Physics 3550, Fall 2011

Angular Momentum.

Relevant Sections in Text: §3.4, 3.5

Angular Momentum

You have certainly encountered angular momentum in a previous class. The importance of angular momentum lies principally in the fact that it is conserved in appropriate circumstances, as we shall explore. It is worth noting that the angular momentum of electrons in atoms is one of the most important physical features of atoms and largely makes them what they are.

The angular momentum of a single particle is defined by*

$$\vec{l} = \vec{r} \times \vec{p},$$

where usually the momentum is just $\vec{p} = m\vec{v}$ (more on this later). Notice that the angular momentum involves the position vector, which is defined relative to a choice of origin. Thus the angular momentum is defined relative to an origin. It is easy to forget this. Don't.

As I mentioned, we focus on angular momentum because of its conservation under appropriate circumstances. What are these circumstances? Well, let the particle move according to Newton's second law, we have

$$\frac{d\vec{l}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F}.$$

The first term involves the cross product of two parallel vectors and so it vanishes. We then get

$$\frac{d\vec{l}}{dt} = \vec{\gamma},$$

where

$$\vec{\gamma} = \vec{r} \times \vec{F}$$

is the *torque*. Like the position vector and the angular momentum, the torque is always defined relative to a choice of origin. We tacitly assume the torque is defined relative to the same origin as the angular momentum. We see then that the angular momentum of a particle is conserved if and only if the torque vanishes.

The torque can vanish – and angular momentum is conserved – in three possible ways that you can easily see from the definition of torque as a cross product. Either the particle

* Why is angular momentum defined this way? Is it always defined that way? Why is the total angular momentum conserved? All these questions will be answered when we explore the Lagrangian form of mechanics.

spends all its time at the origin. Or the net force on the particle is zero. Or the force is parallel to the position vector, *i.e.,* the force points toward the origin. The latter possibility is the most important; such forces are called *central forces.*

What happens if you we use different origins to compute our angular momenta? Let the two origins differ by a vector $\vec{d}$, the two position vectors are related by

$$\vec{r}_1 = \vec{r}_2 + \vec{d}.$$

So the two angular momenta are related by

$$\vec{l}_1 = \vec{l}_2 + \vec{d} \times \vec{p}.$$

Note, in particular, that this means if the angular momentum relative to a particular choice of origin is conserved it may not be conserved relative to another choice of origin:

$$\frac{d\vec{l}_1}{dt} = \frac{d\vec{l}_2}{dt} + \vec{d} \times \frac{d\vec{p}}{dt} = \frac{d\vec{l}_2}{dt} + \vec{d} \times \vec{F}.$$

A classic and important application of angular momentum conservation appears in trying to understand Kepler's second law of planetary motion. Recall that Kepler's first law says that planets move on ellipses, with the sun at one of the foci. (We will prove this later.) The second law of Kepler says that the motion in time of the planets is such that the relative position vector of the planet and the sun sweeps out equal areas in equal times. This result can be understood via conservation of angular momentum. Let's look into this.

First, we must assume that the sun is so much more massive than the planets that the center of mass can be idealized as being at the center of the sun:

$$\vec{R} = \frac{1}{m_{sun} + m_{planet}} \left(m_{sun}\vec{r}_{sun} + m_{planet}\vec{r}_{planet} \right) \approx \vec{r}_{sun}, \quad m_{planet} << m_{sun}.$$

It is this point which the planets orbit around and it is there we place the origin.

To verify that the angular momentum is conserved we note that – in the chosen reference frame – the force on the planet is of the form

$$\vec{F} = \frac{k}{r_{planet}^3}\vec{r}_{planet},$$

where k has Newton's constant and the masses. This is a central force; the torque vanishes and angular momentum is conserved.

Now we prove Kepler's second law using angular momentum conservation. Consider the position vector at two "nearby" times, t and $t + dt$. In the limit as the time interval vanishes, the two position vectors and their relative displacement form a (very skinny!) triangle with edges. $\vec{r}(t)$, $\vec{r}(t+dt)$ and

$$d\vec{r} = \vec{r}(t+dt) - \vec{r}(t).$$

Next we use an important math fact. $\vec{a}$, $\vec{b}$ and $\vec{c} = \vec{a} - \vec{b}$ define a triangle. The area of that triangle is the magnitude of one-half the cross product of any two of the vectors. You can easily check this – and you should check it since this is one of the main geometric intepretations of the cross product. Applying this result to the position vectors we have that the area dA swept out in time dt is

$$dA = \frac{1}{2}||\vec{r} \times d\vec{r}||.$$

Now, divide by dt to get*

$$\frac{dA}{dt} = \frac{1}{2m_{planet}}||\vec{r} \times \vec{p}|| = \frac{1}{2m_{planet}} l.$$

So, assuming $\vec{l}$ (or at least its magnitude) does not change in time, we see that the rate of change of area is constant in time, *i.e.,*

$$A(t_2) - A(t_1) = (const.)(t_2 - t_1),$$

or

$$\Delta A = (const.)\Delta t,$$

which is Kepler's second law. (We got this formula by integrating.)

Let us look at another example. We return to the system consisting of a charged particle (mass m, charge q) moving in a uniform magnetic field, $\vec{B}k$. The force is

$$\vec{F} = q\vec{v} \times \vec{B}.$$

The magnetic field is homogeneous (but not isotropic); we can fix the origin anywhere we like. The torque is

$$\vec{\gamma} = \vec{r} \times \vec{F} = q\vec{r} \times (\vec{v} \times \vec{B}).$$

Next I invoke a very famous vector identity (you should make sure you know this one):

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}.$$

From this identity the torque takes the form:

$$\vec{\gamma} = q(\vec{r} \cdot \vec{B})\vec{v} - (\vec{r} \cdot \vec{v})\vec{B}.$$

* Note this is a nice, generally valid, geometric interpretation of the magnitude of angular momentum.

From this result we can recover the familiar result that for circular motion in a plane perpendicular to $\vec{B}$, the angular momentum is conserved. The angular momentum in this case is parallel to $\vec{B}$ and has magnitude mvr, where v is the particle's speed and r is the radius of the circle. Make sure you can prove this to yourself.

Angular momentum of a system of particles

Let us now consider a system consisting of two particles; the generalization to more than two particles is pretty straightforward. We define the *total angular momentum* as

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2.$$

The time rate of change of this observable is

$$\frac{d\vec{L}}{dt} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = \vec{\gamma}_1 + \vec{\gamma}_2 = \Gamma.$$

We have denoted by Γ the total torque on the system. The force on, say, particle 1 is a superposition of the force $\vec{F}_{21}$ from particle 2 and any external forces $\vec{F}_{1ext}$:

$$\vec{F}_1 = \vec{F}_{21} + \vec{F}_{1ext}.$$

Similarly, for the force on particle 2 we have

$$\vec{F}_2 = \vec{F}_{12} + \vec{F}_{2ext}.$$

Let us assume that the interparticle force obeys Newton's second law. We have

$$\vec{F}_{21} = -\vec{F}_{12} \equiv \vec{F}.$$

We can then write

$$\frac{d\vec{L}}{dt} = (\vec{r}_1 - \vec{r}_2) \times \vec{F} + \vec{\Gamma}_{ext}.$$

We now generalize our notion of *central force*. We say two particles interact via a central force if the mutual force is always directed along a line joining the two particles. In this case, we have

$$\frac{d\vec{L}}{dt} = \vec{\Gamma}_{ext}.$$

If there are no external torques the angular momentum is conserved.

A good example of a central force is the gravitational force, where

$$\vec{F}_{21} = G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_2 - \vec{r}_1).$$

So, for a binary star system, ignoring all external influences, the *total* angular momentum is conserved.

You can easily generalize this discussion to three or more particles. The key result is that, for a system of particles interacting via central forces, the time rate of change of total angular momentum is given by the net external torque.

Motion in a plane

Let us consider a nice application of some of our ideas developed thus far.

Consider the motion of two particles under a mutual central force with no other forces (*e.g.,* gravitational forces only). Again, think of a binary star system, or the Earth-Sun system. We can now show that there always exists a reference frame in which the motion must lie in a plane. The proof is pretty simple. First of all, since the forces are internal and obey Newton't third law, the center of the mass of the system must have no acceleration. Work in the reference frame which is at rest relative to the center of mass. Put the origin at the center of mass. In this reference frame we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0, \quad m_1\vec{v}_1 + m_2\vec{v}_2 = 0.$$

Note that this reference frame has the particles always having diametrically opposed position vectors. Now, consider the conserved total angular momentum; we have

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \mu\vec{r} \times \vec{v},$$

where

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{v} = \vec{v}_1 - \vec{v}_2,$$

are the relative position and velocity and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass.* To prove this, you need to use

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}, \quad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}.$$

The point of this result is the following. The total angular momentum $\vec{L}$ is conserved. This means that, as time runs, positions and velocities change, but $\vec{r} \times \vec{v}$ is a fixed vector in space. In particular, its direction is fixed. Since $\vec{r}$ and $\vec{v}$ are perpendicuar to this fixed vector, as time evolves the relative position (and velocity) moves in a plane orthogonal to $\vec{L}$. But this means that each of the particles moves in that plane. This is why planetary motion is always in a plane – the plane being determined by the angular momentum of the system.

Now for an amusing little paradox. Consider two bodies moving under an attractive central force, such as just described. The system could be the Earth-Sun system, or it could be the hydrogen atom. Under what circumstances is the angular momentum zero? Well, evidently, we can get $\vec{L} = 0$ by either having $\vec{r} = 0$, or $\vec{v} = 0$, or $\vec{r} \,||\, \vec{v}$. Physically these mathematical relations say, respectively, the particles are on top of each other, the particles are moving with the same speed in the same direction, the particles are moving toward each other or away from each other. The first situation is a collision. The second situation only happens when there is no force. The third situation corresponds (in the attractive case) to a collision in the future. Now for the paradox: the ground state of the hydrogen atom has zero angular momentum. How can this be?