s,t)} = 0.$$

We note that the functions $\frac{\partial F^i}{\partial s^A}$ have the geometric meaning of (a basis of) $(n-m)$ tangent vectors to the $(n-m)$-dimensional surface $C_\alpha = 0$. Thus the equations of motion are the statement that the projections of the EL equations along the surface must vanish.

Now we introduce the Lagrange multiplier method. We consider a modified action,

$$\tilde{S}[q,\lambda] = \int_{t_1}^{t_2} dt\, \tilde{L} = S[q] + \int_{t_1}^{t_2} dt\, \lambda^\alpha(t) C_\alpha(q(t),t),$$

in which we have added m new configuration variables λ^α, $\alpha = 1, 2, \ldots, m$; these are the Lagrange multipliers. The variation of the new action is

$$\delta\tilde{S} = \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i + \int_{t_1}^{t_2} dt\, \left(\delta\lambda^\alpha C_\alpha + \lambda^\alpha \frac{\partial C_\alpha}{\partial q^i}\delta q^i\right).$$

The EL equations of motion coming from $\tilde{L}$ are

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \lambda^\alpha \frac{\partial C_\alpha}{\partial q^i} = 0,$$

which come from the variations in q^i and also

$$C_\alpha = 0,$$

which come from variations of λ^α. We have $(n+m)$ equations for $(n+m)$ unknowns. In principle they can be solved to get the q^i and the λ^α as functions of t.

What is the meaning of these equations? Well, the constraints are there, of course. But what about the modified EL expressions? The EL equations you *would* have gotten from L now have a "force term", $\lambda^\alpha\frac{\partial C_\alpha}{\partial q^i}$. The force term is geometrically orthogonal to the surface $C_\alpha = 0$ in configuration space. This you can see from the identity (exercise)

$$0 = \frac{\partial}{\partial s^A}C_\alpha(F(s), t) = \frac{\partial C_\alpha}{\partial q^i}\frac{\partial F^i}{\partial s^A}.$$

(Recall that $\frac{\partial F^i}{\partial s^A}$ represend $n-m$ vectors tangent to the surface defined by $C_\alpha = 0$.) Thus the meaning of the EL equations coming form $\tilde{L}$ is that the EL expressions coming from L no longer have to vanish, they simply have to be orthogonal to the constraint surface since the equations of motion say that

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\lambda^\alpha\frac{\partial C_\alpha}{\partial q^i}.$$

One physically interprets this "force term" as the force required to keep the motion on this surface.

It is easy to verify that these modified equations, $(n + m)$ in number, are equivalent the correct $(n - m)$ equations obtained for s^A earlier. Indeed, we have the m equations of constraint. And, given this constraint, to say the EL expression coming from L is orthogonal to $C_\alpha = 0$ is the same as saying its projection tangent to the surface vanishes, *i.e.*,

$$\frac{\partial F^i}{\partial s^A}\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)_{q=F(s,t)} = \frac{\partial F^i}{\partial s^A}\left(-\lambda^\alpha\frac{\partial C_\alpha}{\partial q^i}\right) = 0,$$

which is precisely the content of the equations for the s^A we obtained above.

Example: Plane pendulum revisited

Let us study the plane pendulum using Lagrange multipliers. We model the system as moving in a plane with coordinates (x, y) subject the constraint

$$C = x^2 + y^2 - l^2 = 0.$$

Without the constraint the Lagrangian would be simply

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy.$$

According to our general prescription for incorporating the constraint, we construct the modified Lagrangian

$$\tilde{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(x^2 + y^2 - l^2).$$

The critical points for the action built from $\tilde{L}$, with the configuration space parametrized by (x, y, λ), should give us the critical points along the surface $C = 0$. To find the critical points we construct the EL equations as usual. We get

$$x^2 + y^2 - l^2 = 0,$$

coming from the variation of λ, and

$$2\lambda x - m\ddot{x} = 0, \quad 2\lambda y - mg - m\ddot{y} = 0,$$

coming from the variations of x and y, respectively.

Here we can see more explicitly how the Lagrange multiplier defines a force term beyond the gravitational force. This "force of constraint" represents the force of the rigid pendulum arm upon the particle and is given by

$$\vec{F}_{constraint} = 2\lambda x\hat{x} + (2\lambda y - mg)\hat{y}.$$

The typical analysis of EL equations involving Lagrange multipliers can now be nicely demonstrated. First, the three EL equations can be solved for λ (exercise)

$$\lambda = \frac{m}{2l^2}\left(x\ddot{x} + y\ddot{y} + gy\right).$$

Next, differentiation of the constraint twice reveals:

$$\ddot{C} = 0 \quad \Longrightarrow \quad x\ddot{x} + y\ddot{y} = -(\dot{x}^2 + \dot{y}^2),$$

so that the multiplier λ can be solved for in terms of the original velocity phase space variables:

$$\lambda = -\frac{m}{2l^2}(\dot{x}^2 + \dot{y}^2 - gy).$$

Substituting this result back into the EL equations for x and y we get the equations of motion for x and y with the effect of the constraint — physically, the tension in the rod — taken into account:

$$m\ddot{x} = -\frac{m}{l^2}(\dot{x}^2 + \dot{y}^2 - gy)x, \quad m\ddot{y} = -\frac{m}{l^2}(\dot{x}^2 + \dot{y}^2 - gy)y - mg.$$

Note we never had to solve the constraint! Still, as a nice exercise you can check that, after solving the constraint with $x = l\cos\phi$, $y = -l\sin\phi$, these remaining 2 equations are equivalent the familiar equation of motion for a plane pendulum, namely,

$$\ddot{\phi} = -\frac{g}{l}\sin\phi,$$

where ϕ is the angular displacement from equilibrium.

Using Lagrangian multipliers, the equations of motion for x and y tell us that the pendulum moves according to a superposition of forces consisting of (i) gravity, (ii) the force of constraint $\vec{F}_{constraint}$ needed to keep the mass moving in a circle of radius l. This latter force is supplied by the Lagrange multiplier terms in the equation of motion. Indeed, thanks to these Lagrange multiplier terms, the radial component of the net force is (exercise)

$$\frac{\vec{r}}{l} \cdot \vec{F} = -\frac{m}{l}(\dot{x}^2 + \dot{y}^2),$$

which is the centripetal force, as it should be.

To summarize: Given a dynamical system with coordinates q^i and Lagrangian L, we can impose constraints $C_\alpha(q,t) = 0$ by the following recipe.

(i) Add variables λ^α – the Lagrange multipliers – to the configuration space,

(ii) Define a Lagrangian on the augmented velocity phase space $\tilde{L} = L + \lambda^\alpha C_\alpha$,

(iii) Compute the usual EL equations from $\tilde{L}$ for the q^i and λ^α degrees of freedom.

The resulting equations will include the constraints themselves as equations of motion coming from variations of λ^α. The equations coming from the variations of the q^i will have extra terms involving the multipliers. For Newtonian systems these terms represent the forces in the system which are necessary to enforce the constraints.

Thus the Lagrange multiplier method has distinct advantages over our previous approach in which we just solve the constraints at the beginning.. Namely, you do not have to explcitly solve the constraints in order to compute the equations of motion, and the equations of motion have additional physical information: the forces of constraint.