Physics 3550, Fall 2011

Introduction. The Particle. Curves, velocity, acceleration. The first law.

Relevant Sections in Text: §1.1, 1.2, 1.4

Introduction.

Mechanics is about how and why things move the way they do. As you can imagine, it is one of the oldest disciplines in physics. In some sense, it is the originial discipline of physics as we know "physics" today (thanks to Newton). This course is meant to be the next step in mechanics sophistication after the mechanics explored in introductory physics. Here we get to use your improved math and physics skills to *really* understand what Newton did. You also get to move out of the 17^{th} century understanding of mechanics (Newton) into the 18^{th} century version (Lagrange), and even into the 19^{th} century version (Hamilton). The twentieth century, as you know, showed that mechanics according to Newton, Lagrange, and Hamilton was inadequate to describe the behavior of matter at atomic distance scales and smaller. Quantum mechanics is needed for that. But, at larger distance scales, "classical mechanics" – suitably amended to replace "Galilean relativity" with Einstein's theory of relativity (also from the early twentieth century) – is an extremely successful framework for studying matter and its interactions. Moreover, one can view classical mechanics (particularly in the Hamiltonian form) as the long-distance imprint of quantum mechanics. As such, one can understand lots of quantum mechanical structure already from classical mechanics structures. Hopefully, we will get to explore this latter point a little toward the end of the course.

The formal goals of this course include the following.

* Acquire a working knowledge of Newtonian, Lagrangian, and Hamiltonian formulations of mechanics.

* Acquire an introductory understanding of variational principles in general and in mechanics in particular.

* Begin to be well-versed in a suite of exactly soluble dynamical systems (e.g., the harmonic oscillator, the 2-body central force system, etc.).

* Begin to have a solid understanding of conservation laws, their utility, and their roots in symmetries of variational principles.

* Begin to be proficient in mapping mechanical systems to mathematical representations and analyzing the resulting mathematical model.

* Begin to be proficient with certain features of analytic geometry, vector analysis and ordinary differential equations.

* Begin to be proficient in scientific/mathematical written and oral exposition.

Yes! All this and more – in only one semester!

We'd better get started.

The ubiquitous "particle".

Probably you have heard "particles" being discussed on plenty of occasions. You have probably used the word yourself. But... what is a particle? In classical mechanics* the "particle" is a very useful, simplified model of ordinary objects, or "bodies". Mathematically, a particle is an object which is completely characterized by its position $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and velocity $\vec{v} = v^x\hat{i} + v^y\hat{j} + v^z\hat{k}$. Of course this is at best an idealization which works well sometimes (*e.g.,* to figure out the motion of the Earth around the Sun) but not so well other times (*e.g.,* to understand the Coriolis effect in our atmosphere, or to understand the tides). The particle concept is also important insofar as we can view extended objects as a collection of particles.

The idea of a particle is almost inevitable in Newtonian mechanics. We have the famous "$F = ma$", which is meant to explain motion ... but where is the force applied, and to what does the acceleration refer? If you are thinking of a baseball falling to the ground under the force of the Earth's gravity, then you can answer these questions pretty easily. Indeed, we almost instinctively model the baseball as a point mass located at the baseball's center of mass. This is a good idea; Newton tells us that acceleration of the ball's center of mass comes from the net gravitational force on the ball acting at its center of mass (as we shall see). But to *prove* that this is true, theoretically speaking, presupposes breaking the ball up into infinitely many "particles" each of which satisfies Newton's second law. And what about something a little more interesting, like the ocean – how do we model that? View it as a continuous distribution of "particles", each of which satisfies "$F = ma$".

Configuration space. Curves. Velocity. Acceleration.

Having informally agreed to use (or break everything up into) particles, we now know that the *configuration* of a particle is a point in Euclidean space. Usually we use a position vector $\vec{r}$, or Cartesian coordinates (x, y, z) to specify this location. Sometimes other coordinates are useful. Like cylindrical coordinates (ρ, ϕ, z) and spherical polar coordinates (r, θ, ϕ). Are you familiar with these *curvilinear coordinate systems*?

* In our most sophisticated physics framework for matter and its interactions, known as *quantum field theory* a "particle" has a pretty precise meaning in terms of excited states of a quantum field. At this point, the notion of "particle" is very different from a tiny little point flying around in space!

For the most part, mechanics is about motion. Motion of a particle is, evidently, a continuous change in its configuration, *i.e.,* its position, in the time t. Mathematically, then, motion can be specified by picking three functions $x(t)$, $y(t)$, and $z(t)$ such that the position of the particle (x, y, z) varies in time according to

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

that is,

$$\vec{r} = \vec{r}(t).$$

These equations define a curve in Euclidean space. For example,

$$x = \cos(t), \quad y = \sin(t), \quad z = t$$

defines a helical motion along the z-axis, much as you would get for a charged particle moving in a uniform magnetic field along the z-axis. In cylindrical coordinates (ρ, ϕ, z) this same curve looks like

$$\rho = 1, \quad \phi = t, \quad z = t.$$

Can you prove this?

So, the configuration of a particle is a point in Euclidean space, and the motion of a particle is a curve in Euclidean space. There are a couple of very important properties of curves which we now need. Both are familiar to you in one guise or another. They are velocity and acceleration. Velocity corresponds to the tangent to the curve. Of course, each point of the curve – each instant of time for a given motion – has a tangent, and that tangent may change from point to point in the curve – that is, change in time. We can compute the velocity vector along the curve by differentiating the formula for the curve with respect to time:

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t).$$

So, we have

$$\vec{v}(t) = v^x(t)\hat{i} + v^y(t)\hat{j} + v^z(t)\hat{k},$$

with

$$v^x(t) = \frac{dx(t)}{dt}, \quad v^y(t) = \frac{dy(t)}{dt}, \quad v^z(t) = \frac{dz(t)}{dt},$$

In our helical motion example you can easily check that

$$\vec{v}(t) = -\sin(t)\hat{i} + \cos(t)\hat{j} + \hat{k}.$$

Does this look right to you?

Exercise: What does the velocity look like in cylindrical coordinates? (Hint: this is slightly subtle and there are two answers.)

As you probably guessed, we can get the acceleration just by differentiating again:

$$\begin{aligned}\vec{a}(t) &= \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) \\ &= \frac{dv^x(t)}{dt}\hat{\imath} + \frac{dv^y(t)}{dt}\hat{\jmath} + \frac{dv^z(t)}{dt}\hat{k} \\ &= \frac{d^2x(t)}{dt^2}\hat{\imath} + \frac{d^2y(t)}{dt^2}\hat{\jmath} + \frac{d^2z(t)}{dt^2}\hat{k}\end{aligned}$$

Returning to the helical motion example, we have

$$\vec{a}(t) = -\cos(t)\hat{\imath} - \sin(t)\hat{\jmath}.$$

Does this make sense?

Note that the conditions for the acceleration vector to be zero are

$$\frac{d^2x(t)}{dt^2} = \frac{d^2y(t)}{dt^2} = \frac{d^2z(t)}{dt^2} = 0.$$

The solution of these differential equations is easy enough:

$$x(t) = x_0 + v_0^x t, \quad y(t) = y_0 + v_0^y t, \quad z(t) = z_0 + v_0^z t,$$

or

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t,$$

where $\vec{r}_0$ and $\vec{v}_0$ are 6 constants.

Exercise: Prove that these equations determine a straight line and that every straight line is defined this way.

Geometrically velocity is the tangent to the curve. What, then, is the geometric interpretation of acceleration? Curvature.

Newton's First Law. Inertial Reference Frames.

If a particle undergoes the helical motion we just described, is it experiencing a force? How about the zero acceleration motion – is there a force there? Trick questions! There are no answers to these questions without more information. To be sure, in the helical motion, for example, one instinctively assumes that some force must be making the particle execute the circular motion about the z axis. But what if the particle is actually experiencing no forces but we are viewing it from a rotating reference frame? Same result. We instinctively see the unaccelerated motion as being force free, but suppose the particle is actually in free fall in a gravitational field and we simply used a clock that happened to speed up in the "real" time so that the motion looked as shown above? Again, without further information one cannot say which is the true situation. You can think of this ambiguity in connecting accelerated motion to forces as a key reason for introducing Newton's first law.

Newton's first law, as Newton put it, is as follows.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

I know you have encountered some version of this law before. Have you ever wondered why it is a law separate from the second law? After all, if we have "$F = ma$", then with $F = 0$ we get $a = 0$ and that is uniform straight motion, right? The reason we need the first law is because of the ambiguity illustrated above. This is an ambiguity in the choice of *reference frame.* A reference frame is a way of labeling *events*, which are defined as things which happen at a given time and place. Put differently, a reference frame fixes points of space and instants of time. A reference frame depicts *spacetime.* The two interpretations of helical motion stem from labeling points of space in two different ways, differing by a steady rotation. The two interpretations of the straight line, uniform motion stem from labeling time in two different ways. *Newton's first law postulates that there exists a reference frame in which a body, isolated from external forces, will move with constant velocity.* It is tacitly assumed that the remaining laws apply to such reference frames. Indeed, "$F = ma$" doesn't hold in a rotating reference frame, nor in a reference frame with a non-uniform clock. We are meant to apply Newton's second law in inertial reference frames, which are defined by the first law.

Newton's first law is a rather deep statement about the nature of space and time itself. It asserts that spacetime events can be organized into *inertial reference frames* where Newton's second law holds. It is not too hard to see that if there exists one inertial reference frame then there exists infinitely many others. In particular, given an inertial reference frame, defined by a choice of (t, x, y, z) one is free to (i) rotate to a new set of axes, (ii) shift the origin of space to a new location, (iii) shift the origin of time to a new instant, (iv) use a reference frame which is moving at constant relative velocity. This set of transformations defines the totality of all inertial reference frames. The fact that there are so many of them reflects the homogeneity and isotropy of space and the homogeneity of time which is fundamental to Netwon's mechanics.

Assuming we are in an inertial reference frame, defined by the first law, we an now assert that the helical motion is due to a force and that the uniform, rectilinear motion is force-free. These are, of course, applications of the second law, to which we now turn.