We demonstrate the magnetic interactions (both forces and torques) between two uniformly magnetized spheres are identical to the magnetic interactions between two point magnetic dipoles.

I. INTRODUCTION

Let’s consider two permanent magnets of arbitrary shape. If each magnet has a nonzero dipole moment, then the dipole moments of these magnets will dominate their interactions at separations that are large compared with their sizes, and each magnet may be treated as if it were a point magnetic dipole. Dipolar fields and forces are often used as the starting point for analytical and numerical approximations of the forces between permanent magnets of various shapes [1–10].

A uniformly magnetized sphere produces a magnetic field that is identical to its dipole field, not just at large distances, but everywhere outside of the sphere [11, 12]. One is thus naturally lead to ask whether the forces and torques between two uniformly magnetized spheres are identical to those between two point dipoles, independent of their separation. Here we show this is indeed the case. This result has practical applications. Dipolar fields and forces have been used to approximate the interactions among assemblies of spherical nanoparticles [13] and magnetic microspheres [14]. Our results enable simple dipole interactions to be used to model the dynamical interactions between these magnets [20, 21].

Previous calculations of the force between two uniformly magnetized spheres have been carried out in three limiting geometries: (i) for magnetizations that are perpendicular to the line through the sphere centers [22], (ii) for parallel magnetizations that make an arbitrary angle with this line [23], and (iii) for configurations with one magnetization parallel to this line and the other in an arbitrary direction [24]. All three calculations yield forces that are identical to the field of an equivalent point dipole. Unlike the previous calculations, ours rely on simple symmetry arguments and pertain to spheres of arbitrary sizes, positions, magnetizations, and magnetic orientations.

The force between two dipoles is noncentral. Namely, this force is not generally directed along the line through the dipoles. While other noncentral magnetic forces violate Newton’s third law [31–33], we show that the paired forces between magnet dipoles obeys this law. These forces therefore exert a net \( \mathbf{r} \times \mathbf{F} \) torque on an isolated two-dipole system. This torque cancels the paired \( \mathbf{m} \times \mathbf{B} \) torques, which are not equal and opposite. Thus angular momentum is conserved.

II. POINT DIPOLE INTERACTIONS

In this section we review the interactions between point magnetic dipoles. The magnetic field at position \( \mathbf{r} \) produced by a point dipole \( \mathbf{m} \) located at the origin is given by [34, 35]

\[
\mathbf{B}(\mathbf{m}; \mathbf{r}) = \frac{\mu_0}{4\pi} \left( \frac{3\mathbf{m} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{m}}{r^3} \right),
\]

for \( r = |r| \) satisfying \( r > 0 \). This field can be obtained from the scalar potential

\[
\varphi(\mathbf{m}; \mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{r}}{r^3}
\]

via

\[
\mathbf{B}(\mathbf{m}; \mathbf{r}) = -\nabla \varphi(\mathbf{r}).
\]

Because \( \nabla \cdot \mathbf{B} = 0 \), \( \varphi \) satisfies Laplace’s equation,

\[
\nabla^2 \varphi = 0.
\]

We consider two dipoles, \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \), which are respectively located at positions \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). From Eq. (1), the field produced by dipole \( \mathbf{m}_i \) is

\[
\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i),
\]

where \( i = 1, 2 \), and where \( \mathbf{r} - \mathbf{r}_i \) is the position vector relative to dipole \( \mathbf{m}_i \) (Fig. 1). Accordingly, the field of
\( \mathbf{m}_i \) evaluated at the location of \( \mathbf{m}_j \) is

\[
\mathbf{B}_i(\mathbf{r}_j) = \frac{\mu_0}{4\pi} \left( \frac{3\mathbf{m}_i \cdot \mathbf{r}_{ij} \mathbf{r}_{ij} - \mathbf{m}_i}{r_{ij}^3} \right),
\]

where \( \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \) is the position of \( \mathbf{m}_j \) relative to \( \mathbf{m}_i \), and \( r_{ij} = |\mathbf{r}_j - \mathbf{r}_i| \).

The interaction energy between \( \mathbf{m}_j \) and the magnetic field of \( \mathbf{m}_i \) is

\[
U_{ij} = -\mathbf{m}_j \cdot \mathbf{B}_i(\mathbf{r}_j).
\]

Inserting Eq. (6) gives

\[
U_{ij} = \frac{\mu_0}{4\pi} \left[ \frac{\mathbf{m}_i \cdot \mathbf{m}_j}{r_{ij}^3} - \frac{3}{5} \left( \frac{\mathbf{m}_i \cdot \mathbf{r}_{ij}}{r_{ij}^4} \right) (\mathbf{m}_j \cdot \mathbf{r}_{ij}) \right].
\]

The force of \( \mathbf{m}_i \) on \( \mathbf{m}_j \) follows from

\[
\mathbf{F}_{ij} = -\nabla U_{ij},
\]

where \( \nabla_j \) is the gradient with respect to \( \mathbf{r}_j \). Inserting Eq. (8) yields

\[
\mathbf{F}_{ij} = \frac{3\mu_0}{4\pi r_{ij}^3} \left[ (\mathbf{m}_i \cdot \mathbf{r}_{ij}) \mathbf{m}_j + (\mathbf{m}_j \cdot \mathbf{r}_{ij}) \mathbf{m}_i + \frac{5}{3} \left( \frac{\mathbf{m}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} \right) (\mathbf{m}_j \cdot \mathbf{r}_{ij}) \right].
\]

The first two terms in the square brackets are respectively parallel to \( \mathbf{m}_j \) and \( \mathbf{m}_i \). Consequently, \( \mathbf{F}_{ij} \) is not central, namely, it is not generally parallel to the vector \( \mathbf{r}_{ij} \) between the dipoles.

Equations (8) and (10) imply that

\[
U_{21} = U_{12}
\]

and

\[
\mathbf{F}_{21} = -\mathbf{F}_{12},
\]

confirming that Newton’s third law applies to the magnetic force between point magnetic dipoles, and ensuring that linear momentum is conserved in an isolated two-dipole system.

We now investigate the torque of \( \mathbf{m}_i \) on \( \mathbf{m}_j \), which has two contributions [36],

\[
\tau_{ij} = \tau^A_{ij} + \tau^B_{ij}.
\]

The first arises from \( \mathbf{m}_j \) residing in the field of \( \mathbf{m}_i \),

\[
\tau^A_{ij} = \mathbf{m}_j \times \mathbf{B}_i(\mathbf{r}_j).
\]

The sum of paired torques

\[
\tau^A_{12} + \tau^A_{21} = \frac{3\mu_0}{4\pi r_{12}^3} \left[ \left( \mathbf{m}_1 \cdot \mathbf{r}_{12} \right) \mathbf{m}_2 \times \mathbf{r}_{12} \right]
+ \left( \mathbf{m}_2 \cdot \mathbf{r}_{12} \right) \mathbf{m}_1 \times \mathbf{r}_{12}
\]

is generally unequal to zero. Therefore, unlike the paired forces \( \mathbf{F}_{12} \) and \( \mathbf{F}_{21} \), the paired torques \( \tau^A_{12} \) and \( \tau^A_{21} \) are not generally equal and opposite.

The second contribution to the torque arises from the force of \( \mathbf{m}_i \) on \( \mathbf{m}_j \),

\[
\tau^B_{ij} = \mathbf{r}_j \times \mathbf{F}_{ij},
\]

for which the sum

\[
\tau^B_{12} + \tau^B_{21} = \mathbf{r}_{12} \times \mathbf{F}_{12}
\]

is also generally unequal to zero.

Equations (15) and (17) and a little algebra reveal that the net torque on an isolated pair of dipoles vanishes identically;

\[
\tau^A_{12} + \tau^A_{21} + \tau^B_{12} + \tau^B_{21} = 0.
\]

The torque supplied by \( \tau^A_{12} \) and \( \tau^A_{21} \) therefore cancels the torque supplied by \( \tau^B_{12} \) and \( \tau^B_{21} \), and angular momentum is conserved. Thus, an isolated dipole-dipole system does not spontaneously rotate, and there is no exchange between mechanical and electromagnetic momentum. Such exchanges have been the subject of considerable study [32, 37–40].

\section{III. FORCE BETWEEN SPHERES}

We now come to the crux of this paper, the force between two uniformly magnetized spheres. We present four separate arguments that show that the force between uniformly magnetized spheres is identical to the force between point dipoles. While each proof is sufficient to show this equivalence, each utilizes different concepts from mechanics and electromagnetism, and each has pedagogical value.

As seen in Fig. 1, we take sphere \( i \) to have position \( \mathbf{r}_i \), radius \( a_i \), magnetization \( \mathbf{M}_i \), and total dipole moment

\[
\mathbf{m}_i = \frac{4}{3} \pi a_i^3 \mathbf{M}_i.
\]
its magnetic field is given by $B = \frac{1}{2\mu_0} \mathbf{M} / 3$ inside the sphere (for $|\mathbf{r} - \mathbf{r}_i| < a_i$) and is given by Eq. (5) outside the sphere [11, 12]. We treat $\mathbf{M}_i$ as spatially uniform and constant in time, neglecting any demagnetization by external fields. This assumption is appropriate for high coercivity materials [41, 42].

A. Newton’s third law

A five-step argument involving Newton’s third law shows that the force between two spheres with uniform magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$ is identical to the force between two point dipoles with corresponding equivalent magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$, located at the same positions as the spheres. Figure 2 illustrates this argument.

Fig. 2(a): As in the preceding discussion, $\mathbf{F}_{12}$ represents the force of dipole 1 on dipole 2. This force is produced by the field $\mathbf{B}_1$ of dipole 1.

Fig. 2(b): Sphere 1 produces the same field $\mathbf{B}_1$, and therefore exerts the same force $\mathbf{F}_{12}$ on dipole 2.

Fig. 2(c): Newton’s third law gives the force $\mathbf{F}_{21} = -\mathbf{F}_{12}$ of dipole 2 on sphere 1. This force is produced by the field $\mathbf{B}_2$ of dipole 2.

Fig. 2(d): Sphere 2 produces the same field $\mathbf{B}_2$, and therefore exerts the same force $\mathbf{F}_{21}$ on sphere 1.

Fig. 2(e): To complete the argument, we again apply Newton’s third law to show that the force $\mathbf{F}_{12}$ of dipole 1 on dipole 2 (Fig. 2a).

B. Direct integration

The force between two uniformly magnetized spheres can be determined by integrating the energy density $-\mathbf{M}_j \cdot \mathbf{B}_i(\mathbf{r})$ associated with sphere $j$ sitting in the magnetic field $\mathbf{B}_i(\mathbf{r})$, giving the total interaction energy

$$U_{ij} = -\int_j \mathbf{M}_j \cdot \mathbf{B}_i \, dV = -\mathbf{M}_j \cdot \int_j \mathbf{B}_i \, dV. \quad (20)$$

Here, the integral is over the volume of sphere $j$ and the second equality exploits the uniformity of $\mathbf{M}_j$. If all sources of a magnetic field lie outside a particular sphere, then the spatial average of the field over the sphere is given by the value of the field at the sphere center [43]. That is,

$$\int_j \mathbf{B}_i \, dV = \frac{4}{3} \pi a_j^3 \mathbf{B}_i(\mathbf{r}_j). \quad (21)$$

Equations (19), (20), and (21) give $U_{ij} = -\mathbf{m}_j \cdot \mathbf{B}_i(\mathbf{r}_j)$, which replicates Eq. (7). Thus, the energy of interaction between the two spheres is identical to the energy of interaction between two point dipoles. The associated force $\mathbf{F}_{ij} = -\nabla U_{ij}$ of sphere $i$ on sphere $j$ is therefore identical to the force between two point dipoles, and obeys Newton’s third law.

C. Field energy

We can also show the force equivalence by integrating the magnetic energy density $B^2/2\mu_0$ over all space, giving the total magnetic energy of interaction

$$U(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\mu_0} \int (\mathbf{B}_1 + \mathbf{B}_2)^2 \, dV$$

$$= \frac{1}{2\mu_0} \int \left( B_1^2 + 2 \mathbf{B}_1 \cdot \mathbf{B}_2 + B_2^2 \right) \, dV. \quad (22)$$

Because the magnetic energy of a single dipole does not depend on its location in space, the self-energy integrals $(2\mu_0)^{-1} \int B_i^2 \, dV$ do not depend on $\mathbf{r}_i$. Therefore the force
\[ \mathbf{F}_{ij} = -\nabla_j U \] on sphere \( j \) depends only on the interaction energy

\[ U_{\text{int}} = \frac{1}{\mu_0} \int \mathbf{B}_1 \cdot \mathbf{B}_2 \, dV \]

\[ = \frac{1}{\mu_0} \left( \int + \int_2 + \int_{\text{outside}} \right) \nabla \varphi_1 \cdot \nabla \varphi_2 \, dV, \quad (23) \]

where we have inserted Eq. (3), and we have separated the integral over all space into integrals over sphere 1, sphere 2, and the region outside of both spheres.

For the integral over sphere 1, we use \( \nabla^2 \varphi_2 = 0 \) and the divergence theorem to write

\[ \int_1 \nabla \varphi_1 \cdot \nabla \varphi_2 \, dV = \int \nabla \cdot (\varphi_1 \nabla \varphi_2) \, dV = \int_{S_1} \varphi_1 \nabla \varphi_2 \cdot \hat{n}_1 \, dA, \quad (24) \]

where \( S_1 \) denotes the integral over the surface of sphere 1, and \( \hat{n}_1 \) is the unit vector directed normally outward from the sphere’s surface. The quantity \( \varphi_1 \nabla \varphi_1 \) is continuous across this surface. The surface values of \( \varphi_1 \) and \( \varphi_2 \) are identical to the potentials of point dipoles, and so the integral over sphere 1 is the same as it would be if the spheres were replaced by equivalent point dipoles. Similarly, the integral over sphere 2 is the same as for equivalent point dipoles. Since the fields outside of both spheres match the fields of equivalent point dipoles, all three integrals Eq. (23) are the same for point dipoles as for uniformly magnetized spheres. Therefore, the force \( \mathbf{F}_{ij} = -\nabla_j U_{\text{int}} \) must also be the same.

D. Stress tensor

The total force on an object can be calculated by integrating the Maxwell stress tensor over an arbitrary surface surrounding the object [44, 45]. The magnetic stress tensor depends only on the magnetic field. The field outside a uniformly magnetized sphere is the same as the field of a point dipole located at its center. The total field produced by and outside of two uniformly magnetized spheres, and the associated stress tensor, is the same as that of two point dipoles. Therefore, the force between spheres must be the same as the force between point dipoles.

IV. TORQUE BETWEEN SPHERES

Here we calculate the torque \( \tau_{ij} \) of sphere \( i \) on sphere \( j \). This torque has two contributions, as before.

The first arises from sphere \( j \) residing in the field of sphere \( i \), and can be obtained by integrating the torque density \( \mathbf{M}_j \times \mathbf{B}_i \) over the volume of sphere \( j \), giving

\[ \tau_{ij}^A = \int_j \mathbf{M}_j \times \mathbf{B}_i \, dV = \mathbf{M}_j \times \int_j \mathbf{B}_i \, dV. \quad (25) \]

Invoking Eqs. (19) and (21) gives \( \tau_{ij}^A = \mathbf{m}_j \times \mathbf{B}_i(r_j) \), which replicates Eq. (14). Thus, \( \tau_{ij}^A \) between two uniformly magnetized spheres is identical to the torque between two point magnetic dipoles.

The second contribution arises from the force of sphere \( i \) on sphere \( j \), which can be obtained from the interaction energy \( u_{ij} = -\mathbf{M}_j \cdot \mathbf{B}_i \) according to

\[ \mathbf{F}_{ij} = -\int_j \nabla u_{ij} \, dV. \quad (26) \]

This gives the same force as \( \mathbf{F}_{ij} = -\nabla_j U_{ij} \), with \( U_{ij} \) given by Eq. (7). The torque follows by integrating the torque density \( \mathbf{r} \times (-\nabla u_{ij}) \) over the volume of sphere \( j \);

\[ \tau_{ij}^B = -\int_j \mathbf{r} \times \nabla u_{ij}(r) \, dV. \quad (27) \]

Rewriting using the position \( r' = r - r_j \) relative to the center of sphere \( j \) gives

\[ \tau_{ij}^B = -\int_j (r' + r_j) \times \nabla u_{ij}(r) \, dV \]

\[ = -\int_j r' \times \nabla \Psi(r') \, dV - r_j \times \int_j \nabla u_{ij}(r) \, dV, \quad (28) \]

where \( \Psi(r') = u_{ij}(r' + r_j) \). Converting the first integral into a surface integral gives

\[ \int_j r' \times \nabla \Psi \, dV' = -\int_j \nabla \times (\Psi r') \, dV' \]

\[ = -\int_{S_j} \hat{n}_j \times r' \Psi \, dA'. \quad (30) \]

Since \( r' \) is parallel to the unit normal vector \( \hat{n}_j \), their cross product is zero and the integral vanishes. Combining Eqs. (26) and (28) gives

\[ \tau_{ij}^B = r_j \times \mathbf{F}_{ij}, \quad (31) \]

which is identical to Eq. (16). Thus, the torque associated with the force on a uniformly magnetized sphere in a dipole field is the same as the torque on the corresponding point dipole. These results ensure that the net torque on an isolated two-sphere system is zero, as seen earlier for point dipoles.

V. SUMMARY

We have demonstrated that the forces and torques between two uniformly magnetized spheres are identical to the forces and torques between two point magnetic dipoles.

This equivalence extends to uniformly polarized spheres and point electric dipoles. This is because the fields, forces, and torques of electric dipoles have the same mathematical forms as their magnetic counterparts, and the field outside of a uniformly polarized sphere is identical to the electric dipole field.
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[18] Typing “Zen Magnets” into the YouTube search field at https://www.youtube.com identifies over 90,000 videos describing various (accessed Mar. 23, 2016). As of August 22, 2014, the most popular of these had a total view count exceeding 145 million (Ref. [16], Appendix D).
[19] Several countries, including the United States, have banned the sale of 5-mm diameter nickel-coated neodymium magnets marketed as a desk toys under the trade names BuckyBalls, Zen Magnets, Neoballs, etc. following reports of intestinal injuries from ingestion of these magnets. But 2.5-mm magnets may be still be purchased at http://micromagnets.com (accessed Feb. 16, 2016), and magnet spheres of various sizes, including 5-mm spheres, may still be purchased from industrial suppliers (http://www.kjmagnetics.com/, http://www.alibaba.com, http://www.magnet-fless.com, accessed Feb. 16, 2016).
[21] John M. Edwards, MagPhyx Simulation and Visualization Software, http://www2.cose.isu.edu/edwa john/MagPhyx (accessed Mar. 11, 2016). This web-based software simulates the 2D motion of a magnet sphere in response to the forces and torques supplied by a second magnet sphere, held fixed. It is provided freely to the physics community for education and exploration.
[23] D. Vokoun and M. Beleg gia, “Forces between arrays of permanent magnets of basic geometric shapes,” J. Magn. and Magn. Mater. 350, 174-178 (2014). This publication quotes the result of an unpublished calculation of the magnetic interaction between two identical spheres with parallel magnetizations. Details of this calculation were sent privately to us by D. Vokoun on Feb. 8, 2016.


Richard P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* Vol. II, (Addison-Wesley, Boston, Massachusetts, 1964), p. 26-5. Here, Feynman discusses the example of the non-central force between two positively charged particles, one moving in the +x direction and the other moving in the +y direction. At the instant that particle 1 is at the coordinate origin and particle 2 is on the +y axis, the electric forces are central and repulsive, but particle 1 exerts a non-central magnetic force on particle 2 in the +x direction, while particle 2 exerts no magnetic force on particle 1. Thus, Newton’s third law does not apply, raising a paradox about the origin of the extra mechanical momentum.

Adam Caprez and Herman Batelaan, “Feynman’s Relativistic Electrodynamics Paradox and the Aharonov-Bohm Effect,” Found. Phys. **39**, 295-306 (2009). Here, the authors show that the extra mechanical momentum of Feynman’s paradox is hidden in the electromagnetic fields.


We ignore the torque on a sphere due to the electric dipole moment that it acquires from its motion. Point magnetic dipoles in uniform motion through a uniform magnetic field acquire an electric dipole moment that interacts with the magnetic field to give an additional torque on the dipole [38]. But this torque is proportional to \(v^2/c^2\), and we ignore it for our non-relativistic investigations.


