Physics 3700Carnot Cycle. Heat Engines. Refrigerators.Relevant sections in text: §4.1, 4.2

## **Heat Engines**

A heat engine is any device which, through a cyclic process, absorbs energy via heat from some external source and converts (some of) this energy to work. Many real world "engines" (*e.g.*, automobile engines, steam engines) can be modeled as heat engines. There are some fundamental thermodynamic features and limitations of heat engines which arise irrespective of the details of how the engines work. Here I will briefly examine some of these since they are useful, instructive, and even interesting.

The principal piece of information which thermodynamics gives us about this simple — but rather general — model of an engine is a limitation on the possible *efficiency* of the engine. The efficiency of a heat engine is defined as the ratio of work performed  $\mathcal{W}$  to the energy absorbed as heat  $Q_h$ :

$$e = \frac{\mathcal{W}}{Q_h}.$$

Let me comment on notation and conventions. First of all, note that I use a slightly different symbol for work here. Following standard conventions (including the text's) for discussions of engines (and refrigerators), we deal with the work done by the engine on its environment. So, in the context of heat engines if W > 0 then energy is *leaving* the thermodynamic system in the form of work. Secondly, the subscript "h" on  $Q_h$  does not signify "heat", but "hot". The idea here is that in order for energy to flow from the environment into the system the source of that energy must be at a higher temperature than the engine.

Evidently, from our definition, if we manage to convert all the energy absorbed as heat into work then the efficiency of the engine is unity. This is reasonable in light of conservation of energy; an efficiency of one is the best we could possibly do in light of the first law. To see this more clearly, recall that a heat engine is by definition a cyclic process. "Cyclic" means that the engine periodically returns to the same state. The change in internal energy of the system (and any other observable, for that matter) after each cycle is zero. The first law then tells us that after a cycle the net heat absorbed by the system is the energy transferred out of the system by work. So, just taking account of the first law, we could imagine a cyclic process where heat goes in and an equal amount of work goes out — this is the case e = 1. Clearly e > 1 is not possible. All of this we learn from the first law. We can summarize the effect of the first law as: "You can't win".

The second law of thermodynamics implies that an efficiency of unity is in fact not possible; we must have e < 1. One of our primary goals is to see how this comes about. The main reason is that the heat transfer of energy into the engine involves an increase in entropy of the engine. Recall our result that  $dS \ge \delta Q/T$ . Assuming the heat is absorbed at constant temperature, we have in fact

$$\Delta S \ge \frac{Q_h}{T}.$$

This entropy must be removed by the time the engine completes one cycle of operation since all observables must return to their original values if the system is back in its original state. So whatever entropy increase accompanied the absorption of energy by heat must be undone by the end of the cycle. This removal of entropy requires expelling some heat from the engine. In light of this heat transfer out of the system, conservation of energy tells us that the work done is, at best, the difference of the heat absorbed and the heat expelled.

To make all this precise, we let  $Q_h$  be the heat *absorbed* by the engine from some hotter reservoir at temperature  $T_h$  and we let  $Q_c$  be the heat *expelled* by the engine to some colder reservoir at temperature  $T_c$ .<sup>\*</sup> We model these heat transfers of energy as occurring at constant temperatures  $T_h$  and  $T_c$ . Note that (as we did with the work) we have changed the convention for the sign of heat at the colder temperature. If  $Q_c > 0$ then energy has been transferred *from* the engine *to* the colder reservoir. At the end of any cyclic process the internal energy has not changed. Conservation of energy – the first law – then tells us

$$Q_h - Q_c - \mathcal{W} = 0.$$

It is now easy to see that

$$e = \frac{\mathcal{W}}{Q_h} = 1 - \frac{Q_c}{Q_h}.$$

This result follows from the definition of efficiency and the first law only. Next we bring in the second law.

The second law asserts that after the cyclic process the change in entropy of the "universe" – here modeled as the engine and its hot and cold reservoirs of energy – must not decrease. At the end of one engine cycle, the entropy of the engine has not changed. Hence the change in entropy of the environment must be greater than or equal to zero by the second law. But we know (from our discussion of the thermodynamic identity) that the change in entropy of the environment is

$$\Delta S_{env} = \frac{Q_c}{T_c} - \frac{Q_h}{T_h}$$

<sup>&</sup>lt;sup>\*</sup> Why hot and cold reservoirs? If energy is to flow into the engine as heat, the source of the energy must be hotter. Similarly, if energy is to flow from the engine as heat, it must flow to a colder place.

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And the second law then tells us

$$\frac{Q_c}{T_c} - \frac{Q_h}{T_h} \ge 0.$$

This means: (1)  $Q_c$  cannot be zero — some heat must be expelled from the engine; (2)

$$\frac{Q_c}{Q_h} \ge \frac{T_c}{T_h},$$

so that

$$e \le 1 - \frac{T_c}{T_h}.$$

Consequently e < 1. This is pretty neat: just assuming a cyclic process and the first and second laws we get a very simple, explicit upper limit on the efficiency of an engine!

The first law says you can not get efficiency greater than unity. The second law forbids an efficiency of unity – not all energy absorbed as heat can be converted into work. Better efficiency comes by making the ratio  $\frac{T_c}{T_h}$  as small as possible. We can summarize the effect of the second law as: "You can't break even."

We have found a limit on the efficiency of an engine. How good is this limit? In other words, is the upper bound on efficiency attainable? From our derivation of the upper bound it is clear that to attain it one must create no net entropy in the environment after each cycle. In particular, the entropy gained/lost by the engine via  $Q_h/Q_c$  must be exactly the entropy lost/gained by the reservoirs  $T_h/T_c$ . This can be done via the *Carnot cycle*. This quasi-static cycle involving compressional work has 4 steps: (a) isothermal expansion from  $V_1$  to  $V_2$  at  $T_h$ ; (b) adiabatic expansion from  $(V_2, T_h)$  to  $(V_3, T_c)$ ; (c) isothermal compression from  $V_3$  to  $V_2$  at  $T_c$ ; (d) adiabatic compression from  $(V_2, T_c)$  to  $(V_1, T_h)$ . You can compute the efficiency of this quasi-static process for an ideal gas and check that it is the maximum possible efficiency (your homework problem 4.5). This is guaranteed by the fact that the net change in entropy of the engine plus environment is zero. To see this, you have to observe that steps (b) and (d) involve no heat transfer and hence no exchange of entropy. Steps (a)/(c) have the entropy gained/lost by the engine exactly the same as the entropy lost/gained by the environment. At the end of the cycle, the entropy change of the engine is zero and then so is that of the environment.

Now for the fine print. Strictly speaking, what the argument above shows is that you can get arbitrarily close to the maximum efficiency via the Carnot cycle. The reason is that you will not really be able to get heat transfer to occur in steps (a) or (c) unless the temperature of the engine in these steps is less or greater than that of the hot or cold reservoir, respectively. But, as we have seen, when energy is transferred by heat from a hotter to a colder system the net entropy of the combined system increases. Thus, insofar as the engine is cooler than  $T_h$  during step (a) and hotter than  $T_c$  during step (c), the entropy of the universe (the engine and the reservoirs) will increase and the efficiency will

be less than its theoretical maximum. The idea of the Carnot cycle is that as you let the engine temperatures approach those of the hot and cold reservoirs the efficiency approaches its maximum. Of course, this means that the engine will work *very slowly* (during steps (a) and (c)) when it is near its maximum efficiency.

As an example of all this stuff, let's have a look at the cyclic process featuring in Figure 1.10b on page 23 of your text (Problem 1.34). It is not hard to compute that the cycle has a *net* absorption of heat

$$Q = (P_2 - P_1)(V_2 - V_1),$$

with work done by the system of

$$\mathcal{W} = (P_2 - P_1)(V_2 - V_1).$$

This does not mean the efficiency is unity since  $Q = Q_h - Q_c$  and the efficiency is  $\mathcal{W}/Q_h$ .

Let us suppose the gas is diatomic and that

$$V_2 = 3V_1, \quad P_2 = 2P_1.$$

We will seek to answer two questions: (i) What is the efficiency of this process? (ii) How does the efficiency compare with the theoretical limit derived above? (These questions consitute problem 4.1.)

To determine the efficiency of this cycle we can compute the ratio of  $\mathcal{W}$  and  $Q_h$ . Keeping in mind our convention for work, and noting that this quantity is just the area of the rectangle, we have

$$\mathcal{W} = (P_2 - P_1)(V_2 - V_1) = 2P_1V_1$$

During each of the four processes there is heat transfer. During processes A and B the heat transfer is into the engine – energy is absorbed. Using  $U = \frac{5}{2}NkT$ , the work done, and the first law, the heat absorbed is found to be

$$Q_h = \frac{5}{2}V_1(P_2 - P_1) + \frac{7}{2}P_2(V_2 - V_1) = \frac{33}{2}P_1V_1.$$

We then get

$$e = \frac{\mathcal{W}}{Q_h} = \frac{4}{33} = 0.12$$

To estimate the upper limit of the efficiency we compare with an "ideal" cycle in which all the heat absorption takes place at the hottest temperature of the cycle and the heat expulsion takes place at the lowest temperature of the cycle. These temperatures can be found from PV = NkT. Evidently,

$$T_h = \frac{P_2 V_2}{Nk}, \quad T_c = \frac{P_1 V_1}{Nk}.$$

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We then have

$$e \le 1 - \frac{P_1 V_1}{P_2 V_2} = \frac{5}{6} = 0.83$$

Evidently, the cycle shown in figure 1.10b is not very efficient relative to the Carnot type of cycle.

## Refrigerators

An ideal heat engine absorbs heat  $Q_h$  from a hot reservoir, turns some of this energy into work  $\mathcal{W}$ , and expels the rest of the energy as heat  $Q_c$  to a cold reservoir. A *refrigerator* is essentially this process in reverse and models, well, your refrigerator as well as your air conditioner. The reverse process consists of (i) heat transfer of energy  $Q_c$  from a cold reservoir (inside of fridge, your house), (ii) heat  $Q_h$  expelled to a hot reservoir (outside); all this is possible because (iii) work W is done on the refrigerator. Note the sign conventions on the heats and work. Heat transferred to the system from a cold reservoir is positive. Heat transferred from the system to a hot reservoir is positive. Note also that in physics the refrigerator is whatever system is moving the energy from place to place. It is often some kind of compressor and fluid/gas. This is a little different from everyday usage where the "refrigerator" (where you keep your food) is actually the cold reservoir.

In this cyclic refrigeration process the energy for work is supplied by the environment (e.g., your wall socket and the compressor) to the system (the refrigerant – used to be freon, now HFC-134) with the goal being the energy transfer  $Q_c$ . Thus the useful measure of efficiency is the *coefficient of performance* (COP) defined by\*

$$COP = \frac{Q_c}{W}.$$

Even though it is a bit of a rehash, I think it is instructive to briefly repeat the exercise of using the first and second laws to put a limit on the COP.

As before, the first law tells us that at the end of a cyclic process:

$$\Delta U = 0 \quad \Longrightarrow \quad Q_c - Q_h + W = 0,$$

so that

$$COP = \frac{Q_c}{Q_h - Q_c} = \frac{1}{\frac{Q_h}{Q_c} - 1}.$$

The change in entropy of the refrigerator engine is, of course, zero after a cycle. The change in entropy of the environment (including the inside of the fridge, or the inside of your house) is

$$\Delta S = \frac{Q_h}{T_h} - \frac{Q_c}{T_c}.$$

<sup>\*</sup> Note that we use the original notation for the work W. In the context of refrigerators our original convention, where W is the work done *on* the system, is being used.

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The second law then tells us that

$$\frac{Q_h}{T_h} - \frac{Q_c}{T_c} \ge 0.$$

Hence

$$\frac{Q_h}{Q_c} \ge \frac{T_h}{T_c},$$

and

$$COP \le \frac{1}{\frac{T_h}{T_c} - 1}$$

Alternatively,

$$COP \le \frac{T_c}{T_h - T_c}$$

You can see that the COP can be greater than one – indeed, it is unbounded as  $T_h \to T_c$ ! The refrigerator can be most efficient when  $T_h$  and  $T_c$  are near each other. Of course, if  $T_h \approx T_c$  not much cooling gets done!

For a given  $T_h$  and  $T_c$  one can use a Carnot cycle, reversed from what we did with the heat engine, to get the COP arbitrarily close to its upper bound (at the expense of cooling things down more and more slowly).

Let us estimate the maximum possible COP for an air conditioner operating in a typical home. We assume the inside of the house – the cold reservoir – is kept at 20 C, and the outside of the house – the hot reservoir is at 35 C. We then get

$$COP \le \frac{293}{15} = 19.5.$$

This means that a "perfect" air conditioner would remove 19.5 joules of energy for every joule it cost you to run it.

It is perhaps worth noting that non-ideal processes modeling engines and real refrigerators are not necessarily the reverses of each other. For example, we studied the rectangular process of problem 1.34, figure 1.10 b, which was an engine: heat was absorbed in steps A and B, heat was expelled in steps C and D, work was done in steps B and D. Note that the temperature is climbing as you move from A to B, then declining as you move from C to D. Running this in reverse, heat is absorbed from a cold reservoir in steps D and C, and expelled to a hot reservoir in steps B and A. As you go through the process DCBA the temperature is climbing, eventually the temperature of the gas upon which work is done has to be bigger than that of the cold reservoir (e.g. the inside of the fridge) this won't do if your goal is to cool it down.