

FOUNDATIONS OF WAVE PHENOMENA

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READ ME

This text provides an introduction to some of the foundations of wave phenomena. Wave phenomena appear in a wide variety of physical settings, for example, electrodynamics, quantum mechanics, fluids, plasmas, atmospheric physics, seismology, and so forth. Of course, there are already a number of fine texts on the general subject of “waves” and related physical phenomena, and some of these texts are very comprehensive. So it is natural to ask why you might want to work through this rather short, condensed treatment which is largely devoid of detailed applications. The answer is that this course has a slightly different — and perhaps more general — aim than found in the more conventional courses on waves. Indeed, an alternative title for this course might be something like “Introduction to Mathematical Physics with Applications to Wave Phenomena”. So, while one of the principal goals here is to introduce you to many of the features of waves, an equally — if not more — important goal is to get you up to speed with the plethora of mathematical techniques that you will encounter as you continue your studies in the physical sciences.

It is often said that “mathematics is the language of physics”. Unfortunately for you — the student — you are expected to learn the language as you learn the concepts. In physics courses any new mathematical tools are introduced only as needed and usually in the context of the current application. Compare this to the traditional progression of a course in mathematics (which you have surely encountered by now), where a branch of the subject is given its theoretical development from scratch, mostly in the abstract, with applications used to illustrate the key mathematical points. Both ways of introducing the mathematics have their advantages. The mathematical approach has the virtue of rigor and completeness. The physics approach — while usually less complete and less rigorous — is very efficient and helps to keep clear precisely why/how this or that mathematical idea is being developed. Moreover, the physics approach implements a style of instruction that many students in science and engineering find accessible: abstract concepts are taught in the context of concrete examples. Still, there are definite drawbacks to the usual physics approach to the introduction of mathematical ideas. The student cannot be taught *all* the math that is needed in a physics course. This is exacerbated by the fact that (prerequisites notwithstanding) the students in a given class will naturally have some variability in their mathematics background. Moreover, if mathematics tools are taught only as needed, the student is never fully armed with the needed arsenal of mathematical tools until very late in his/her studies, *i.e.*, until enough courses have been taken to introduce and gain experience with the majority of the mathematical material that is needed. Of course, a curriculum in science and/or engineering includes prerequisite mathematical courses which serve to mitigate these difficulties. But you may have noticed already that these mathematics courses, which are designed not just for scientists and engineers but also for mathematicians, often involve a lot of material that simply is not needed by the typical

scientist and the engineer. For example, the scientist may be interested in what the theorems are and how to apply them but not so interested in the details of the proofs of the theorems, which are of course the bread and butter of the mathematician. And, there is always the well-known but somewhat mysterious difficulty that science/engineering students almost always seem to have when translating what they have learned in a pure mathematics course into the context of the desired application.

The traditional answer to this dilemma is to offer some kind of course in “Mathematical Physics”, designed for those who are interested more in applications and less in the underlying theory. The course you are about to take is, in effect, a Mathematical Physics course for undergraduates – but with a twist. Rather than just presenting a litany of important mathematical techniques, selected for their utility in the sciences, as is often done in the traditional Mathematical Physics course, this course tries to present a (slightly shorter) litany of techniques, always framed in the context of a single underlying theme: wave phenomena. This topic was chosen for its intrinsic importance in science and engineering, but also because it allows for a treatment of a wide variety of mathematical concepts. The hope is that this way of doing things combines some of the advantages of both the mathematician’s and the physicist’s ways of learning the language of physics. In addition, unlike many mathematical physics texts which try to give a more comprehensive “last word” on the subject. This text only aspires to give you an introduction to the key mathematical ideas. The hope is that when you encounter these ideas again at a more sophisticated level you will find them much more palatable and easy to work with, having already played with them in the context of wave phenomena.

This text is designed to accomodate a range of student backgrounds and needs. But, at the very least, it is necessary that a student has had an introductory (calculus-based) physics course, and hopefully a modern physics course. Mathematics prerequisites include: multivariable calculus and linear algebra. Typically, one can expect to cover most (if not all) of the material presented here in one semester.

How to use this text

Here are some suggestions about how you (the student) should use this text. The author, having been a student himself in the distant past, knows quite well how the typical science/engineering student uses a text. Usually, the text is given a quick, superficial read (perhaps “glance” is more accurate) to get acquainted with the “big picture”, the location of the key results and equations, *etc.* The homework problems get assigned and the student reads the text a bit more closely with the pragmatic goal of finding just what is needed to solve the problems. Finally, the text and the problems are frantically reviewed prior to the agony of the periodic exams. If a sufficiently large number of very diverse homework problems (and exams!) could be created and assigned, the usual way of doing

things might work out reasonably well. Unfortunately, this “sufficiently large number” of problems to solve is usually too high a number to be practical for the student, for the instructor, and (most importantly) for the author. *The best way to understand the material that is presented here is simply to spend time carefully working through it.* Some of this will be done for you, at the blackboard, by your instructor. But it will take a super-human instructor to get you to assimilate all that you need to know via the few hours of lecture you get each week. It’s an unpleasant fact, but we learn the most not by listening to well-crafted lectures, but by working things out for ourselves. With that in mind, there are two key tools in the text that will help you along the way.

- * *Exercises:* The exercises typically involve providing intermediate steps, arguments, computations, and additional derivations in the various developments. They represent an attempt to help you focus your attention as you work through the material. They should not take much effort and can often times be done in your head. If you are really stumped and/or are filling up your trash can with unsuccessful computations, then you are missing a key elementary fact — get help!
- * *Problems:* The problems represent more substantial endeavors than the exercises, although none should be terribly painful. Some problems fill in important steps in the main text, some provide key applications/extensions. Watch out — some problems will assume you are aware of basic math/physics results from other classes.

1. Harmonic Oscillations

Everyone has seen waves on water, heard sound waves and seen light waves. But, what exactly is a wave? Of course, the goal of this course is to answer this question for you. But for now you can think of a wave as a traveling or oscillatory disturbance in some continuous medium (air, water, the electromagnetic field, etc.). As we shall see, waves can be viewed as a collective effect resulting from a combination of many harmonic oscillations. So, to begin, we review the basics of harmonic motion.

Harmonic motion of some quantity (*e.g.*, displacement, intensity, current, angle,...) means the quantity exhibits a sinusoidal time dependence. In particular, this means that the quantity oscillates in value with a frequency which is independent of both amplitude and time. Generically, we will refer to the quantity of interest as the *displacement* and denote it by the symbol q . The value of the displacement at time t is denoted $q(t)$; for harmonic motion it can be written in one of the equivalent forms

$$q(t) = B \sin(\tilde{\omega}t + \phi) \tag{1.1}$$

$$= C \cos(\tilde{\omega}t + \psi) \tag{1.2}$$

$$= D \cos(\tilde{\omega}t) + E \sin(\tilde{\omega}t). \tag{1.3}$$

Here $B, C, D, E, \tilde{\omega}, \phi, \psi$ are all constants. The relationship between the various versions of harmonic motion shown above is obtained using the trigonometric identities

$$\sin(a + b) = \sin a \cos b + \sin b \cos a, \tag{1.4}$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b. \tag{1.5}$$

You will be asked to explore these relationships in the Problems.

The parameter $\tilde{\omega}$ represents the (angular) frequency of the oscillation and is normally determined by the nature of the physical system being considered.* So, typically, $\tilde{\omega}$ is fixed once and for all. (Recall that the angular frequency is related to the physical frequency f of the motion via $\tilde{\omega} = 2\pi f$.) For example, the time evolution of angular displacement of a pendulum is, for small displacements, harmonic with frequency determined by the length of the pendulum and the acceleration due to gravity. In all that follows we assume that $\tilde{\omega} > 0$; there is no loss of generality in doing so (*exercise*). The other constants B, C, D, E, ϕ, ψ , which represent amplitudes and phases, are normally determined by the initial conditions for the problem at hand. For example, it is not hard to check that $D = q(0)$ and $E = \frac{1}{\tilde{\omega}}v(0)$, where $v(t) = \frac{dq(t)}{dt}$ is the velocity at time t . Given $\tilde{\omega}$, no matter

* We have put a tilde over the traditional symbol for angular frequency since we shall use the unadorned ω to represent the frequency of a wave. If you don't mind using the same symbol for more than one thing in different contexts, you can mentally erase the tilde.

which form of the harmonic motion is used, you can check that one must pick two real numbers to uniquely specify the motion (*exercise*).

1.1 The Harmonic Oscillator equation

To say that the displacement exhibits harmonic motion is equivalent to saying that $q(t)$ satisfies the *harmonic oscillator equation*:

$$\frac{d^2q(t)}{dt^2} = -\tilde{\omega}^2 q(t), \quad (1.6)$$

for some real constant $\tilde{\omega} > 0$. This is because each of the functions (1.1)–(1.3) satisfy (1.6) (*exercise*) and, in particular, every solution of (1.6) can be put into the form (1.1)–(1.3) – a fact you can understand using techniques from a basic course in differential equations. (See also the discussion in §1.2, below.)

It is via the harmonic oscillator equation that one usually arrives at harmonic motion when modeling a physical system. For example, if q is a displacement of a particle with mass m whose dynamical behavior is controlled (at least approximately – see below) by a Hooke’s force law,

$$F = -kq, \quad k = \text{const.}, \quad (1.7)$$

then Newton’s second law reduces to the harmonic oscillator equation with $\tilde{\omega} = \sqrt{\frac{k}{m}}$ (*exercise*). More generally, we have the following situation. Let the *potential energy* function describing the variable q be denoted by $V(q)$. This means that the equation governing q takes the form*

$$\frac{d^2q(t)}{dt^2} = -V'(q(t)). \quad (1.8)$$

Suppose the function $V(q)$ has a local minimum at the point q_0 , that is, suppose q_0 is a point of stable equilibrium. Let us choose our origin of coordinates at this point, so that $q_0 = 0$. Further, let us choose our reference point of potential energy (by adding a constant to V if necessary) so that $V(0) = 0$. These choices are not necessary; they are for convenience only. In the vicinity of this minimum we can write a Taylor series approximation to $V(q)$ (*exercise*):

$$\begin{aligned} V(q) &\approx V(0) + V'(0)q + \frac{1}{2}V''(0)q^2 \\ &= \frac{1}{2}V''(0)q^2. \end{aligned} \quad (1.9)$$

* Here we use a prime on a function to indicate a derivative with respect to the argument of the function, *e.g.*,

$$f'(x) = \frac{df(x)}{dx}.$$

(If you would like a quick review of Taylor’s theorem and Taylor series, have a look at Appendix A.) The zeroth and first order terms in the Taylor series are absent, respectively, because (i) we have chosen $V(0) = 0$ and (ii) for an equilibrium point $q = 0$ we have $V'(0) = 0$. Because the equilibrium point is a minimum (stable equilibrium) we have that

$$V''(0) > 0. \quad (1.10)$$

Incidentally, the notation $V'(0)$, $V''(0)$, *etc.* , means “take the derivative(s) and evaluate the result at zero”:

$$V'(0) \equiv V'(x) \Big|_{x=0}.$$

If we use the Taylor series approximation (1.9), we can approximate the force on the system near a point of stable equilibrium as that of a harmonic oscillator with “spring constant” (*exercise*)

$$k = V''(0). \quad (1.11)$$

The equation of motion of the system can thus be approximated by

$$m \frac{d^2 q(t)}{dt^2} = -V'(q(t)) \approx -kq(t), \quad (1.12)$$

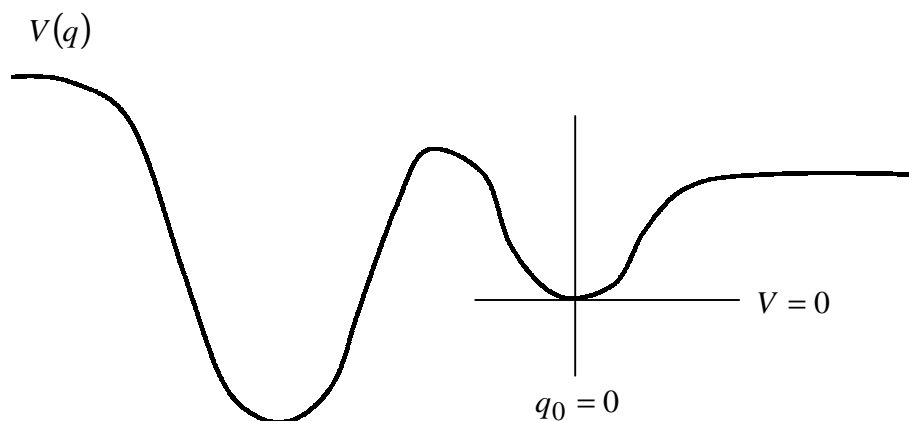


Figure 1. One dimensional potential $V(q)$.

which leads back to (1.6). When one approximates, as we just did, the potential energy as a quadratic function in the neighborhood of some point one is using the *harmonic*

approximation. No matter what the physical system is, or the meaning of the variable $q(t)$, the harmonic approximation leads to the harmonic motion (1.1)–(1.3).

The harmonic oscillator equation is a second-order, linear, homogeneous ordinary differential equation with constant coefficients. Let us pause for a moment to explain this terminology. The general form of a linear, second-order ordinary differential equation is

$$a(t)q''(t) + b(t)q'(t) + c(t)q(t) = d(t), \quad (1.13)$$

where the *coefficients* a, b, c some given functions of t . There is one *dependent variable*, q , and one *independent variable*, t . Because there is only one independent variable, all derivatives are *ordinary* derivatives rather than partial derivatives, which is why the equation is called an *ordinary* differential equation. The harmonic oscillator has *constant coefficients* because a, b, c do not in fact depend upon time. The harmonic oscillator equation is called *homogeneous* because $d = 0$, making the equation homogeneous of the first degree in the dependent variable q .[†] If $d \neq 0$, then the equation is called *inhomogeneous*. As you can see, this equation involves no more than 2 derivatives, hence the designation *second-order*. We say that this equation is *linear* because the differential operator*

$$L = a \frac{d^2}{dt^2} + b \frac{d}{dt} + c \quad (1.14)$$

appearing in equation (1.13) via

$$Lq = d, \quad (1.15)$$

has the following property (exercise)

$$L(rq_1 + sq_2) = rL(q_1) + sL(q_2), \quad (1.16)$$

for any constants r and s .

Indeed, to be more formal about this, following the general discussion in Appendix B, we can view the set of all real-valued functions $q(t)$ as a real vector space. So, in this case the “vectors” are actually functions of t .[‡] The addition rule is the usual pointwise addition of functions. The scalar multiplication is the usual multiplication of functions by real numbers. The zero vector is the zero function, which assigns the value zero to all t . The additive inverse of a function $q(t)$ is the function $(-q(t))$, and so forth. We can then view L as a linear operator, as defined in Appendix B (exercise).

[†] A function $f(x)$ is homogeneous of degree p if it satisfies $f(ax) = a^p f(x)$ for any constant a .

* In this context, a differential operator is simply a rule for making a function (or functions) from any given function $q(t)$ and its derivatives.

[‡] There are a *lot* of functions, so this vector space is infinite dimensional. We shall make this a little more precise when we discuss Fourier analysis.

The linearity and homogeneity of the harmonic oscillator equation has a very important consequence. It guarantees that, given two solutions, $q_1(t)$ and $q_2(t)$, we can take any linear combination of these solutions with constant coefficients to make a new solution $q_3(t)$. In other words, if r and s are constants, and if $q_1(t)$ and $q_2(t)$ are solutions to the harmonic oscillator equation, then

$$q_3(t) = rq_1(t) + sq_2(t) \quad (1.17)$$

also solves that equation.

Exercise: Check that $q_3(t)$ is a solution of (1.6) if $q_1(t)$ and $q_2(t)$ are solutions. Do this explicitly, using the harmonic oscillator equation, and then do it just using the linearity and homogeneity L .

1.2 The General Solution to the Harmonic Oscillator Equation

The *general solution* to the harmonic oscillator equation is a solution depending upon two arbitrary constants (two because the equation is second-order) such that all possible solutions can be obtained by making suitable choices of the constants. Physically, we expect to need two arbitrary constants in the general solution since we must have the freedom to adjust the solution to match any initial conditions, *e.g.*, initial position and velocity. To construct the most general solution it is enough to find two “independent” solutions (which are not related by linear combinations) and take their general linear combination. For example, we can take $q_1(t) = \cos(\tilde{\omega}t)$ and $q_2(t) = \sin(\tilde{\omega}t)$ and the general solution is (*c.f.* (1.3)):

$$q(t) = A \cos(\tilde{\omega}t) + B \sin(\tilde{\omega}t). \quad (1.18)$$

While we won’t prove that all solutions can be put into this form, we can easily show that this solution accomodates any initial conditions. If we choose our initial time to be $t = 0$, then this follows from the fact that (exercise)

$$q(0) = A, \quad \frac{dq(0)}{dt} = B\tilde{\omega}.$$

You can now see that the choice of the constants A and B is equivalent to the choice of initial conditions.

It is worth noting that the solutions to the oscillator equation form a *vector space* (see Appendix B for the definition of a vector space). The underlying set is the set of solutions to the harmonic oscillator equation. So, here again the “vectors” are functions of t . (But this time the vector space will be finite-dimensional because only the vectors which are mapped to zero by the operator L are being considered.) As we have seen, the solutions

can certainly be added and multiplied by scalars (*exercise*) to produce new solutions. This is because of the linear, homogeneous nature of the equation. So, in this vector space, “addition” is just the usual addition of functions, and “scalar multiplication” is just the usual multiplication of functions by real numbers. The function $q = 0$ is a solution to the oscillator equation, so it is one of the “vectors”; it plays the role of the zero vector. As discussed in Appendix B, a *basis* for a vector space is a subset of linearly independent elements out of which all other elements can be obtained by linear combination. The number of elements needed to make a basis is the *dimension* of the vector space. Evidently, the functions $q_1(t) = \cos \tilde{\omega}t$ and $q_2(t) = \sin \tilde{\omega}t$ form a basis for the vector space of solutions to the harmonic oscillator equation (*c.f.* equation (1.18)). Indeed, both $q_1(t)$ and $q_2(t)$ are vectors, *i.e.*, solutions of the harmonic oscillator equation, and all solutions can be built from these two solutions by linear combination. Thus the set of solutions to the harmonic oscillator equation constitutes a two-dimensional vector space.

It is possible to equip the vector space of solutions to the harmonic oscillator equation with a scalar product. While we won’t need this result in what follows, we briefly describe it here just to illustrate the concept of a scalar product. Being a scalar product, it should be defined by a rule which takes any two vectors, *i.e.*, solutions of the harmonic oscillator equation, and returns a real number (see Appendix B for the complete definition of a scalar product). Let $q_1(t)$ and $q_2(t)$ be solutions of the harmonic oscillator equation. The scalar product is defined by

$$(q_1, q_2) = q_1 q_2 + \frac{1}{\tilde{\omega}^2} q_1' q_2'. \quad (1.19)$$

In terms of this scalar product, the basis of solutions provided by $\cos \tilde{\omega}t$ and $\sin \tilde{\omega}t$ is orthonormal (*exercise*).

1.3 Complex Representation of Solutions

In any but the simplest applications of harmonic motion it is very convenient to work with the *complex representation* of harmonic motion, where we use complex numbers to keep track of the harmonic motion. Before showing how this is done, we’ll spend a little time reviewing the basic properties of complex numbers.

Recall that the *imaginary unit* is defined formally via

$$i^2 = -1$$

so that

$$\frac{1}{i} = -i. \quad (1.20)$$

A *complex number* z is an ordered pair of real numbers (x, y) which we write as

$$z = x + iy. \quad (1.21)$$

The variable x is called the *real part* of z , denoted $Re(z)$, and y is called the *imaginary part* of z , denoted by $Im(z)$. We apply the usual rules for addition and multiplication of numbers to z , keeping in mind that $i^2 = -1$. In particular, the sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are equal if and only if their real parts are equal and their imaginary parts are equal:

$$z_1 = z_2 \iff x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Given $z = x + iy$, we define the *complex conjugate* $z^* = x - iy$. It is straightforward to check that (exercise)

$$Re(z) = x = \frac{1}{2}(z + z^*); \quad Im(z) = y = \frac{1}{2i}(z - z^*). \quad (1.22)$$

Note that (exercise)

$$z^2 = x^2 - y^2 + 2ixy, \quad (1.23)$$

so that the square of a complex number z is another complex number.

Exercise: What are the real and imaginary parts of z^2 ?

A complex number is neither positive or negative; these distinctions are only used for real numbers. Note in particular that z^2 , being a complex number, cannot be said to be positive (unlike the case with real numbers). On the other hand, if we multiply z by its complex conjugate we get a non-negative real number:

$$zz^* \equiv |z|^2 = x^2 + y^2 \geq 0. \quad (1.24)$$

We call the non-negative real number $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ the *absolute value* of z .

A complex number defines a point in the x - y plane — in this context also called the *complex plane* — and *vice versa*. We can therefore introduce polar coordinates to describe any complex number. So, for any $z = x + iy$, let

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta, \quad (1.25)$$

where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. (Note that, for a fixed value of $r > 0$, $\theta = 0$ and $\theta = 2\pi$ represent the same point. Also, θ is not defined when $r = 0$.) You can check that (exercise)

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right). \quad (1.26)$$

It now follows that for any complex number z there is a radius r and an angle θ such that

$$z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta). \quad (1.27)$$

A famous result of Euler is that

$$\cos \theta + i \sin \theta = e^{i\theta}, \quad (1.28)$$

so we can write

$$z = re^{i\theta}. \quad (1.29)$$

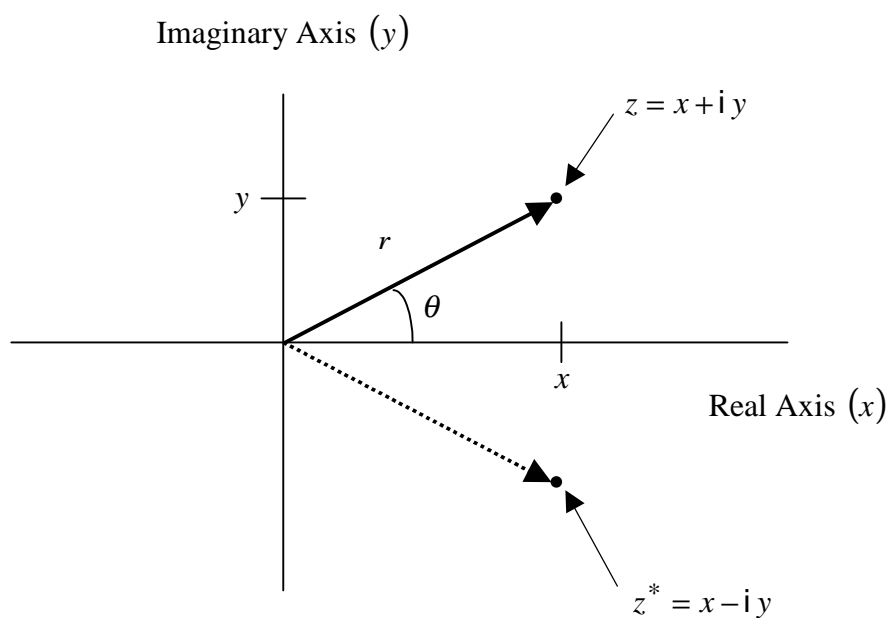


Figure 2. Illustration of a point z in the complex plane and its complex conjugate z^* .

In the polar representation of a complex number z the radius r corresponds to the absolute value of z while the angle θ is known as the *phase* or *argument* of z . Evidently, a complex number is completely characterized by its absolute value and phase.

As an exercise you can show that

$$\cos \theta = \operatorname{Re}(e^{i\theta}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin \theta = \operatorname{Im}(e^{i\theta}) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

We also have (exercise)

$$z^2 = r^2 e^{2i\theta} \quad \text{and} \quad zz^* = r^2. \quad (1.30)$$

These last two relations summarize (1) the double angle formulas for sine and cosine,

$$\sin 2\theta = 2 \cos \theta \sin \theta, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta, \quad (1.31)$$

and (2) the identity $\cos^2 \theta + \sin^2 \theta = 1$. As a good exercise you should verify (1) and (2) using the complex exponential.

Complex numbers are well-suited to describing harmonic motion because the real and imaginary parts involve cosines and sines. Indeed, it is easy to check that both $e^{i\tilde{\omega}t}$ and $e^{-i\tilde{\omega}t}$ solve the oscillator equation (1.6) (exercise). (Here you should treat i as just another constant.) Hence, for any complex numbers α and β , we can solve the harmonic oscillator equation via

$$q(t) = \alpha e^{i\tilde{\omega}t} + \beta e^{-i\tilde{\omega}t}. \quad (1.32)$$

Note that we have added, or superposed, two solutions of the equation. We shall refer to (1.32) as the *complex solution* since, as it stands, $q(t)$ is not a real number.

Normally, q represents a real-valued quantity (displacement, temperature,...). In this case, we require

$$\text{Im}(q(t)) = 0 \quad (1.33)$$

or (exercise)

$$q^*(t) = q(t). \quad (1.34)$$

Substituting our complex form (1.32) for $q(t)$ into (1.34) gives

$$\alpha e^{i\tilde{\omega}t} + \beta e^{-i\tilde{\omega}t} = \alpha^* e^{-i\tilde{\omega}t} + \beta^* e^{i\tilde{\omega}t}. \quad (1.35)$$

In this equation, for each t , the real parts of the left and right-hand sides must be equal as must be the imaginary parts. In a homework problem you will see that this implies that $\beta = \alpha^*$. Real solutions of the harmonic oscillator equation can therefore be written as

$$q(t) = \alpha e^{i\tilde{\omega}t} + \alpha^* e^{-i\tilde{\omega}t}. \quad (1.36)$$

We shall refer to (1.36) as the *complex form* or *complex representation* of the *real* solution of the harmonic oscillator equation. From (1.36) it is clear that $q(t)$ is a real number because it is a sum of a number and its complex conjugate, *i.e.*, is the real part of a complex number. Equation (1.36) is equivalent to the general (real) solution to the harmonic oscillator equation (1.18) (exercise). As it should, the solution depends on two free constants: the real and imaginary parts of α .

It is not hard to relate the complex representation of harmonic motion to any of the sinusoidal representations given in (1.1)–(1.3). Let us see how to recover (1.2); the

others are obtained in a similar fashion. Since α is a complex number, we use the polar representation of the complex number to write

$$\alpha = ae^{i\psi}, \quad (1.37)$$

where a and ψ are both real numbers ($a = a^*$, $\psi = \psi^*$). We then have that

$$\begin{aligned} q(t) &= ae^{i(\tilde{\omega}t+\psi)} + ae^{-i(\tilde{\omega}t+\psi)} \\ &= a[e^{i(\tilde{\omega}t+\psi)} + e^{-i(\tilde{\omega}t+\psi)}] \\ &= 2a \cos(\tilde{\omega}t + \psi), \end{aligned} \quad (1.38)$$

which matches our representation of harmonic motion (1.2) with the identification $2a = C$.

So, we see that a specific harmonic motion is determined by a choice of the complex number α through (1.36). Using (1.36), you can check that $q(0) = 2\text{Re}(\alpha)$ and $v(0) = -2\tilde{\omega}\text{Im}(\alpha)$ (*exercise*). Hence the real and imaginary parts of α encode the initial conditions. From the point of view of the polar representation of α , the absolute value of the complex number α determines the amplitude of the oscillation. The phase of the complex number α determines where (*i.e.*, , what “phase”) the oscillation is in its cycle relative to $t = 0$.

Note that we can also write the general solution (1.36) as

$$q(t) = 2\text{Re}(\alpha e^{i\tilde{\omega}t}). \quad (1.39)$$

When you see a formula such as this you should remember that it works because (1) if $q(t)$ is a complex solution, then so is $q^*(t)$ (*exercise*), and (2) the superposition of any two (complex solutions) to the harmonic oscillator equation is another (in general, complex) solution (*exercise*). Because the real part of the complex solution $q(t)$ is proportional to $q(t) + q^*(t)$ we get a real solution to the harmonic oscillator equation by taking the real part of a complex solution. As a nice exercise you should check that we can also get a real solution of the harmonic oscillator equation by taking the imaginary part of a complex solution. The relation between these two forms of a real solution is explored in the problems.

Thus we have obtained a useful strategy that we shall take advantage of from time to time: When solving a linear equation with harmonic solutions, we will frequently work with complex solutions to simplify various manipulations, and only take the real part at the end of the day. This is allowed because any real linear equation which admits a complex solution will also admit the complex conjugate as a solution (see homework problems). By linearity and homogeneity of the harmonic oscillator equation, any linear combination of these two solutions will be a solution, and in particular the real part will be a solution.

PROBLEM SET 1

Problem 1.1

Verify that each of the three forms of harmonic motion

$$\begin{aligned}q(t) &= B \sin(\tilde{\omega}t + \phi) \\&= C \cos(\tilde{\omega}t + \psi) \\&= D \cos \tilde{\omega}t + E \sin \tilde{\omega}t.\end{aligned}$$

satisfy the harmonic oscillator equation. Give formulas relating the constants B, C, D, E, ϕ, ψ in each case, *i.e.*, given B and ϕ in the first form of the motion, how to compute C, ψ and D, E ?

Problem 1.2

Consider a potential energy function $V(x) = ax^2 + bx^4$. Discuss the possible equilibrium positions for various choices of a and b . For $a > 0, b > 0$ show that the frequency for oscillations near equilibrium is independent of b .

Problem 1.3

Consider the motion of an oscillator which is started from rest from the initial position q_0 . Give the motion in each of the 3 trigonometric forms from the text. Do the same for the case where the initial velocity v_0 is non-zero but the initial position is zero.

Problem 1.4

Suppose that we changed the sign in the harmonic oscillator equation so that we consider the equation

$$\frac{d^2q}{dt^2} = \tilde{\omega}^2 q. \tag{1.40}$$

- (a) What shape must the potential energy graph have in the neighborhood of an equilibrium point to lead to this (approximate) equation?
- (b) Find the general solution to this equation. In particular, show that the solutions can grow exponentially with time (rather than oscillate), so that this equation permits solutions which depart from the initial values by arbitrarily large amounts.

- (c) For what initial conditions will the exponential growth found in (b) *not* occur.
- (d) Show that (1.40) and its solutions can be obtained from the general complex solution to the harmonic oscillator equation by letting the oscillator frequency become imaginary.

Problem 1.5

Find the absolute value and phase of the following complex numbers:

- (a) $3 + 5i$
- (b) 10
- (c) $10i$.

Problem 1.6

Using the relation between the (r, θ) and (x, y) parametrizations of a complex number show that

$$z^2 = r^2 e^{2i\theta} \quad \text{and} \quad zz^* = r^2$$

agree with the formulas for z^2 and zz^* obtained by using $z = x + iy$.

Problem 1.7

Prove that two complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal.

Problem 1.8

Show that $q(t) = Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}$ is a real number for all values of t if *and only if* $A^* = B$.

Problem 1.9

In the text it is argued that one can find the general *real* solution to the harmonic oscillator equation by first finding the general *complex* solution and then taking its real part at the end of the day. How does the general real solution thus obtained compare to the real solution that is obtained if, instead, we take the *imaginary part* of the general complex solution?

Problem 1.10

With $q(t) = 2\text{Re}(\alpha e^{i\tilde{\omega}t})$, show that the initial position and velocities are given by $q(0) = 2\text{Re}(\alpha)$ and $v(0) = -2\tilde{\omega}\text{Im}(\alpha)$.

Problem 1.11

Using Euler's formula (1.28), prove the following trigonometric identities.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

and

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

(Hint: $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$.)

Problem 1.12

Generalize equation (1.9) to allow for an arbitrary equilibrium point q_0 and an arbitrary reference point for the potential, $V(q_0) = V_0$. Show that (1.12) becomes the harmonic oscillator equation for the displacement from equilibrium ($q - q_0$).

Problem 1.13

Show that the set of complex numbers forms a complex vector space.

Problem 1.14

Using the complex form of a real solution to the harmonic oscillator equation,

$$q(t) = \alpha e^{i\tilde{\omega}t} + \alpha^* e^{-i\tilde{\omega}t},$$

show that the solution can be expressed in the forms (1.1) and (1.3). (Hint: Make use of the polar representation of the complex number α .)

Problem 1.15

The *energy* of a harmonic oscillator can be defined by

$$E = \gamma \frac{1}{2} \left\{ \left(\frac{dq}{dt} \right)^2 + \tilde{\omega}^2 q^2 \right\},$$

where γ is a constant (needed to get the units right). The energy E is conserved, that is, it doesn't depend upon time. Prove this in the following two distinct ways.

- (i) Substitute one of the general forms (1.1) – (1.3) of harmonic motion into E and show that the time dependence drops out.
- (ii) Take the time derivative of E and show that it vanishes provided (1.6) holds (without using the explicit form of the solutions).

Problem 1.16

It was pointed out in §1.2 that the functions $\cos \tilde{\omega}t$ and $\sin \tilde{\omega}t$ are orthogonal with respect to the scalar product (1.19). This implies they are linearly independent when viewed as elements of the vector space of solutions to the harmonic oscillator equation. Prove directly that these functions are linearly independent, *i.e.*, prove that if a and b are constants such that (for all values of t)

$$a \cos \tilde{\omega}t + b \sin \tilde{\omega}t = 0,$$

then $a = b = 0$.

(*Hint: This one is really easy!*)

2. Two Coupled Oscillators.

Our next step on the road to a *bona fide* wave is to consider a more interesting oscillating system: two coupled oscillators. Suppose we have two identical oscillators, both characterized by an angular frequency $\tilde{\omega}$. Let the displacement of each oscillator from equilibrium be q_1 and q_2 , respectively. Of course, if the two oscillators are uncoupled, that is, do not interact in any way, each of the displacements satisfies a harmonic oscillator equation

$$\frac{d^2 q_1}{dt^2} = -\tilde{\omega}^2 q_1, \quad (2.1)$$

$$\frac{d^2 q_2}{dt^2} = -\tilde{\omega}^2 q_2. \quad (2.2)$$

For example, the oscillators could each consist of a mass m connected by a spring (with having spring constant $k = m\tilde{\omega}^2$) to a wall (see figure 3a). Now suppose that the masses are joined by a spring characterized by spring constant k' . With a little thought, you can see that the forces on the masses are such that their equations of motion take the form:

$$\frac{d^2 q_1}{dt^2} + \tilde{\omega}^2 q_1 - \tilde{\omega}'^2 (q_2 - q_1) = 0, \quad (2.3)$$

$$\frac{d^2 q_2}{dt^2} + \tilde{\omega}^2 q_2 + \tilde{\omega}'^2 (q_2 - q_1) = 0, \quad (2.4)$$

where $\tilde{\omega}' = \sqrt{\frac{k'}{m}}$. Notice that while the equations are still linear and homogeneous with constant coefficients they are now *coupled*, that is, the equation for $q_1(t)$ depends on $q_2(t)$ and *vice versa*.

2.1 Normal Modes

The motion of the masses described by (2.3) and (2.4) can be relatively complicated but, remarkably enough, it can be viewed as a superposition of harmonic motions! One quick way to see this is to simply take the sum and difference of (2.3) and (2.4). You will find (*exercise*) that the quantity $q_1(t) + q_2(t)$ satisfies the harmonic oscillator equation with frequency* $\tilde{\omega}$, while the quantity $q_2(t) - q_1(t)$ satisfies the harmonic oscillator equation with frequency $\Omega = \sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}$. One can therefore solve the harmonic oscillator equations with the indicated frequencies for the combinations $q_2(t) \pm q_1(t)$ and then reconstruct the motion of the individual oscillators, $q_1(t)$ and $q_2(t)$.

To see how to do this systematically, let us define new position variables,

$$Q_1 = \frac{1}{2}(q_1 + q_2), \quad Q_2 = \frac{1}{2}(q_2 - q_1). \quad (2.5)$$

* Reminder: the frequencies are always chosen positive.

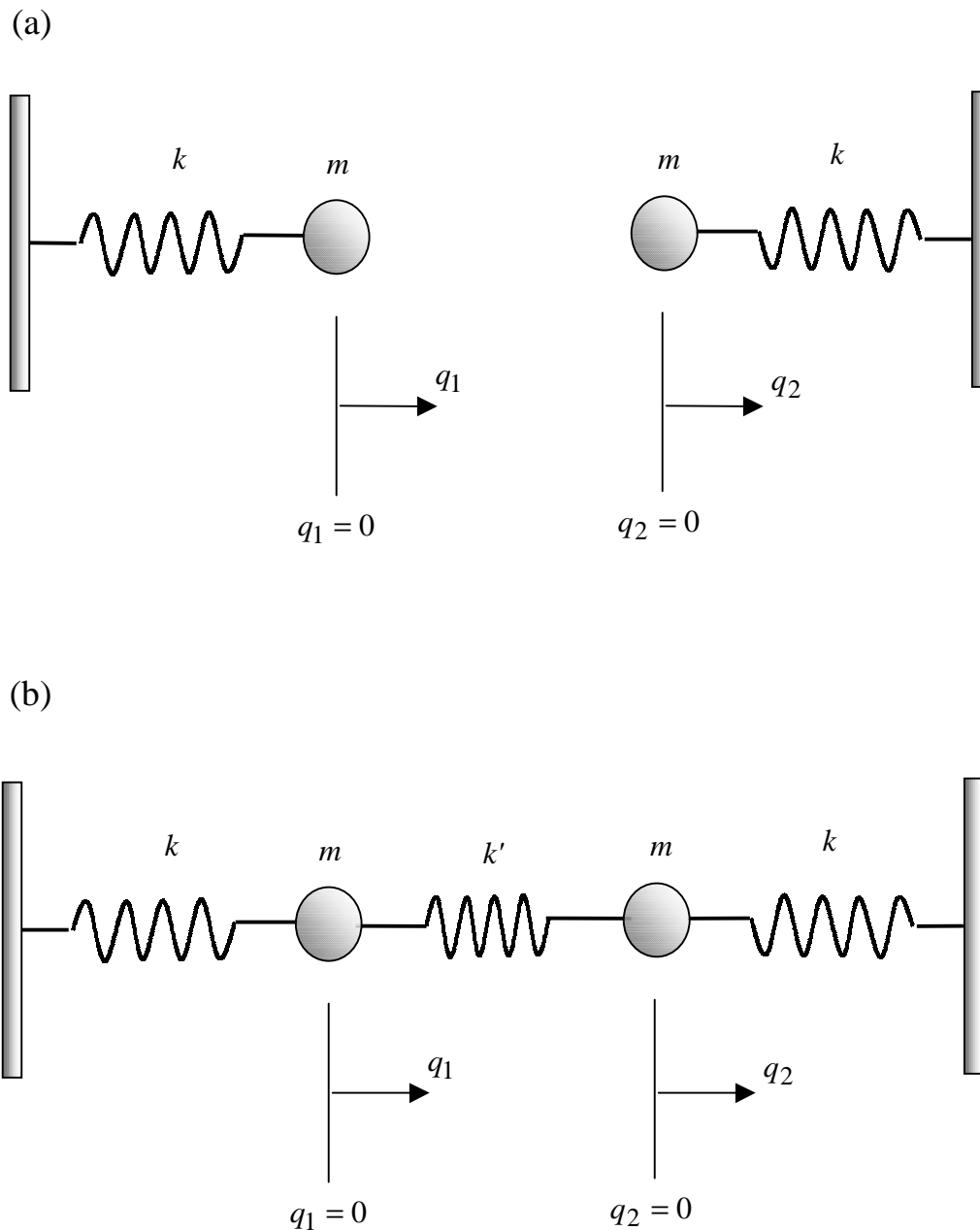


Figure 3. (a) Two uncoupled oscillators. (b) Two coupled oscillators. Coupling occurs through the spring k' .

This defines the Q 's in terms of the q 's. (The factors of $1/2$ are there for later convenience.) The Q 's are examples of *generalized coordinates*. A physical interpretation of these variables is not hard to come by. The variable Q_1 carries information about the center of

mass of the system (Q_1 differs from the center of mass by an additive constant – *exercise*). The variable Q_2 carries information about the relative separation of the two masses. Thus the new variables correspond to the procedure of separation of the motion of a two body system into its relative and center of mass parts. We can invert the definitions of the Q 's to get the q 's in terms of the Q 's (*exercise*):

$$q_1 = Q_1 - Q_2, \quad q_2 = Q_1 + Q_2. \quad (2.6)$$

So, if we can solve the differential equations for the Q 's, then we can also get the solutions for the q 's from (2.6). To get the equations satisfied by the Q 's, we use (2.6) in (2.3)–(2.4); after a little simplification, we find that the Q 's satisfy (*exercise*)

$$\frac{d^2 Q_1}{dt^2} = -\tilde{\omega}^2 Q_1 \quad (2.7)$$

$$\frac{d^2 Q_2}{dt^2} = -(\tilde{\omega}^2 + 2\tilde{\omega}'^2) Q_2. \quad (2.8)$$

The equations for Q_1 and Q_2 are *decoupled*. Moreover, as expected, Q_1 executes harmonic motion with angular frequency $\Omega_1 = \tilde{\omega}$ and Q_2 executes harmonic motion with frequency $\Omega_2 = \sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}$.

The generalized coordinates Q_1 and Q_2 , which satisfy uncoupled harmonic equations of motion, are called *normal coordinates*. The solutions in which

$$(1) \quad Q_1 = Q_1(t), \quad Q_2 = 0$$

and

$$(2) \quad Q_1 = 0, \quad Q_2 = Q_2(t)$$

are called the two *normal modes* of vibration. The corresponding frequencies Ω_1 and Ω_2 are called the *resonant frequencies*, or the *natural frequencies* or the *characteristic frequencies* of the normal modes (1) and (2) of vibration. Note that case (1) corresponds to a solution to the equations of motion given by

$$q_1(t) = q_2(t) = \text{Re}(A_1 e^{i\Omega_1 t}), \quad (2.9)$$

while case (2) corresponds to a solution of the form

$$q_1(t) = -q_2(t) = \text{Re}(A_2 e^{i\Omega_2 t}). \quad (2.10)$$

Here A_1 and A_2 are any complex numbers.

We can now write down the general solution for the motion of the system. In normal coordinates we have

$$Q_1 = \text{Re}(A_1 e^{i\Omega_1 t}), \quad Q_2 = \text{Re}(A_2 e^{i\Omega_2 t}), \quad (2.11)$$

where A_1 and A_2 are any complex numbers. We can thus write the general solution in the original variables as (*exercise*)

$$\begin{aligned} q_1 &= \operatorname{Re} \left(A_1 e^{i\Omega_1 t} - A_2 e^{i\Omega_2 t} \right) \\ q_2 &= \operatorname{Re} \left(A_1 e^{i\Omega_1 t} + A_2 e^{i\Omega_2 t} \right) \end{aligned} \quad (2.12)$$

From (2.9) and (2.10) you can see that the complex numbers A_1 and A_2 control the amplitude and phase of the normal modes of vibration which are being superposed to get the solution for q_1 and q_2 . Note that we have (2 complex) = (4 real) numbers at our disposal in (2.12), namely, A_1 and A_2 . From a mathematical point of view this is exactly what is needed to give the general solution to a pair of linear, second-order ordinary differential equations. Physically, we need 4 real numbers to specify the initial positions and velocities of the 2 oscillators. Indeed, each specific solution uniquely is determined by the choice of A_1 and A_2 , which can be specified by giving the initial conditions (*exercise*, see also §2.2). See figure 4 for some graphical descriptions of the general solution.

Let us summarize the results of this section thus far. The normal modes of vibration correspond to having either $Q_1(t)$ or $Q_2(t)$ zero for all time. These modes of vibration are harmonic motions at the frequencies Ω_2 and Ω_1 , respectively. In general, the motion of the system is a superposition of the normal modes of vibration. The particular superposition that arises is determined by initial conditions.

Once again the basic results we have encountered for the system of two coupled oscillators can be given a vector space interpretation (see Appendix B for information about vector spaces). We choose the underlying set to be the space of all possible solutions $(q_1(t), q_2(t))$ to the equations (2.3), (2.4). Let us write these solutions as a column vector of functions:

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}. \quad (2.13)$$

Since the equations for \mathbf{q} are linear and homogeneous we can add solutions with numerical factors to make new solutions. More precisely, if we have 2 solutions,

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}, \quad \tilde{\mathbf{q}}(t) = \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix}, \quad (2.14)$$

then for any real constants a and b

$$\hat{\mathbf{q}}(t) = a\mathbf{q}(t) + b\tilde{\mathbf{q}}(t) = \begin{pmatrix} aq_1(t) + b\tilde{q}_1(t) \\ aq_2(t) + b\tilde{q}_2(t) \end{pmatrix} \quad (2.15)$$

is also a solution to the coupled equations. We thus define the vector space addition and scalar multiplication of two solutions as indicated in (2.15). Note that we are taking advantage of the familiar vector space structure admitted by the set of column vectors.

From the explicit form (2.12) of the solutions it follows that the following is a *basis* for the real vector space of solutions to the coupled oscillator equations:

$$\begin{aligned}\mathbf{b}_1(t) &= \cos(\Omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \mathbf{b}_2(t) &= \sin(\Omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \mathbf{b}_3(t) &= \cos(\Omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & \mathbf{b}_4(t) &= \sin(\Omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.\end{aligned}\tag{2.16}$$

(You will be asked to show this in the Problems.) In other words, every solution to the coupled equations can be uniquely expressed as a superposition

$$\mathbf{q}(t) = a_1 \mathbf{b}_1(t) + a_2 \mathbf{b}_2(t) + a_3 \mathbf{b}_3(t) + a_4 \mathbf{b}_4(t).\tag{2.17}$$

Note that \mathbf{b}_1 and \mathbf{b}_2 give the part of the solution oscillating at frequency Ω_1 . You can check that this part of the basis defines the part of the solution corresponding to the normal mode labeled by Q_1 . Similarly, \mathbf{b}_3 and \mathbf{b}_4 give the part of the motion coming from normal mode labeled by Q_2 . Because there are 4 basis vectors, the vector space of solutions is 4-dimensional. As we have already mentioned, this should match your physical intuition: it should take four real numbers – *e.g.*, initial positions and velocities – to specify a solution.

It is possible to generalize the scalar product (1.19) to our current example, in which case the basis (2.17) is orthonormal. If you are interested, you might try to work out the details. We won't do it here.

2.2 Physical Meaning of the Normal Modes

The normal mode of vibration corresponding to $Q_1 = Q_1(t)$, $Q_2 = 0$ is a motion of the system in which the displacement of each oscillator is equal and *in phase* (exercise).^{*} In other words, the masses oscillate together (with a constant separation) at an angular frequency of $\Omega_1 = \tilde{\omega}$. Recall that the normal coordinate Q_1 represented, essentially, the center of mass of the system. The normal mode of vibration $Q_1(t)$ is a harmonic oscillation of the center of mass with frequency $\tilde{\omega}$. Because the two oscillators keep the same relative distance, there is no compression of the spring which couples the oscillators and so it is easy to see why the frequency of this normal mode is controlled by $\tilde{\omega}$ alone. To “excite” the normal mode associated with Q_1 , we start the system off at $t = 0$ such that

$$q_1(0) = q_2(0) \quad \text{and} \quad v_1(0) = v_2(0),\tag{2.18}$$

which forces $A_2 = 0$ in (2.12), *i.e.*, $Q_2(t) = 0$ (exercise). Note that the initial conditions (2.18) correspond to giving each mass the same initial displacement and velocity.

^{*} The use of the term “phase” in this context refers to the phase of the cosine functions that describe the displacement of each of the two masses in this normal mode. To say that the two masses are “in phase” is to say that arguments of the cosines are the same for all time. Physically, the two masses are always in the same part of their cycle of oscillation.

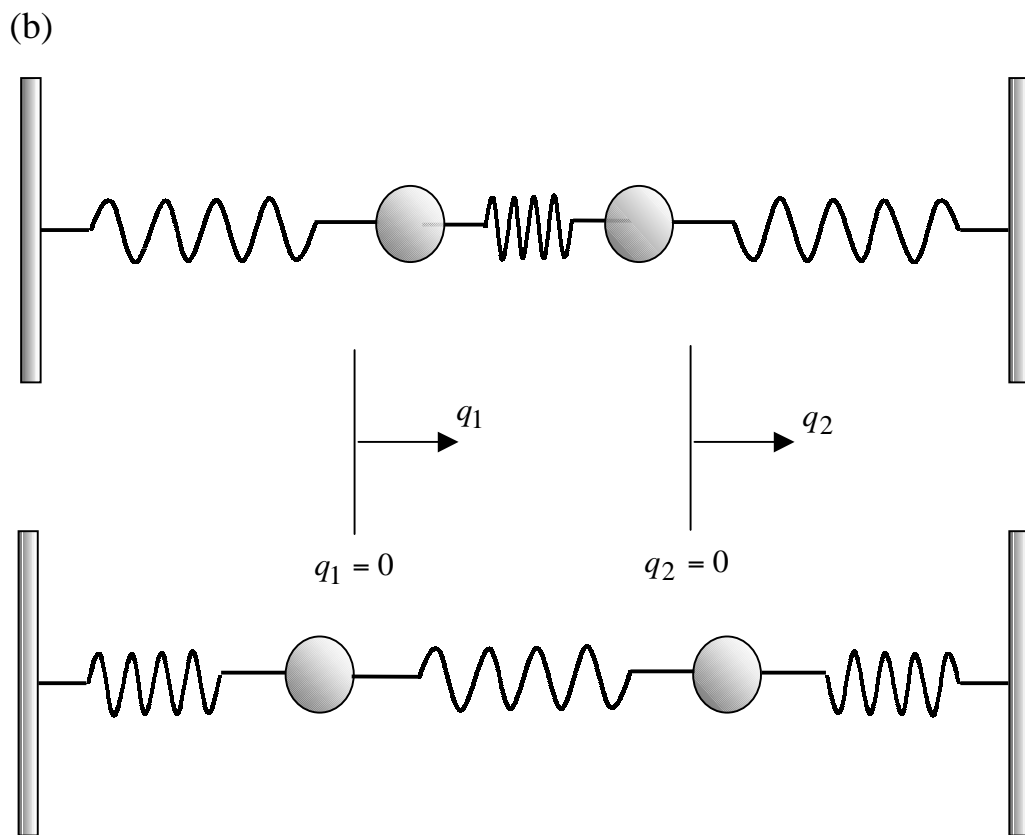
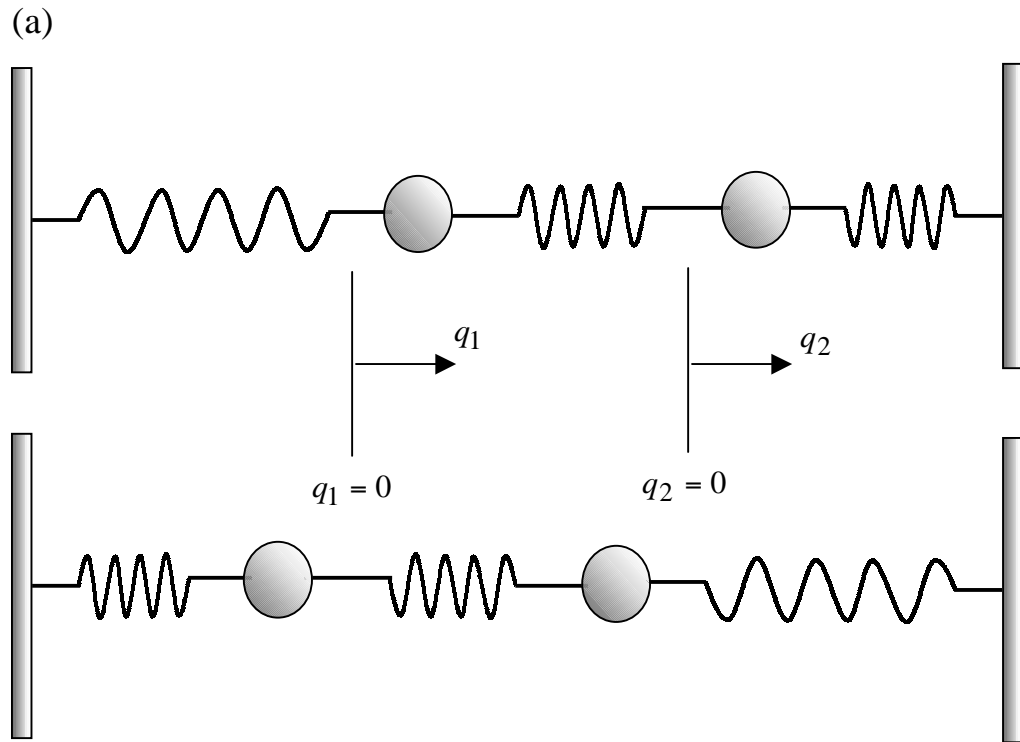
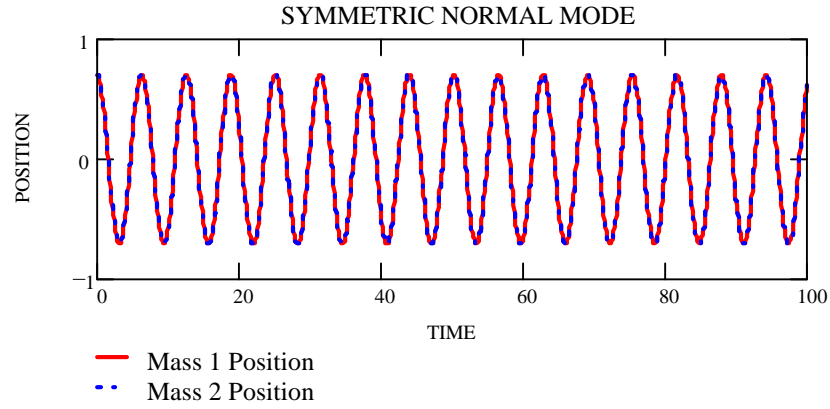


Figure 5. Illustration of the normal modes for two coupled oscillators. (a) Symmetric mode where $q_1 = q_2$. (b) Antisymmetric mode where $q_1 = -q_2$.

(a)



(b)

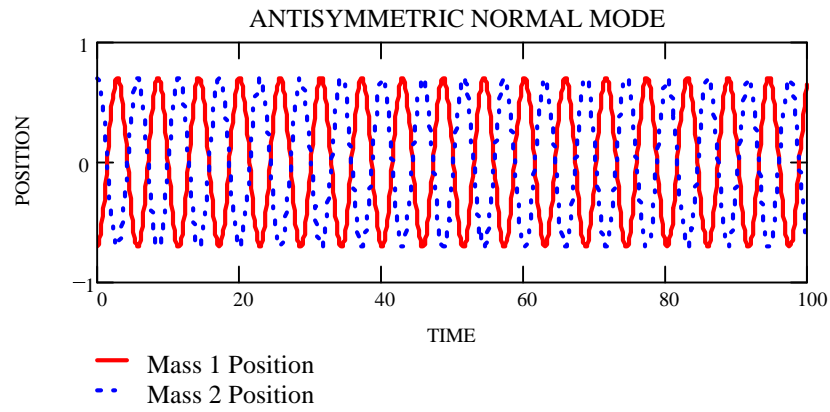
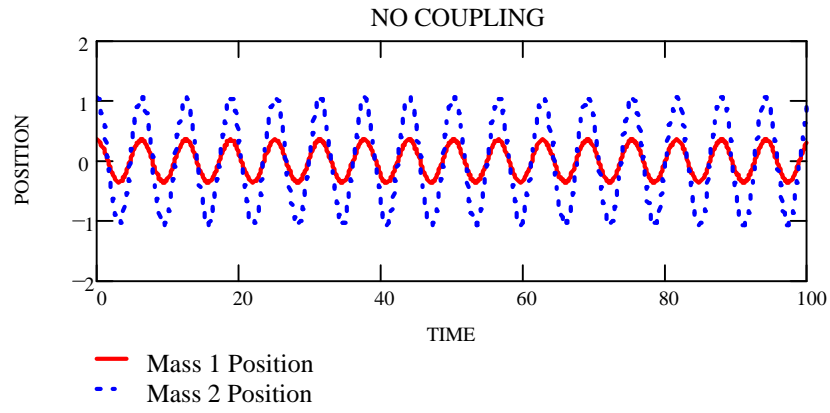
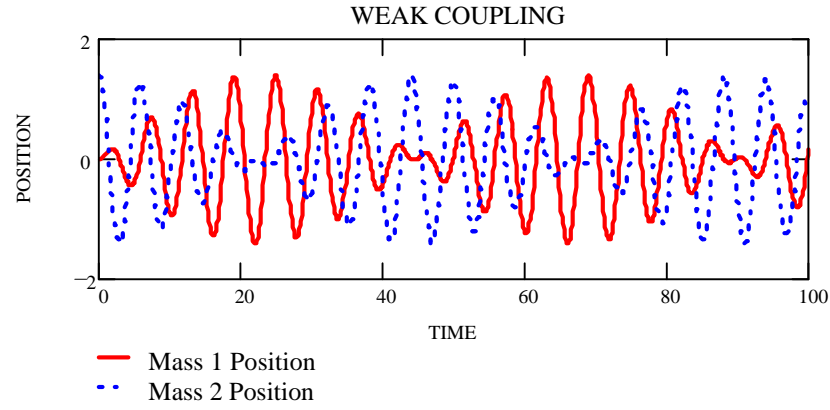


Figure 6. Time dependence of coupled oscillator positions q_1 and q_2 for (a) oscillation in the symmetric normal mode ($q_1 = q_2$), and (b) oscillation in antisymmetric normal mode ($q_1 = -q_2$). For all graphs $M = 1, k = 1$. For (a) $A_1 = 1, A_2 = 0$. For (b) $A_1 = 0, A_2 = 1$.

(a)



(b)



(c)

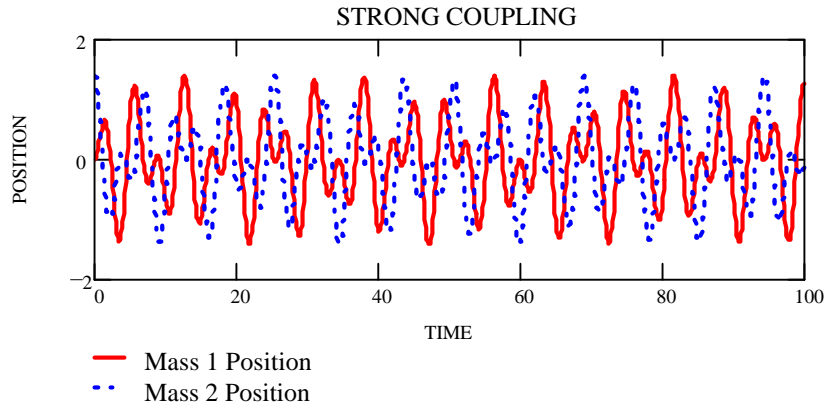


Figure 4. Time dependence of coupled oscillator positions q_1 and q_2 for (a) no coupling ($k' = 0$), (b) weak coupling ($k' = 0.1$), and (c) strong coupling ($k' = 1$). For all graphs $M = 1$, $k = 1$. For (a) $A_1 = 1$, $A_2 = 0.5$. For (b) and (c) $A_1 = A_2 = 1$.

If we start the system out so that

$$q_1(0) = -q_2(0) \quad \text{and} \quad v_1(0) = -v_2(0), \quad (2.19)$$

then this forces $A_1 = 0$ in (2.12), so that $Q_1(t) = 0$ (*exercise*), and we get the other normal mode of vibration. Note that these initial conditions amount to displacing each mass in the opposite direction by the same amount and giving each mass a velocity which is the same in magnitude but oppositely directed.[†] In this mode the particles oscillate oppositely, or completely *out of phase* (*i.e.*, the phases of the cosine functions that describe the oscillations of each mass differ by π radians). This is consistent with the interpretation of Q_2 as the relative position of the particles. Clearly the spring which couples the particles (the one characterized by $\tilde{\omega}'$) is going to play a role here – this spring is going to be compressed or stretched – which is why the (higher) frequency of oscillation of this mode, Ω_2 , involves $\tilde{\omega}'$. From (2.12), all other motions of the system are particular superpositions of these two basic kinds of motion and are obtained by using initial conditions other than (2.18) or (2.19). This is the meaning of the basis (2.16) and the form (2.17) of the general solution.

At this point you cannot be blamed if you feel that, aside from its interest as a step on the road to waves, the system of coupled oscillators is not particularly relevant in physics. After all, how useful can a system be that consists of a couple of masses connected by springs? Actually, the mathematics used in this section (and generalizations thereof) can be fruitfully applied to vibrational motions of a variety of systems. Perhaps the most outstanding of such applications are provided by molecules. For example, one can use the above normal mode analysis to find the possible vibrational motion of a linear triatomic molecule, such as ozone (O_3). The vibrational motion of such a molecule can be excited by an oscillating electric field (*e.g.*, an electromagnetic wave), hence normal mode calculations are common in optical spectroscopy. Simple variations on these calculations occur when the masses are not equal (*e.g.*, CO_2), when the molecule is not linear (*e.g.*, NO_2), or when there are more atoms in the molecule (*e.g.*, methane (CH_4) or ammonia (NH_3)).

3. How to find normal modes.

How do we find the normal modes and resonant frequencies without making a clever guess? Well, you can get a more complete explanation in an upper-level mechanics course, but the gist of the trick involves a little linear algebra. The idea is the same for any number of coupled oscillators, but let us stick to our example of two oscillators.

[†] Note that in both (2.18) and in (2.19) one can have vanishing initial displacements or velocities.

To begin, we again assemble the 2 coordinates, q_i , $i = 1, 2$, into a column vector \mathbf{q} ,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \quad (3.1)$$

Let K be the 2×2 symmetric matrix

$$K = \begin{pmatrix} \tilde{\omega}^2 + \tilde{\omega}'^2 & -\tilde{\omega}'^2 \\ -\tilde{\omega}'^2 & \tilde{\omega}^2 + \tilde{\omega}'^2 \end{pmatrix}. \quad (3.2)$$

The coupled oscillator equations (2.3), (2.4) can be written in matrix form as (exercise)

$$\frac{d^2 \mathbf{q}}{dt^2} = -K\mathbf{q}. \quad (3.3)$$

The fact that the matrix K is not diagonal corresponds to the fact that the equations for $q_i(t)$ are coupled.

Exercise: Check that the matrix form of the uncoupled equations (2.1), (2.2) gives a diagonal matrix K .

Our strategy for solving (3.3) is to find the *eigenvalues* λ and *eigenvectors* \mathbf{e} of K . These are the solutions to the equation

$$K\mathbf{e} = \lambda\mathbf{e}, \quad (3.4)$$

where λ is a scalar and \mathbf{e} is a (column) vector. We will make two assumptions about the solutions to this well-known type of linear algebra problem. First, we assume that the two (possibly equal) eigenvalues, λ_1 and λ_2 are all positive. Second, we assume that the corresponding eigenvectors, \mathbf{e}_1 and \mathbf{e}_2 , are linearly independent.* This means that any column vector \mathbf{v} can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2,$$

for some real numbers v_1 and v_2 , which are the *components* of \mathbf{v} . We shall soon see why we need these assumptions and when they are satisfied.

Given the solutions $(\lambda_1, \mathbf{e}_1)$, $(\lambda_2, \mathbf{e}_2)$ to (3.4), we can build a solution to (3.3) as follows. Write

$$\mathbf{q}(t) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2. \quad (3.5)$$

Exercise: Why can we always do this?

* In other words, the eigenvectors form a *basis* for the vector space of 2-component column vectors.

Using

$$\frac{d^2 \mathbf{q}}{dt^2} = \frac{d^2 \alpha_1}{dt^2} \mathbf{e}_1 + \frac{d^2 \alpha_2}{dt^2} \mathbf{e}_2, \quad (3.6)$$

and[†]

$$\begin{aligned} K \mathbf{q} &= \alpha_1(t) K \mathbf{e}_1 + \alpha_2(t) K \mathbf{e}_2 \\ &= \lambda_1 \alpha_1(t) \mathbf{e}_1 + \lambda_2 \alpha_2(t) \mathbf{e}_2, \end{aligned} \quad (3.7)$$

you can easily check that (3.5) defines a solution to (3.3) if and only if

$$\left(\frac{d^2 \alpha_1}{dt^2} + \lambda_1 \alpha_1(t) \right) \mathbf{e}_1 + \left(\frac{d^2 \alpha_2}{dt^2} + \lambda_2 \alpha_2(t) \right) \mathbf{e}_2 = 0. \quad (3.8)$$

Using the linear independence of the eigenvectors, this means (*exercise*) that α_1 and α_2 each solves the harmonic oscillator equation with frequency $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, respectively:

$$\frac{d^2 \alpha_n}{dt^2} = -\lambda_n \alpha_n(t), \quad n = 1, 2. \quad (3.9)$$

The general solution to (3.3) can then be written as (*exercise*)

$$\mathbf{q}(t) = \text{Re}(A_1 e^{i\sqrt{\lambda_1}t} \mathbf{e}_1 + A_2 e^{i\sqrt{\lambda_2}t} \mathbf{e}_2), \quad (3.10)$$

where A_1 and A_2 are any complex numbers. Thus, by finding the eigenvalues and eigenvectors we can reduce our problem to the harmonic oscillator equation, which we already know how to solve.

Now you can see why we made those assumptions about the eigenvalues and eigenvectors. Firstly, if the eigenvectors don't form a basis, we can't assume \mathbf{q} takes the form (3.5) nor that (3.8) implies (3.9). It is an important theorem from linear algebra that for any *symmetric* matrix with real entries, such as (3.2), the eigenvectors will form a basis, so this assumption is satisfied in our current example. Secondly, the frequencies $\sqrt{\lambda_n}$ will be real numbers if and only if the eigenvalues λ_n are always positive. While the aforementioned linear algebra theorem guarantees the eigenvalues of a symmetric matrix will be real, it doesn't guarantee that they will be positive. However, as we shall see, for the coupled oscillators the eigenvalues are positive definite, which one should expect on physical grounds. (*Exercise*: How would you interpret the situation in which the eigenvalues are negative?).

Comparing our general solution (3.10) with (2.12) we see that the resonant frequencies ought to be related to the eigenvalues of K via

$$\Omega_i = \sqrt{\lambda_i}, \quad i = 1, 2$$

and the normal modes should correspond to the eigenvectors \mathbf{e}_i . Let us work this out in detail.

[†] Note that here we use the fact that matrix multiplication is a linear operation.

The eigenvalues of K are obtained by finding the two solutions λ to the equation (3.4). This equation is equivalent to

$$(K - \lambda I)\mathbf{e} = 0,$$

where I is the identity matrix. A standard result from linear algebra is that this equation has a non-trivial solution[†] \mathbf{e} if and only if λ is a solution of the *characteristic (or secular) equation*:

$$\det[K - \lambda I] = 0.$$

You can easily check that the secular equation for (3.2) is

$$\lambda^2 - 2(\tilde{\omega}^2 + \tilde{\omega}'^2)\lambda - \tilde{\omega}'^4 + (\tilde{\omega}^2 + \tilde{\omega}'^2)^2 = 0.$$

This quadratic equation is easily solved to get the two roots (*exercise*)

$$\begin{aligned}\lambda_1 &= \tilde{\omega}^2 \\ \lambda_2 &= \tilde{\omega}^2 + 2\tilde{\omega}'^2.\end{aligned}\tag{3.11}$$

Note that we have just recovered the (squares of the) resonant frequencies by finding the eigenvalues of K .

To find the eigenvectors \mathbf{e}_i of K we substitute each of the eigenvalues λ_i , $i = 1, 2$ into the eigenvalue equation (3.4) and solve for the components of the \mathbf{e}_i using standard techniques. As a very nice exercise you should check that the resulting eigenvectors are of the form

$$\begin{aligned}\mathbf{e}_1 &= a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{e}_2 &= b \begin{pmatrix} -1 \\ 1 \end{pmatrix}\end{aligned}\tag{3.12}$$

where a and b are any constants, which can be absorbed into the definition of A_1 and A_2 in (3.10) (*exercise*).

Exercise: Just from the form of (3.4), can you explain why the eigenvectors are only determined up to an overall multiplicative factor?

Using these eigenvectors in (3.10) we recover the expression (2.12) – you really should verify this yourself. In particular, it is the eigenvectors of K that determine the column vectors appearing in (2.16) (*exercise*).

Note that the eigenvectors are linearly independent as advertised (*exercise*). Indeed, using the usual scalar product on the vector space of column vectors \mathbf{v} and \mathbf{w} ,

$$(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \mathbf{w},$$

[†] *Exercise:* what is the trivial solution?

you can check that \mathbf{e}_1 and \mathbf{e}_2 are orthogonal (see Problems).

To summarize: The resonant frequencies of a system of coupled oscillators, described by the matrix differential equation

$$\frac{d^2}{dt^2}\mathbf{q} = -K\mathbf{q},$$

are determined by the eigenvalues of the matrix K . The normal modes of vibration are determined by the eigenvectors of K .

4. Linear Chain of Coupled Oscillators.

As an important application and extension of the foregoing ideas, and to obtain a first glimpse of wave phenomena, we consider the following system. Suppose we have N identical particles of mass m in a line, with each particle bound to its neighbors by a Hooke's law force, with "spring constant" k . Let us assume the particles can only be displaced in one-dimension; label the displacement from equilibrium for the j^{th} particle by q_j , $j = 1, \dots, N$. Let us also assume that particle 1 is attached to particle 2 on the right and a rigid wall on the left, and that particle N is attached to particle $N - 1$ on the left and another rigid wall on the right. The equations of motion then take the form (*exercise*):

$$\frac{d^2 q_j}{dt^2} + \tilde{\omega}^2(q_j - q_{j-1}) - \tilde{\omega}^2(q_{j+1} - q_j) = 0, \quad j = 1, 2, \dots, N. \quad (4.1)$$

For convenience, in this equation and in all that follows we have extended the range of the index j on q_j to include $j = 0$ and $j = N + 1$. You can pretend that there is a particle fixed to each wall with displacements labeled by q_0 and q_{N+1} . Since the walls are rigid, to obtain the correct equations of motion we must set

$$q_0 = 0 = q_{N+1}. \quad (4.2)$$

These are *boundary conditions*.

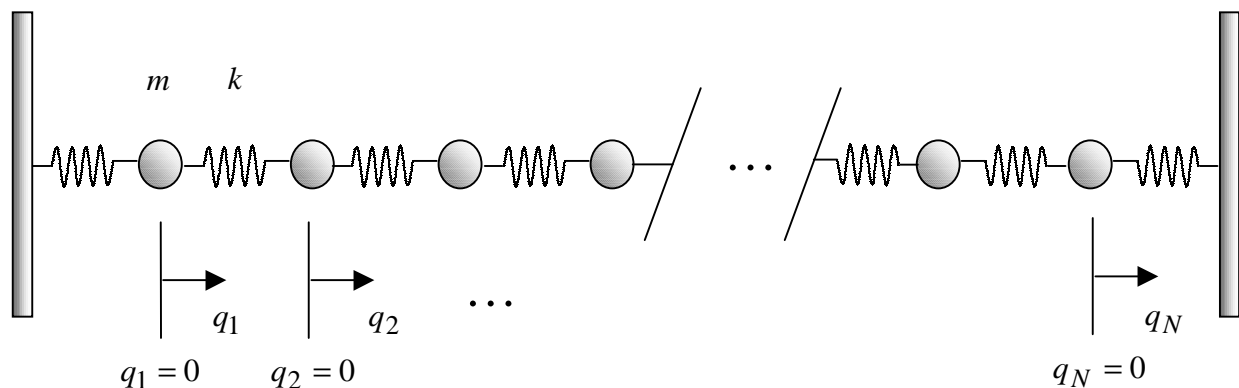


Figure 7. Linear chain of coupled oscillators. Each oscillator of mass m is coupled to its nearest neighbor with a spring with spring constant k . As in the case of the two-coupled oscillator problem, displacement from equilibrium q_i is restricted to be along the chain of oscillators, as illustrated.

The equations of motion (4.1) are, mathematically speaking, a system of N coupled, linear, homogeneous, ordinary differential equations with constant coefficients. Note that each oscillator is coupled only to its “nearest neighbors” (exercise). As it turns out, the system of coupled oscillators described by (4.1) exhibits resonant frequencies and normal modes of vibration. To see this we could set up (4.1) as a matrix equation (see Problems) and use the linear algebraic techniques discussed above. In particular, the generalization of the matrix K from the last section will be symmetric and hence will admit N linearly independent eigenvectors, which define the normal modes and whose eigenvalues define the characteristic frequencies. While this is a perfectly reasonable way to proceed, as you will see below, we can reduce the analysis considerably by employing a shortcut.

This picture of a linear chain of coupled oscillators (and its three-dimensional generalization) is used in solid state physics to model the vibrational motion of atoms in a solid. The masses represent the atomic nuclei that make up the solid and the spacing between the masses is the atomic separation. The “springs” coupling the masses represent a harmonic approximation to the forces binding the nuclei into the solid. In the context of applications to solid state physics the normal modes are identified with *phonons*. After incorporating quantum mechanics, this phonon picture of vibrational modes of a solid is used to describe thermal conductivity, specific heat, propagation of sound, and other properties of the solid.

Our goal will be to obtain the normal modes and characteristic frequencies of vibration defined by (4.1). Recall that each of the normal modes of vibration for a pair of coupled

oscillators has the masses oscillating harmonically, all at the same frequency (cf. (2.9) and (2.10)). Let us therefore look for a complex solution to (4.1) of the form

$$q_j(t) = \text{Re} \left(A_j e^{i\Omega t} \right). \quad (4.3)$$

By convention we assume that the frequency Ω is non-negative. Substituting this into our equations yields a *recursion relation**(exercise):

$$-\Omega^2 A_j = \tilde{\omega}^2 (A_{j-1} - 2A_j + A_{j+1}), \quad j = 1, 2, \dots, N, \quad (4.4)$$

still subject to the boundary conditions

$$A_0 = 0 = A_{N+1}.$$

We can solve this relation via a trial solution

$$A_j = a \sin(j\phi), \quad (4.5)$$

where ϕ is some real number and a can be complex.† Note that this trial solution satisfies the boundary condition $q_0 = 0$, but we still have to take care of the condition $q_{N+1} = 0$ — we shall do this below by specifying the parameter ϕ . We plug (4.5) into the recursion relation to get (exercise)

$$-\Omega^2 a \sin(j\phi) = \tilde{\omega}^2 \left\{ a \sin[(j-1)\phi] - 2a \sin[j\phi] + a \sin[(j+1)\phi] \right\}. \quad (4.6)$$

Note that a will drop out of this condition, that is, a is not determined by (4.4)

Exercise: What property of the equations (4.1) and/or (4.4) guarantees that a will drop out of (4.6)?

To analyze (4.6) we use the trigonometric identity (exercise),

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

to write

$$\sin[(j \pm 1)\phi] = \sin(j\phi) \cos(\phi) \pm \cos(j\phi) \sin(\phi).$$

* A recursion relation for a set of variables A_j , $j = 1, 2, \dots, n$, is a sequence of equations which allows one to determine A_k from the set A_1, A_2, \dots, A_{k-1} .

† This form of the trial solution is certainly not obvious. It can be motivated by studying several special cases with N small. Alternatively, one can consider (4.4) for very large values of j , in which case one can pretend that A_j is a function of the continuous variable j . One can then interpret the recursion relation as (approximately) saying that the second derivative of this function is proportional to the function itself. Using $A_0 = 0$ one arrives at (4.5) (exercise).

This gives for (4.6) (*exercise*)

$$\Omega^2 \sin(j\phi) = 2\tilde{\omega}^2 [1 - \cos(\phi)] \sin(j\phi).$$

Given (4.3) and (4.5), we naturally assume that $\sin(j\phi)$ does not vanish identically for all j . Thus the recursion relation (and hence the equations of motion (4.1)) are satisfied by (4.3) and (4.5) if and only if

$$\Omega^2 = 2\tilde{\omega}^2 [1 - \cos(\phi)] = 4\tilde{\omega}^2 \sin^2(\phi/2), \quad (4.7)$$

that is,

$$\Omega = 2\tilde{\omega} |\sin(\phi/2)|. \quad (4.8)$$

Note that we are adhering to our convention that Ω be non-negative.

We still must enforce the boundary condition $q_{N+1} = 0$, which is now $A_{N+1} = 0$. This condition means

$$\sin[(N+1)\phi] = 0, \quad (4.9)$$

so that

$$(N+1)\phi = n\pi, \quad n = 1, 2, \dots, N. \quad (4.10)$$

In (4.10) we take the maximum value for n to be N to avoid redundant solutions; if $n > N$ then we obtain solutions for A_j that were already found when $n \leq N$ (see below and also the homework problems). We exclude the solution corresponding to $n = 0$ because this solution has $\phi = 0$, which forces $A_j = 0$, *i.e.*, this is the trivial solution $q_j(t) = 0$ (for all values of j) of the coupled oscillator equations.

Exercise: What property of (4.1) guarantees that $q_j = 0$ is a solution?

To summarize thus far, there are N distinct resonant frequencies, which we label by an integer n , where $n = 1, 2, \dots, N$. They take the form

$$\Omega_n = 2\tilde{\omega} \left| \sin \left(\frac{n\pi}{2N+2} \right) \right|, \quad n = 1, 2, \dots, N. \quad (4.11)$$

Compare this with the case of two coupled oscillators, treated earlier, where there were 2 resonant frequencies.

We can now return to our trial solution for the complex amplitudes A_j . For each resonant frequency there will be a corresponding set of complex amplitudes. (In the case of two coupled oscillators there were two resonant frequencies and two sets of amplitudes, representing the normal modes.) For the resonant frequency Ω_n (for some choice of n)

we denote the corresponding complex amplitudes by $A_{(n)j}$, $j = 1, 2, \dots, N$. We have (*exercise*)

$$A_{(n)j} = a_n \sin\left(\frac{n\pi j}{N+1}\right), \quad (4.12)$$

where a_n is any complex number. Let us pause to keep track of our notation: j labels the masses, N is the total number of masses, and n labels the normal modes of vibration and their resonant frequencies. If you view the N amplitudes for each n , $A_{(n)j}$, $j = 1, 2, \dots, N$, n fixed, as forming the entries of a column vector, then the totality of the column vectors (obtained by letting $n = 1, 2, \dots, N$) would form a basis for the N -dimensional space of column vectors with N entries. This basis is in fact the basis of eigenvectors defined by the matrix K which we mentioned (but didn't explicitly write down) at the beginning of this section. As guaranteed by general results in linear algebra, all the vectors in this basis are orthogonal. You will investigate this in the Problems.

The n^{th} normal mode is (*exercise*)

$$\begin{aligned} q_{(n)j} &= \text{Re} \left[a_n \sin\left(\frac{n\pi j}{N+1}\right) e^{i\Omega_n t} \right], \\ &= |a_n| \sin\left(\frac{n\pi j}{N+1}\right) \cos(\Omega_n t + \alpha_n), \quad n = 1, 2, \dots, N, \end{aligned} \quad (4.13)$$

where we have written $a_n = |a_n|e^{i\alpha_n}$.

For each normal mode we have the following behavior. By considering (4.13) for a fixed value of j , you can see that each mass is undergoing a harmonic oscillation at frequency Ω_n . The amplitude of each oscillator depends sinusoidally on the location of the mass (and has an overall scale set by a_n) according to (4.12). In particular, for the n^{th} mode, as you move from one mass to the next the displacement of each mass advances in phase by $\frac{n\pi}{N+1}$, leading to the patterns shown in figure 8. Another point of view on these patterns is as follows. Let us suppose that the equilibrium positions of the masses are separated by a distance d , and that the first ($j = 1$) and last ($j = N$) masses are separated from their walls also by d when in equilibrium. Then the j^{th} mass, in its equilibrium position, will be a distance $x = jd$ from the wall attached to q_1 (*exercise*). According to (4.13), if you examine the system at a fixed time t , *i.e.*, take a photograph of the system at time t , then the displacement from equilibrium as a function of location on the chain of oscillators will be a function of the form $P \sin(Qx)$ (*exercise*), where P and Q are some real constants. Thus the displacement is a discrete form of a *standing wave*. Recall that a standing wave in a continuous medium (*e.g.*, a guitar string) is a motion of the medium in which each point of the medium oscillates harmonically (*i.e.*, sinusoidally) in time from its equilibrium position, while the amplitude of the oscillation varies sinusoidally from point to point in the medium.

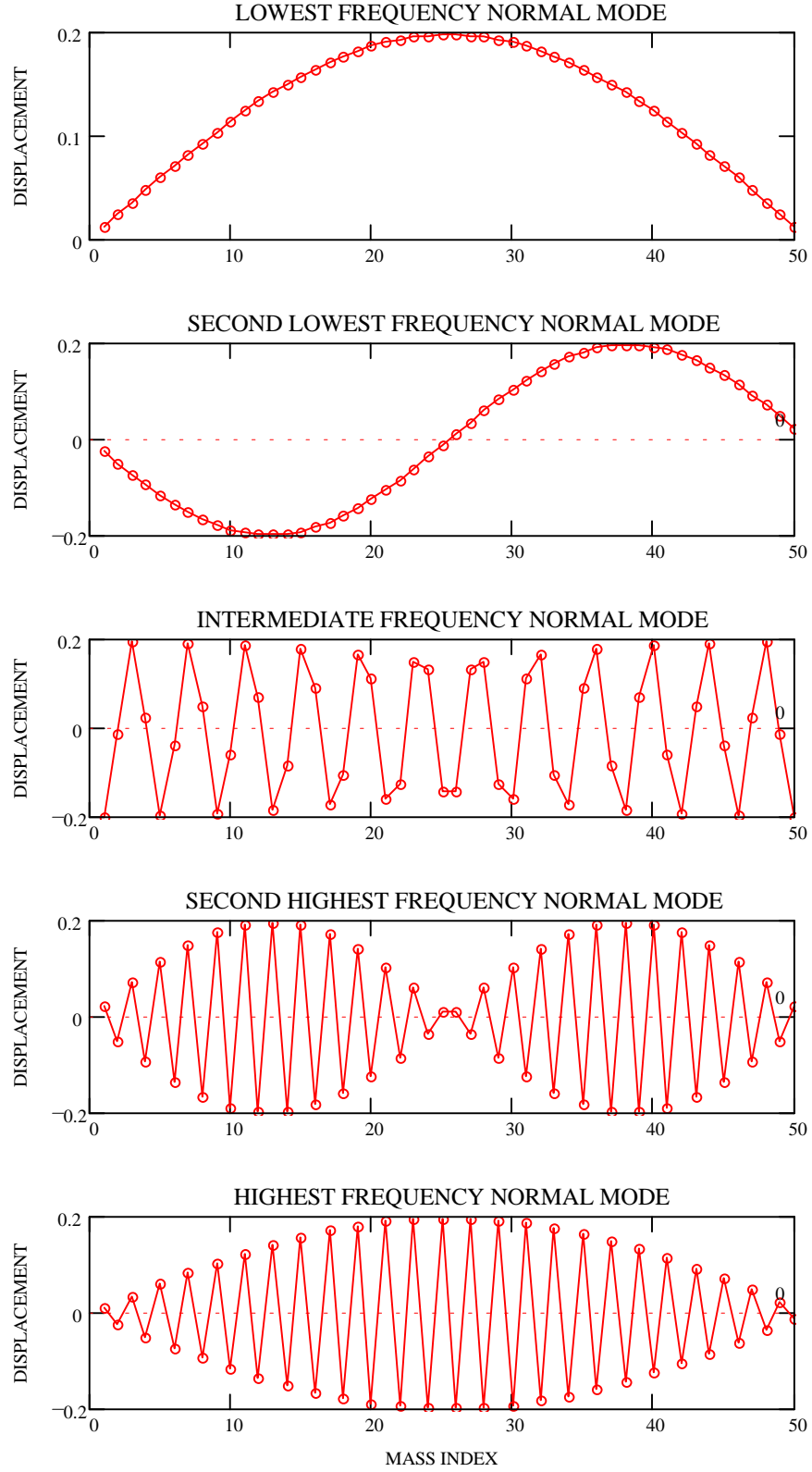


Figure 8. Selected normal modes for an $N = 50$ linear chain of coupled oscillators.

Also recall that standing waves have *nodes*, which are points which have zero oscillation amplitude, that is, they do not move at all. For our linear chain of coupled oscillators nodes will occur where the sine vanishes, that is, where

$$j = \frac{(N+1)}{n}l, \quad l = 0, 1, 2, \dots, n. \quad (4.14)$$

Note that we include the cases $j = 0$ and $j = N + 1$, which are always nodes (*exercise*) corresponding to the (pretend) masses fixed on the walls. Of course, (4.14) only applies when j works out to be an integer, or else there is no mass at the putative node. Indeed, the standing wave picture must be augmented by the knowledge that the wave is only “sampled” at the points $x = jd$, $j = 0, 1, 2, \dots, N + 1$, which is why the displacement profiles in figure 8 are somewhat more intricate than one would expect when thinking of a sine function.

Equation (4.11) is a relation between the frequency of vibration of the (discrete) standing wave and the mode number n . Using the interpretation for (4.12) given above, the wavelength of the discrete standing wave is inversely proportional to n . Thus one can also view this relation as between frequency and wavelength of the standing wave and hence as a relation between wavelength and wave speed. We will find such a relation in each instance of wave phenomena. For reasons we shall discuss later, such a relation is called a *dispersion relation*.

Exercises: Show that when $n \ll N$ the frequency is approximately proportional to n , and when $n \approx N \gg 1$ the frequency is approximately $2\tilde{\omega}$.

The general solution to the equations of motion (4.1) is a superposition of all the normal modes:

$$q_j(t) = \text{Re} \left\{ \sum_{n=1}^N a_n \sin \left(\frac{n\pi j}{N+1} \right) e^{i\Omega_n t} \right\}, \quad (4.15)$$

where a_n is a complex constant for each n . Note we can take the real part before or after the summation (*exercise*). An equivalent form of the general solution is (*exercise*)

$$q_j(t) = \left\{ \sum_{n=1}^N |a_n| \sin \left(\frac{n\pi j}{N+1} \right) \cos(\Omega_n t + \alpha_n) \right\}, \quad (4.16)$$

where $|a_n|$ and α_n are real numbers. In any case, the solution depends on $2N$ real constants via the complex numbers a_n in (4.15) or the real numbers $(|a_n|, \alpha_n)$ in (4.16). You should not be surprised by this. There are N particles, each obeying Newton’s second law. Each particle will require specification of an initial position (displacement) and initial velocity to

uniquely determine its motion. This is the same as giving an initial *displacement profile* and *velocity profile* along the chain. Specifying the initial conditions is equivalent to specifying the amplitudes $|a_n|$ and phases α_n . Thus one can accommodate every possible set of initial conditions using (4.15) or (4.16) and so one is indeed justified in claiming these formulas provide the general solution to the coupled oscillator problem.

Let us note two key results here that we can glean from our analysis. First, it is the *boundary conditions* (requirements at a fixed location for all time, *i.e.*, the rigid wall conditions) which serve to fix the form of the characteristic frequencies and the normal modes and hence the form of the general solution. Second, it is the *initial conditions* (requirements for all space at a fixed time, *e.g.*, initial displacement and velocity profiles) that pick out specific solutions of the equations of motion from the general solution, *i.e.*, determine the constants a_n . In other words, it is the initial conditions which determine the specific linear combination of normal modes that should describe a given situation.

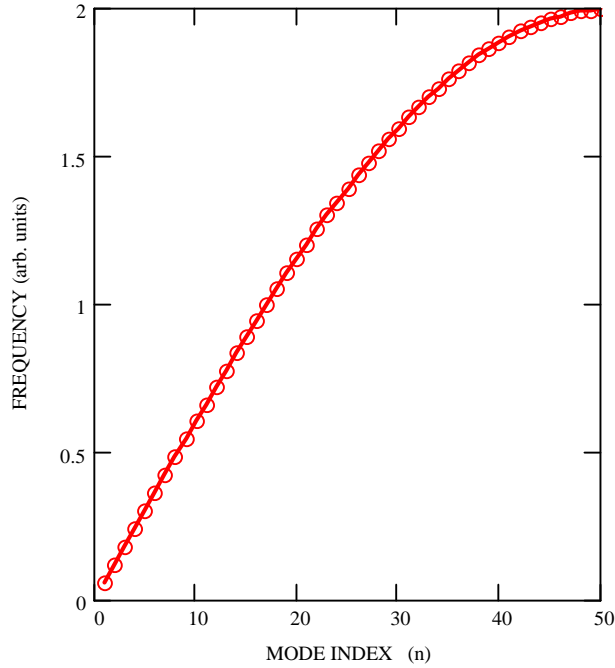


Figure 9. Dispersion relation Ω_n for the $N = 50$ linear chain of coupled oscillators.

4.1 Other Boundary Conditions

As it turns out, the normal modes have the form of (discrete) standing waves because we have fixed the ends of our chain of oscillators to rigid walls, *i.e.*, $q_0 = 0 = q_{N+1}$. If we

change our boundary conditions we can obtain discrete versions of traveling wave solutions. Let us briefly have a look at this.

To begin, let us consider what happens if there are no boundary conditions at all. To do this with a minimum of fuss, we assume that the chain of oscillator extends “to infinity”. Of course, no such thing exists. Rather, this is just a convenient mathematical *model* for a situation where we have a long chain of many oscillators and we are only interested in the behavior of oscillators far from the ends of the chain. The idea is that near the center of a very long chain the effect of the boundary conditions should be negligible.* In this model we still have the equations of motion (4.1) for the displacements q_l , but we let l run over all integer values. We can still use the *ansatz* (4.3) and we obtain (4.4). Since we don’t have to satisfy the rigid wall boundary conditions (4.2), we try a solution of the form

$$A_l = ae^{il\phi}. \quad (4.17)$$

This gives (*exercise*)

$$-\Omega^2 e^{il\phi} = \tilde{\omega}^2 \left\{ e^{i(l-1)\phi} - 2e^{il\phi} + e^{i(l+1)\phi} \right\}, \quad (4.18)$$

from which it follows (again!) that

$$\Omega(\phi) = 2\tilde{\omega} |\sin(\phi/2)|.$$

This time, however, there are no boundary conditions and hence no conditions upon ϕ . The normal mode solutions are determined/labeled by ϕ ; they take the form

$$q_{\phi,l} = \text{Re} \left\{ a(\phi) e^{i(l\phi + \Omega t)} \right\}. \quad (4.19)$$

Notice that the exponential in (4.19) is unchanged if $\phi \rightarrow \phi + 2\pi$. This means that a non-redundant description of the normal modes of vibration is achieved by restricting ϕ to a region of size 2π , *e.g.*, $0 < \phi \leq 2\pi$. Notice also that when $\phi = 0$ the normal mode has zero frequency. What can this mean? Evidently, in this case all of the displacements are equal and constant in time. It might help to picture a chain of masses connected by springs and free to move only in one dimension (parallel to the chain). Now visualize the chain of oscillators displaced rigidly as a whole (in one dimension, along its length) with no compression or stretching of the springs. This is the zero frequency mode. In our previous example, the fixed-wall boundary conditions prevented this mode from appearing.† For $\phi > 0$, the form of the normal modes given in (4.19) is a discrete version of a *traveling*

* This sort of model (suitably generalized to 3-dimensions) is used to describe the bulk properties of crystalline solids.

† In fact, there is another zero frequency mode which appears here, corresponding to a displacement with constant velocity.

sinusoidal wave. In particular, at each time t the displacement profile is a (discretely sampled) sinusoidal pattern which moves with velocity $\vec{v} = -\frac{\Omega d}{\phi}$ (*exercise*). (Here d is the equilibrium separation of the oscillators.)

Aside from rigid displacements of the chain, the general motion of the chain is obtained by a superposition of the normal modes with non-zero frequency. This is an integral of the form:

$$q_l = Re \int_0^{2\pi} d\phi a(\phi) e^{i(l\phi + \Omega(\phi)t)}. \quad (4.20)$$

The behavior we have found for the infinite chain of oscillators can be understood by considering the previous example with fixed walls and considering the situation where there are many oscillators and we study the motion near the center of the chain. This will be explored in the Problems.

Let us now consider a different type of boundary condition — periodic boundary conditions. Imagine we have $N + 1$ oscillators, as before, but now we identify the first and the last oscillators, that is, we assume that they always have the same displacement:

$$q_1(t) = q_{N+1}(t). \quad (4.21)$$

This could be done by physically identifying the two oscillators — you might try imagining the chain of oscillators connected into a circle — or by some other means. Our analysis goes through as above in the case of no boundary conditions. In particular, we have the normal modes

$$q_{\phi,l} = Re \left\{ a(\phi) e^{i(l\phi + \Omega(\phi)t)} \right\}, \quad (4.22)$$

with

$$\Omega(\phi) = 2\tilde{\omega} |\sin(\phi/2)|. \quad (4.23)$$

but the periodic boundary conditions (4.21) mean that (*exercise*)

$$e^{i\phi} = e^{i(N+1)\phi}, \quad (4.24)$$

so that (*exercise*)

$$\phi = \frac{2\pi}{N} n, \quad n = 0, 1, 2, \dots, N-1. \quad (4.25)$$

As in the case of fixed wall boundary conditions, the periodic boundary conditions force the normal modes to come in a discrete set. We have limited the range of n so that we have a non-redundant set of modes (*exercise*). As in the case of no boundary conditions, there is a zero frequency mode corresponding to a rigid displacement of all the oscillators. The normal modes are again in the form of (discretely sampled) traveling waves. Aside from rigid displacements, the general motion of the oscillators is a superposition of the normal modes.

PROBLEM SET 2

Problem 2.1

Consider a particle moving in two dimensions (x, y) under a force

$$\mathbf{F} = -k\mathbf{r} = -kx\mathbf{i} - ky\mathbf{j}.$$

- (a) Show that the equations of motion for $\mathbf{r}(t) = (x(t), y(t))$ are equivalent to those describing a pair of uncoupled oscillators.
- (b) Show that the equations of motion admit a solution in which the particle moves in a circle at constant speed.

Problem 2.2

Consider a general linear, homogeneous equation $M(v) = 0$, where M is a matrix and v is a column vector. Suppose that all matrix elements of M are real: $M = M^*$. Show that if v is a complex solution to the linear equation (*i.e.*, the elements of the column vector are complex numbers), then v^* , $Re(v)$, and $Im(v)$ are also solutions to the same equation. (If you like, you can just let M be a 2×2 matrix, but it is not necessary to assume this in order to prove this result.)

Problem 2.3

When finding the resonant frequencies Ω_1, Ω_2 for a pair of coupled oscillators we must solve the characteristic equation. Suppose that one of the frequencies, Ω_1 say, turns out to be a *complex* number $\Omega_1 = \alpha + i\beta$. Discuss the physical behavior of the (putative) normal modes.

Problem 2.4

When we analyzed the chain of coupled oscillators we set $A_j = a \sin(j\phi)$ and then found that ϕ must satisfy $\sin[(N+1)\phi] = 0$. Show that there is no loss in generality by assuming the solution for ϕ is $\phi = \frac{n\pi}{N+1}$, $n = 1, 2, \dots, N$. In particular, why can we choose the range $1 \leq n \leq N$? (*Hint*: Show that a solution A_j with $n > N$ is a constant multiple of a solution in which $n \leq N$.)

Problem 2.5

Show that the normal modes and resonant frequencies we obtained for the chain of n oscillators reduce to our previous results for a pair of coupled oscillators when $N = 2$.

Problem 2.6

Consider 3 coupled oscillators all with the same “spring constant”; use rigid wall boundary conditions.

- (i) Compute the equations of motion and derive the matrix K analogous to (3.2).
- (ii) Compute the eigenvectors and eigenvalues of K . Show that the eigenvectors are orthogonal.
- (iii) From (ii) find the normal modes and characteristic frequencies.
- (iv) Show that your result in (iii) agrees with the specialization of the chain of oscillator formulas in §4 to the case $N = 3$.

Problem 2.7

Show that (3.12) is equivalent to (2.12).

Problem 2.8

Derive equations (2.3) and (2.4).

Problem 2.9

Check that the eigenvectors appearing in (3.12) are linearly independent.

Problem 2.10

Show that equations (4.15) and (4.16) are equivalent representations of the general solution for the chain of oscillators.

Problem 2.11

Show that (2.16) is indeed a basis for the vector space of solutions to (2.3), (2.4). (*Hint:* Write the general solution (2.12) in terms of sines and cosines.)

Problem 2.12

Show that the equations (4.1) can be written in the matrix form

$$\ddot{\mathbf{q}} = K\mathbf{q},$$

where K is an $N \times N$ *symmetric* matrix and \mathbf{q} is a column vector.

5. The Continuum Limit and the Wave Equation.

Our example of a chain of oscillators is nice because it is easy to visualize such a system, namely, a chain of masses connected by springs. But the ideas of our example are far more useful than might appear from this one simple mechanical model. Indeed, many materials (including solids, liquids and gases) have some aspects of their physical response to (usually small) perturbations behaving just as if they were a bunch of coupled oscillators — at least to a first approximation. In a sense we will explore later, even the electromagnetic field behaves this way! This “harmonic oscillator” response to perturbations leads — in a continuum model — to the appearance of wave phenomena in the traditional sense. We caught a glimpse of this when we examined the normal modes for a chain of oscillators with various boundary conditions. Because the harmonic approximation is often a good first approximation to the behavior of systems near equilibrium, you can see why wave phenomena are so ubiquitous. The key difference between a wave in some medium and the examples of §4 is that wave phenomena are typically associated with propagation media (stone, water, air, etc.) which are modeled as *continuous* rather than *discrete*. As mentioned earlier, our chain of oscillators in §4 can be viewed as a discrete model of a continuous (one-dimensional) material. We now want to introduce a phenomenological description of the material in which we ignore the atomic discreteness of matter.

The basic physical idea is reasonably simple. Often times we are interested in certain macroscopic properties of the material (*e.g.*, the behavior of a plucked guitar string as a function of time and space) and we want to ignore most of the details of the microscopic make-up of the material since they should be irrelevant for the most part. So long as the length scales associated with the macroscopic behavior of the material (*e.g.*, wavelengths) are much larger than the length scales associated with the microscopic structure (*e.g.*, the interparticle spacing) we can approximate the behavior of the material by taking a limit in which we set the interparticle spacing to zero while letting the number of oscillators become arbitrarily large (“approach infinity”).

We will have to exercise a little care in this limiting process. Here “care” means that we keep fixed some macroscopic quantities characterizing the material in which the waves are propagating. Some good examples of the materials to keep in mind are: sound waves in an “elastic solid”, *e.g.*, in a metal rod; a vibrating string or rope under tension; sound waves in a gas. Each of these materials will have certain physical parameters which are relevant to the propagation of the wave and which are macroscopic reflections of the effective oscillator parameters which model the microscopic behavior of the material. For example each of the three illustrations just mentioned will be characterized (in part) by their mass density.

Let us emphasize that a continuum approximation, by its very nature, will not have universal validity. For example, if we consider wave phenomena in which the wavelengths are comparable to (or smaller than) the interparticle spacing, then we don’t expect our

model will accurately model what is actually happening physically.

5.1 Derivation of the Wave Equation

As in §4, we suppose that the equilibrium separation of the oscillators is d and we label the equilibrium position of the oscillators by $x = jd$. We can then denote by $q(x, t)$ the displacement of the j^{th} oscillator from its equilibrium position at time t . Our use of the symbol x , usually reserved for a continuous variable, anticipates our implementation of the strategy wherein the interparticle spacing is so small (compared to the typical sizes of macroscopic phenomena) that we can model the particles as forming a continuous mass distribution. We rearrange the equations of motion (4.1) into the form (exercise)

$$\frac{d^2 q(x, t)}{dt^2} = -\tilde{\omega}^2 d \left[\frac{1}{d} \{q(x, t) - q(x - d, t)\} \right] + \tilde{\omega}^2 d \left[\frac{1}{d} \{q(x + d, t) - q(x, t)\} \right]. \quad (5.1)$$

We now study the right-hand side of this equation in the limit where d is very small. In this case we can view $q(x, t)$ as a continuous function of x to a better and better approximation, and we have that[†] (exercises)

$$\frac{1}{d} \{q(x, t) - q(x - d, t)\} \approx \left(\frac{\partial q}{\partial x} \right)_{x=jd-d/2}, \quad (5.2)$$

and

$$\frac{1}{d} \{q(x + d, t) - q(x, t)\} \approx \left(\frac{\partial q}{\partial x} \right)_{x=jd+d/2}. \quad (5.3)$$

In the same manner, the difference of these terms defines, in the limit as $d \rightarrow 0$, the second derivative of q times d :

$$\left(\frac{\partial q}{\partial x} \right)_{x=jd+d/2} - \left(\frac{\partial q}{\partial x} \right)_{x=jd-d/2} \approx d \left(\frac{\partial^2 q(x, t)}{\partial x^2} \right)_{x=jd}. \quad (5.4)$$

We can therefore write the equation of motion in this approximation as:

$$\frac{\partial^2 q(x, t)}{\partial t^2} = \tilde{\omega}^2 d^2 \frac{\partial^2 q(x, t)}{\partial x^2}. \quad (5.5)$$

This is already a wave equation, but to get our final form of it we need to consider the limit as $d \rightarrow 0$. We do this as follows.

First recall that

$$\tilde{\omega} = \sqrt{\frac{k}{m}}, \quad (5.6)$$

[†] Yes, those definitions for derivatives as limits of differences that you learned in calculus class really do come in handy after all!

We express the spring constant k — which represents “microscopic” information — as

$$k = \frac{\kappa}{d}, \quad (5.7)$$

where the physical interpretation of the “macroscopic” constant κ depends upon what we are modeling. For transverse[†] vibrations of a string, κ will represent the tension on the string. For the longitudinal[†] vibrations of an elastic medium (*e.g.*, sound waves in a solid), κ will represent the Young’s modulus, which determines the stiffness of the material making up the medium. For compression (sound) waves in air, κ will be the elastic modulus. The quantity κ is one of two macroscopic quantities that are held fixed when taking the continuum limit. We now have (*exercise*)

$$\tilde{\omega}^2 d^2 = \frac{\kappa d}{m}. \quad (5.8)$$

Next we express the mass of the oscillators — another microscopic quantity — as

$$m = \mu d, \quad (5.9)$$

where μ , which is a macroscopic quantity, represents the mass per unit length of the continuum medium. The mass per unit length is the other macroscopic quantity that is held fixed in the continuum limit. We now have

$$\tilde{\omega}^2 d^2 = \frac{\kappa}{\mu}. \quad (5.10)$$

The continuum limit has the microscopic parameters behaving as $d \rightarrow 0$, $\tilde{\omega} \rightarrow \infty$ (also $m \rightarrow 0$, $k \rightarrow \infty$), with the macroscopic parameters κ and μ — characterizing the continuous material in question — held fixed. Setting $v^2 := \frac{\kappa}{\mu}$,* (5.5) becomes

$$\frac{\partial^2 q(x, t)}{\partial t^2} = v^2 \frac{\partial^2 q(x, t)}{\partial x^2}. \quad (5.11)$$

This is the one-dimensional *wave equation*, which is a fundamental example of a *partial differential equation*. The equation (5.11), and its generalizations, will be the subject of all of our attention from now on.

Note that we are now using *partial* time derivatives, *i.e.*, we hold x fixed when we vary t to take the time derivative and we hold t fixed when we compute derivatives with

[†] Recall that “transverse waves” have a displacement which is orthogonal to the direction of propagation of the displacement, while “longitudinal waves” have a displacement which is parallel to the direction of propagation.

* The notation “ $a := b$ ” indicates that we are making a definition, namely, “ a is defined to be the quantity b ”. Thus we can distinguish between equalities that we should be able to deduce from some other facts (which use “ $=$ ”), and equalities true merely by definition (which use “ $:=$ ”). The notation “ $a := b$ ” is close to, but not quite the same as, $a \equiv b$, which means “ a is identically equal to b ”, as in $5 \equiv 5$.

respect to x . Don't forget: $x = jd$ formerly labeled equilibrium positions of the j^{th} mass and $q_j(t)$ denoted the displacement of that mass from its equilibrium position at time t . In the continuum approximation, the chain of oscillators is represented by a line of fixed mass density. Points on the line are labeled x and the displacement “from equilibrium” of a point at x on the line at time t is denoted by $q(x, t)$. The second time derivative of $q(x, t)$ in the wave equation is just the acceleration that features in Newton's second law. The second spatial derivative of $q(x, t)$ is the continuum limit of the harmonic “nearest neighbor” interaction.

If the continuum is meant to describe an elastic medium undergoing longitudinal vibrations (sound waves) then the displacement $q(x, t)$ represents a compression or rarefaction of the elastic medium the sound is traveling in at the point x and time t , that is, q represents a longitudinal density wave. If the continuum is meant to represent a vibrating string under tension, then $q(x, t)$ represents the deflection of the string at (x, t) from its equilibrium position, that is, q represents a transverse displacement wave.

The parameter v that appears in (5.11) is easily seen to have units of speed (exercise). We shall see that v characterizes the speed of the waves that satisfy (5.11). By the way, an easy way to remember how the velocity factor enters the wave equation is to use dimensional analysis: the $\frac{1}{v^2}$ is needed to balance the units (exercise).

To summarize: *the 1-dimensional wave equation describes a continuum of matter in which displacements of infinitesimal elements of mass experience a Hooke's law restoring force.*

5.2. Boundary Conditions

In our discussion of the chain of oscillators we considered various boundary conditions. Since the wave equation can be viewed as a limiting case of the chain of oscillators, there are corresponding boundary conditions here as well. Let us briefly describe them here.

Of course, the case of no boundary conditions has for the most part been dispensed with in the previous paragraphs — no boundary conditions were imposed there. Typically, one will ignore boundary conditions if one is not near a boundary and one is considering features of the wave which are much smaller than the spatial domain of the problem. One will then speak of, *e.g.*, waves on a “long” string. The usual mathematical model for such a situation is to suppose that the spatial domain x is all of the real numbers, $-\infty < x < \infty$.

Let us consider our original fixed-wall boundary conditions. These are sometimes called *Dirichlet conditions* in this context. Here the spatial domain of interest is finite, say, $0 \leq x \leq L$, and the displacement $q = q(x, t)$ vanishes at the boundaries for all time:

$$q(0, t) = 0 = q(L, t). \quad (5.12)$$

This boundary condition models a string under tension with fixed ends (*e.g.*, a guitar string). It also models sound waves in air in a finite, closed region (in a one-dimensional approximation — *e.g.*, a long closed pipe.)

A vibrating metal rod (whose spatial cross section is small compared to its length L) which is clamped at one end or a vibrating column of air in a pipe which is open at one end could be modeled with just the boundary conditions

$$q(0, t) = 0.$$

Periodic boundary conditions can be handled similarly, on a finite interval $0 \leq x \leq L$ we insist that

$$q(0, t) = q(L, t). \quad (5.13)$$

This condition could be used to model a vibrating loop of string, or sound in a closed circular pipe.

6. Elementary Solutions to the Wave Equation.

Before systematically exploring the wave equation, it is good to pause and contemplate some basic solutions. We are looking for a function q of 2 variables, x and t , whose second x derivatives and second t derivatives are proportional. You can probably guess such functions with a little thought. But our derivation of the equation from the model of a chain of oscillators gives a strong hint. The normal modes for a chain of oscillators look like products of sinusoidal functions of j (now replaced by x) and sinusoidal functions of t . If we try such a combination, say,

$$q(x, t) = A \sin(Bx) \cos(Ct), \quad (6.1)$$

where A , B , and C are real constants, we find that the wave equation is satisfied provided

$$\frac{C^2}{B^2} = v^2, \quad (6.2)$$

which is one restriction on the constants B and C . Note that A is unrestricted by the wave equation. (*Exercise:* what property of the wave equation guarantees that A is unrestricted?) Thus we can let A be any constant, we can let C be any constant, and then set $B = C/v$ to get a solution. It is customary to set

$$\omega = C \quad \text{and} \quad B = \frac{2\pi}{\lambda}, \quad \omega = 2\pi v/\lambda, \quad (6.3)$$

Then our solution takes the form

$$q(x, t) = A \sin\left(\frac{2\pi x}{\lambda}\right) \cos(\omega t). \quad (6.4)$$

We have obtained a *standing wave* with *wavelength* λ and *angular frequency* ω . In detail, at each instant of time, the displacement from equilibrium is a sinusoidal function of x with wavelength λ , and at any given value of x the displacement is a sinusoidal function of time with angular frequency ω .

When you check that the standing wave (6.4) does indeed satisfy the wave equation, you will immediately see we can add a constant to the argument of either sine or cosine function and we still will have a solution.* By adding a constant to ωt , say,

$$\omega t \rightarrow \omega t + c,$$

we are, in effect, shifting the origin of time (*exercise*). This has the effect of shifting the initial conditions to an earlier time if $c > 0$ (*exercise*). By adding a constant to $2\pi x/\lambda$ we are making a rigid displacement of the wave in the negative x directions. In particular, we are shifting the locations of the *nodes*, *i.e.*, the locations of the points of zero displacement. Of course, we are also shifting the locations of the anti-nodes, *i.e.*, the locations of maximum displacement. You can also check that we can interchange the role of sine and cosine in the above solution, or we can have both functions as sines or both as cosines (*exercises*). And, because the wave equation is linear and homogeneous, any linear combination of these possibilities will solve the wave equation.

A particularly interesting linear combination is of the form

$$\begin{aligned} q(x, t) &= A \cos\left(\frac{2\pi x}{\lambda}\right) \cos(\omega t) + A \sin\left(\frac{2\pi x}{\lambda}\right) \sin(\omega t) \\ &= A \cos\left[\frac{2\pi}{\lambda}(x - vt)\right]. \end{aligned} \tag{6.5}$$

This is an example of a *traveling wave*. If you take a photograph of the system at $t = 0$, you will see a nice sinusoidal displacement with maximum displacement at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots$, and zero displacement at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots$. At some later time t a photograph will reveal the same sinusoidal displacement pattern, but shifted to the right by an amount vt (*exercise*). Thus the constant v in the wave equation (5.11) is the speed of the traveling wave.

A complex version of the traveling wave will be handy later. It takes the form

$$q(x, t) = Ae^{i(kx \pm \omega t)}, \tag{6.6}$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = |k|v$. You can easily see that the real and imaginary parts of q as defined in (6.6) will satisfy the wave equation, as does the complex conjugate of q .

* For now we are imposing no boundary or initial conditions. We are simply studying solutions of the differential equation.

7. General Solution of the One-Dimensional Wave Equation.

We will now find the “general solution” to the one-dimensional wave equation (5.11). What this means is that we will find a formula involving some “data” — some arbitrary functions — which provides every possible solution to the wave equation.* We can find the general solution of the (one-dimensional) wave equation as follows.

Make a change of variables $t = t(s, u)$, $x = x(s, u)$:

$$\begin{aligned} t &= \frac{(u - s)}{2v}, \\ x &= \frac{(u + s)}{2}, \end{aligned} \tag{7.1}$$

with inverse $u = u(x, t)$, $s = s(x, t)$:

$$\begin{aligned} u &= x + vt, \\ s &= x - vt. \end{aligned} \tag{7.2}$$

(Note: this change of variables is reminiscent of the change of variables used for our problem involving two coupled oscillators. This is purely coincidence.) The function $q(x, t)$ can now be viewed as a new function $\tilde{q}(u, s)$, which is obtained by substitution:

$$\tilde{q}(u, s) = q(x(s, u), t(s, u)). \tag{7.3}$$

Likewise, given $\tilde{q}(u, s)$ we can reconstruct $q(x, t)$ via

$$q(x, t) = \tilde{q}(u(x, t), s(x, t)). \tag{7.4}$$

For example, if $q(x, t) = xt$, then

$$\tilde{q}(u, s) = \left(\frac{(u + s)}{2} \right) \left(\frac{(u - s)}{2v} \right) = \frac{1}{4v}(u^2 - s^2) \tag{7.5}$$

As an exercise you can take this $\tilde{q}(u, s)$ and reconstruct $q(x, t)$ via

$$q(x, t) = \tilde{q}(u(x, t), s(x, t)) = xt. \tag{7.6}$$

A note on notation:

Using different symbols q and \tilde{q} for the displacement function in different coordinates is not that common in physics texts and so might be viewed as a somewhat fanatical devotion to notational consistency. The reason for using different symbols is that q and \tilde{q} are in

* We will tacitly assume that all solutions are continuous and have continuous derivatives to sufficient order.

general different functions, mathematically speaking, as the example above demonstrates.* On the other hand, one can reasonably take the point of view that we have a single physical quantity — the displacement q — which assigns numbers to each location in space and instant of time, and we can express this same quantity in different coordinate systems. From this last point of view it is reasonable to use the same symbol (q) for the displacement whether expressed in (t, x) or (u, v) coordinates. Because this last point of view is quite common, we will eventually adopt it, *i.e.*, we will eventually use the symbol q to denote wave displacement in any coordinate system. But for the purposes of explaining the chain rule calculation below, it is more instructive and less confusing (if more cumbersome) to keep clear which function is which.

The derivatives of $q(x, t)$ are related to derivatives of $\tilde{q}(u, s)$ by the chain rule of calculus. So, for example, we have (*exercise*)

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{\partial \tilde{q}}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial \tilde{q}}{\partial s} \frac{\partial s}{\partial t} \\ &= v \left(\frac{\partial \tilde{q}}{\partial u} - \frac{\partial \tilde{q}}{\partial s} \right),\end{aligned}\tag{7.7}$$

and

$$\frac{\partial^2 q}{\partial t^2} = v^2 \left(\frac{\partial^2 \tilde{q}}{\partial u^2} - 2 \frac{\partial^2 \tilde{q}}{\partial u \partial s} + \frac{\partial^2 \tilde{q}}{\partial s^2} \right),\tag{7.8}$$

where we used (7.2) and its consequence:

$$\begin{aligned}\frac{\partial u}{\partial t} &= v \\ \frac{\partial s}{\partial t} &= -v,\end{aligned}$$

Similarly, we have

$$\frac{\partial^2 q}{\partial x^2} = \left(\frac{\partial^2 \tilde{q}}{\partial u^2} + 2 \frac{\partial^2 \tilde{q}}{\partial u \partial s} + \frac{\partial^2 \tilde{q}}{\partial s^2} \right).\tag{7.9}$$

It is important to keep in mind when manipulating these chain rule formulas that equality between functions, *e.g.*, as in (7.7), holds provided we express the functions in terms of the *same* coordinates using the relations (7.1)–(7.2).

Take the results (7.7)–(7.9) and substitute them into the wave equation (5.11) to get the corresponding equation for $\tilde{q}(s, u)$. You will find that the wave equation (5.11) then takes the very simple form (*exercise*)

$$\frac{\partial^2 \tilde{q}}{\partial u \partial s} = 0.\tag{7.10}$$

* For a more blatant example, consider the familiar exponential function $\exp(u)$. Under the change of variables $u = \log(\sin(v))$ we get $\exp(u(v)) = \sin(v)$. We certainly would not want to retain the original notation (\exp) in this case!

This equation for $\tilde{q}(u, s)$ is completely equivalent to the original wave equation (5.11) for $q(x, t)$. What this means is that if you have a solution $q(x, t)$ to (5.11) then it defines, via the change of independent variables $(x, t) \rightarrow (u, s)$, a solution to (7.10). Conversely, if you have a solution to (7.10) then it defines a solution (via the inverse coordinate transformation) to (5.11).

We can easily solve equation (7.10). It says that $\frac{\partial \tilde{q}}{\partial u}$ is independent of s , *i.e.*,

$$\frac{\partial \tilde{q}}{\partial u} = h(u), \quad (7.11)$$

where $h(u)$ is any function of u . You can view this result as coming from integrating an equation that says the derivative with respect to s of $\frac{\partial \tilde{q}}{\partial u}$ is zero, in which case $h(u)$ is the “integration constant”. Equation (7.11) can be viewed as a first-order differential equation for \tilde{q} with a given function $h(u)$ and is easily integrated. Hold s fixed and integrate both sides of the equation with respect to u to find

$$\tilde{q}(s, u) = \int h(u) du + g(s).$$

Here $g(s)$ is an arbitrary function. The integral of $h(u)$ is just some other arbitrary function of u ; call it $f(u)$. The solution to the wave equation in terms of $\tilde{q}(s, u)$ is thus of the form:

$$\tilde{q}(s, u) = f(u) + g(s), \quad (7.12)$$

where f and g are any functions of one variable. You can easily check that this form for $\tilde{q}(s, u)$ does indeed solve the wave equation expressed in the form (7.10) (*exercise*). We can now go back to our original time and space coordinates to see that (*exercise*)

$$q(x, t) = f(x + vt) + g(x - vt) \quad (7.13)$$

is the solution to the original form of the wave equation.

Because of the equivalence of (5.11) and (7.10) (via the coordinate transformation), and since we have found a formula for all solutions to (7.10), it follows that (7.13) is the general solution to (5.11). We have thus completely solved (or “integrated”) the wave equation in one spatial dimension.

So, to solve the wave equation we only need pick a couple of functions of one variable, call them $f(z)$ and $g(z)$. Aside from requiring them to be suitably differentiable (so we can plug them into the wave equation!) they can be chosen to be any functions you like. We then set $z = x + vt$ in $f(z)$ and $z = x - vt$ in $g(z)$ and add the results together to get a solution to the wave equation. Note that we can always add a constant to f and subtract that same constant from g without changing the form of the solution. Thus f and g , while convenient for specifying a solution, are slightly redundant.

Let us go back and see how our elementary solutions in §6 fit in with our general form for the solution (7.13). The traveling wave solution (6.5) is easy to check; it is a solution in which (*exercise*)

$$f(z) = 0, \quad g(z) = A \cos\left(\frac{2\pi}{\lambda}z\right). \quad (7.14)$$

We see that g leads to “right-moving” solutions, *i.e.*, the cosine wave profile $g(z)$ moves toward increasing x values with speed v (*exercise*). Likewise, f leads to “left-moving” solutions. The standing wave solution (6.4) is a superposition of a left-moving and right-moving sinusoidal traveling wave solution to (5.11). To obtain this solution we set

$$f(z) = \frac{A}{2} \sin\left(\frac{2\pi}{\lambda}z\right) = g(z). \quad (7.15)$$

Evidently, given (7.13), every solution to the one-dimensional wave equation can be viewed as a superposition of a left moving and right moving wave profile. These profiles are determined by the choice of the functions f and g . Each of these component wave profiles move to the left and right, respectively, without changing their shape. The actual wave of interest is, of course, obtained by superposing the displacements defined by the left-moving and right-moving wave profiles.

We have obtained all solutions to the wave equation in one dimension, but not all of these solutions need be appropriate for a given physical situation. This is because one will typically have to impose boundary conditions. Normally, these boundary conditions will involve fixing the value of $q(x, t)$ at the endpoints of the allowed range of x , and this implies restrictions on the functions f and g appearing in (7.13). For fixed endpoints,

$$q(0, t) = 0 = q(x, t), \quad (7.16)$$

we have for all z (*exercise*)

$$f(z) + g(-z) = 0, \quad g(z) = g(z + 2L). \quad (7.17)$$

For periodic boundary conditions,

$$q(0, t) = q(L, t) \quad (7.18)$$

we have (*exercise*)

$$f(z) = f(z + L), \quad g(z) = g(z + L). \quad (7.19)$$

7.1 The Initial Value Formulation

There are evidently quite a few solutions to the wave equation in one dimension: roughly speaking, there are as many solutions as there are pairs of functions of one variable!

Why do we get so many solutions? One way to answer this question is to recall the oscillator model from which the wave equation was derived. Physically, we expect that the motion for N particles is determined by the initial positions and velocities of the particles. Consequently, a viable mathematical description of the particles' motion must be such that the solutions are uniquely determined once we have specified these initial conditions. This means that there must be $2N$ free parameters in the general solution which can be adjusted to meet the $2N$ initial conditions. The number N is the number of “degrees of freedom” in the one-dimensional chain of oscillators. In the continuum limit of our chain of oscillators we are taking a limit as $N \rightarrow \infty$; one sometimes says that the continuum description has an infinite number of degrees of freedom – essentially one degree of freedom for each point on the (one-dimensional) medium of vibration. In the continuum limit we naturally expect that the initial conditions will be the initial displacement at each x , denoted by $q(x, t = 0)$, and the initial rate of displacement at each x , denoted by $\frac{\partial q}{\partial t}(x, t = 0)$. Further, we expect to be able to choose these *initial data* any way we like*, and that the solution to the wave equation will be uniquely determined in terms of these data. This is indeed the case, which we shall demonstrate below. The reason this works is because the general solution depends upon two functions of one variable, and these functions can be chosen/determined by the initial data.

The above argument concerning initial data for the wave equation can be made more explicit and precise as follows. Let us call the initial displacement profile $a(x)$, i.e.,

$$q(x, t = 0) = a(x) \tag{7.20}$$

and call the initial velocity profile $b(x)$ †:

$$\frac{\partial q}{\partial t}(x, t = 0) = b(x). \tag{7.21}$$

We demand that our solution (7.13) to the wave equation matches this initial data at $t = 0$. This means we must choose f and g such that (*exercise*)

$$f(x) + g(x) = a(x) \tag{7.22}$$

and

$$v[f'(x) - g'(x)] = b(x), \tag{7.23}$$

* Up to suitable requirements on the continuity of the initial data functions and their derivatives.

† Here

$$\frac{\partial q}{\partial t}(x, t = 0) \equiv \left(\frac{\partial q}{\partial t} \right)_{t=0}.$$

where the prime indicates differentiation with respect to the argument (x) of the function, that is, $f'(w) = \frac{df}{dw}$.

We can solve (7.23) by integrating:

$$f(x) - g(x) = \frac{1}{v} \int_{x_0}^x b(x') dx', \quad (7.24)$$

where x_0 is any constant. We see that, up to this constant, f and g are uniquely determined by (7.24) and (7.22) (*exercise*):

$$\begin{aligned} f(x) &= \frac{1}{2} \left[a(x) + \frac{1}{v} \int_{x_0}^x b(x') dx' \right] \\ g(x) &= \frac{1}{2} \left[a(x) - \frac{1}{v} \int_{x_0}^x b(x') dx' \right]. \end{aligned} \quad (7.25)$$

The constant x_0 is not determined by the initial data, but x_0 does not contribute to $q(x, t)$ — see the formula below. As we pointed out above, we can always add a constant to f and subtract that constant from g without altering the solution to the wave equation; this redundancy is equivalent to the arbitrariness of x_0 (*exercise*). We have now obtained the general solution to the wave equation expressed in terms of any initial conditions $a(x)$ and $b(x)$. It takes the form:

$$q(x, t) = \frac{1}{2} \left[a(x + vt) + a(x - vt) + \frac{1}{v} \int_{x-vt}^{x+vt} b(x') dx' \right]. \quad (7.26)$$

Notice how the two integrals featuring in the solution for f and g in (7.25) have been combined. You should definitely make sure that you can obtain (7.26) from our previous formulas. You can check that this $q(x, t)$ given in (7.26) does indeed produce the desired initial data at $t = 0$. That this function solves the wave equation is easily seen since this function is of the form (7.13) (*exercise*). This formula is known as d'Alembert's formula for the solution to the wave equation in one spatial dimension.

It is perhaps worth noting that d'Alembert's formula (7.26) proves *existence and uniqueness* of solutions to the wave equation with prescribed initial conditions $(q(x, 0), \frac{\partial q(x, 0)}{\partial t})$. Existence is clear: we exhibited a formula for the general solution, namely, (7.26). Uniqueness is also clear: there is only one solution defined by the initial data. Note in particular that the arbitrary constant x_0 dropped out of the formula for q . If x_0 had survived in (7.26), then the solution would not have been uniquely determined by the initial data.

To summarize, we have shown that the solutions to the one-dimensional wave equation are uniquely determined by the choice of initial data — displacement and displacement velocity at each x — and that we can choose the initial data any way we like. These qualitative features will generalize to higher dimensions, but the explicit form of the general solution we have found is applicable only in one spatial dimension.

7.2 Gaussian Wave Packet

Let us look at a simple example of the initial value problem. Suppose we want to find a wave which has for its initial displacement

$$q(x, 0) \equiv a(x) = A \exp\left(-\frac{x^2}{a_0^2}\right). \quad (7.27)$$

This function is called a *Gaussian*. As an exercise you can show that it is peaked at $x = 0$, where it has its maximum value $q = A$, and falls to $1/e \approx 0.37$ of its maximum at $x = \pm a_0$. We thus say that the *height* of the Gaussian is A , and the *width* of the Gaussian is $2a_0$. So far we have given the initial wave displacement profile; to completely specify the wave we should give its initial velocity as well. Let us suppose that

$$\frac{\partial q(x, 0)}{\partial t} \equiv b(x) = 0. \quad (7.28)$$

This mathematical situation could be a model for a guitar string under tension which has been “plucked” into a Gaussian of height A and width $2a_0$ at $t = 0$, after which the string is released from rest. (Of course, when you pluck a guitar string the initial shape of the string is never really a Gaussian. But you get the idea.) Of course, we are neglecting the boundary condition that the string displacement is zero at the bridges of the guitar. Our simple model is, however, reasonably viable provided the width of the Gaussian wave is much smaller than the length of the string, and provided we only consider wave propagation for times short enough so that reflective effects are not relevant. The solution to the wave equation is easily obtained from our general formula (7.26); we have (exercise)

$$q(x, t) = \frac{A}{2} \left[\exp\left\{-\frac{(x + vt)^2}{a_0^2}\right\} + \exp\left\{-\frac{(x - vt)^2}{a_0^2}\right\} \right]. \quad (7.29)$$

At each t , that is, if we take a photograph of the resulting wave, we find a superposition of Gaussian pulses, each of width $2a_0$ and height $A/2$ centered at $x = \pm vt$. In effect, the initial Gaussian pulse “splits” into two similar Gaussians with half the original height, which move off in opposite directions. Of course, for a real guitar string the Gaussian waves reflect repeatedly from the ends and set up a standing wave.

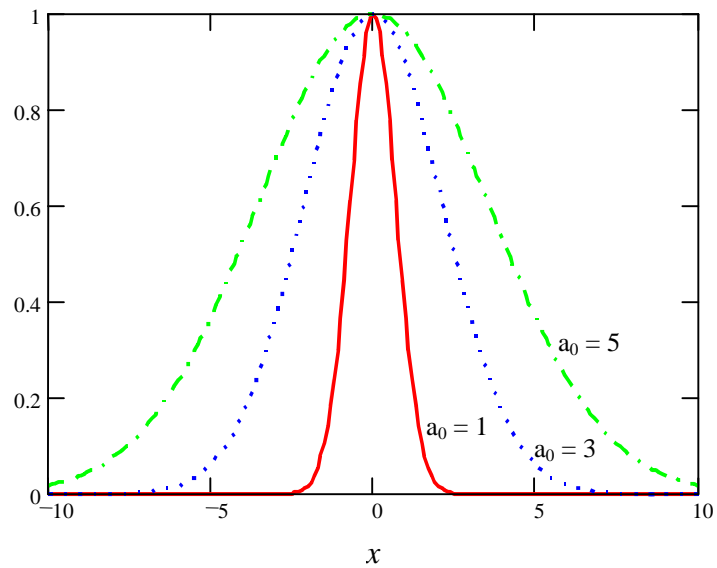


Figure 10. Gaussian function $\exp\left(-(x/a_0)^2\right)$ for three different values of the width parameter a_0 .

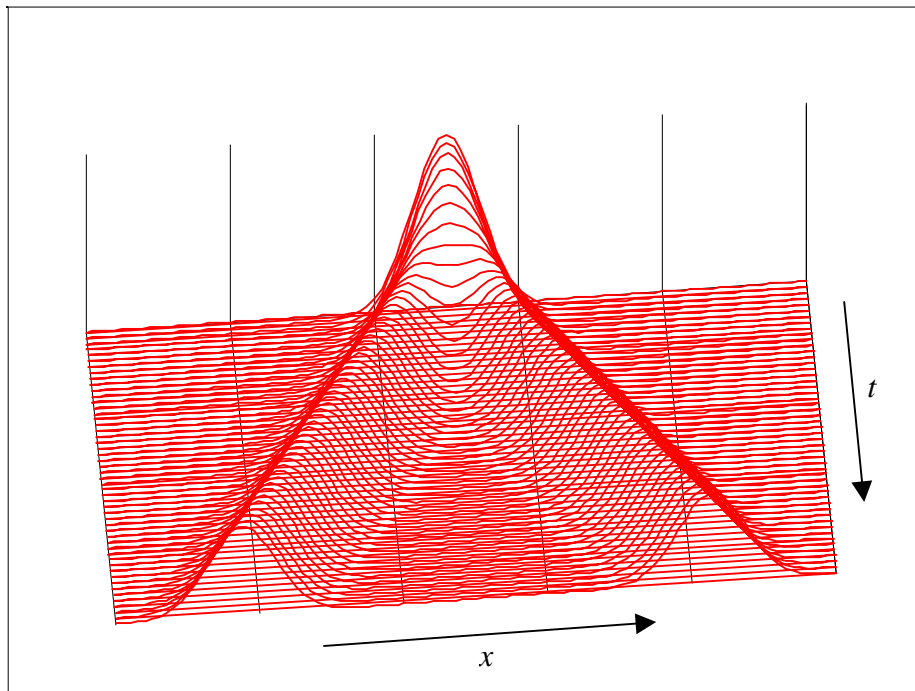


Figure 11. Solution to wave equation for initial Gaussian pulse displacement [$a(x)$] and zero initial velocity [$b(x) = 0$].

7.3 Linearity and superposition

Just as the oscillator equation(s) were linear and homogeneous, so is the wave equation. What this means is the following. If $q_1(x, t)$ and $q_2(x, t)$ are solutions of the wave equation, then

$$q_3(x, t) = c_1 q_1(x, t) + c_2 q_2(x, t)$$

is also a solution, where c_1 and c_2 are any two constants. We say that q_3 is obtained by a *linear superposition* of q_1 and q_2 . Because solutions to the wave equation are completely determined by their initial data it is natural to ask how the initial data for the superposition are related to the data for q_1 and q_2 . As an exercise you can check that the initial data for q_3 are obtained by taking the same superposition of initial data for q_1 and q_2 .

It is worth pointing out that this “superposition property” is a signal that the objects under consideration (solutions to the one-dimensional wave equation) form a *vector space* (see Appendix B for a review of vector spaces). Recall that a vector space is (i) a set of objects (the vectors), (ii) a set of “scalars” (real or complex numbers) which can be used via a commuting “scalar multiplication” to make new vectors from old vectors, (iii) a rule for addition of the vectors that is closed, commutative, associative, and distributive with respect to the scalar multiplication, and (iv) a zero vector (additive identity). In the present example, let us take the set to be all functions $q(x, t)$ which solve the wave equation. If $q(x, t)$ is a solution, then so is any constant multiple of $q(x, t)$, consequently scalar multiplication can be defined by simply multiplying $q(x, t)$ by numbers. As we have noted, the sum of any two solutions is another solution, so the rule for addition is the usual pointwise addition of functions. Finally, the function $q(x, t) = 0$ is a solution, *i.e.*, a member of the set, and plays the role of the zero vector. All of this mathematical finery may seem like overkill but, as you know, vector spaces have many nice properties (existence of a basis, *etc.*) and in more sophisticated applications it is good to know when the objects under consideration form a vector space.

Exercise: Show that the fixed endpoint and periodic boundary conditions can be imposed without destroying the superposition (vector space) property.

PROBLEM SET 3

Problem 3.1

Verify that

$$q(x, t) = A \cos\left(\frac{2\pi}{\lambda}x\right) \cos(\omega t) + B \sin\left(\frac{2\pi}{\lambda}x\right) \sin(\omega t)$$

solves the 1-d wave equation. Here A and B are constants, and $\omega = \frac{2\pi v}{\lambda}$. If $A = B$, show that this solution can be written in the form

$$q(x, t) = A \cos\left[\frac{2\pi}{\lambda}(x - vt)\right].$$

Problem 3.2

Let $q(x, t) = x + te^x$. Define new independent variables: $u = x + vt$, $s = x - vt$ and find the form of $\tilde{q}(u, s) = q(x(u, s), t(u, s))$ in the new variables. Compute $\frac{\partial q}{\partial t}$ and $\frac{\partial q}{\partial x}$ using (a) direct calculation and (b) the chain rule via $\tilde{q}(u, s)$ and its u, s derivatives. Show that you get the same result.

Problem 3.3

Verify by direct computation using the chain rule that $q(x, t) = f(x + vt) + g(x - vt)$ solves the one-dimensional wave equation.

Problem 3.4

Find the solution to the wave equation corresponding to the following initial data:

- (a) $q(x, 0) = 0$, $\frac{\partial q(x, 0)}{\partial t} = 0$,
- (b) $q(x, 0) = ax$, $\frac{\partial q(x, 0)}{\partial t} = 0$, $a = \text{constant}$,
- (c) $q(x, 0) = 0$, $\frac{\partial q(x, 0)}{\partial t} = bx$, $b = \text{constant}$.

Problem 3.5

In (7.26) we present the general form of the solution to the wave equation in terms of arbitrary initial data. Verify directly that (a) this is a solution to the wave equation, and (b) it matches the initial data as advertised.

Problem 3.6

Determine the functions f and g in (7.13) associated with the standing wave solution

$$q(x, t) = A \sin kx \cos kvt.$$

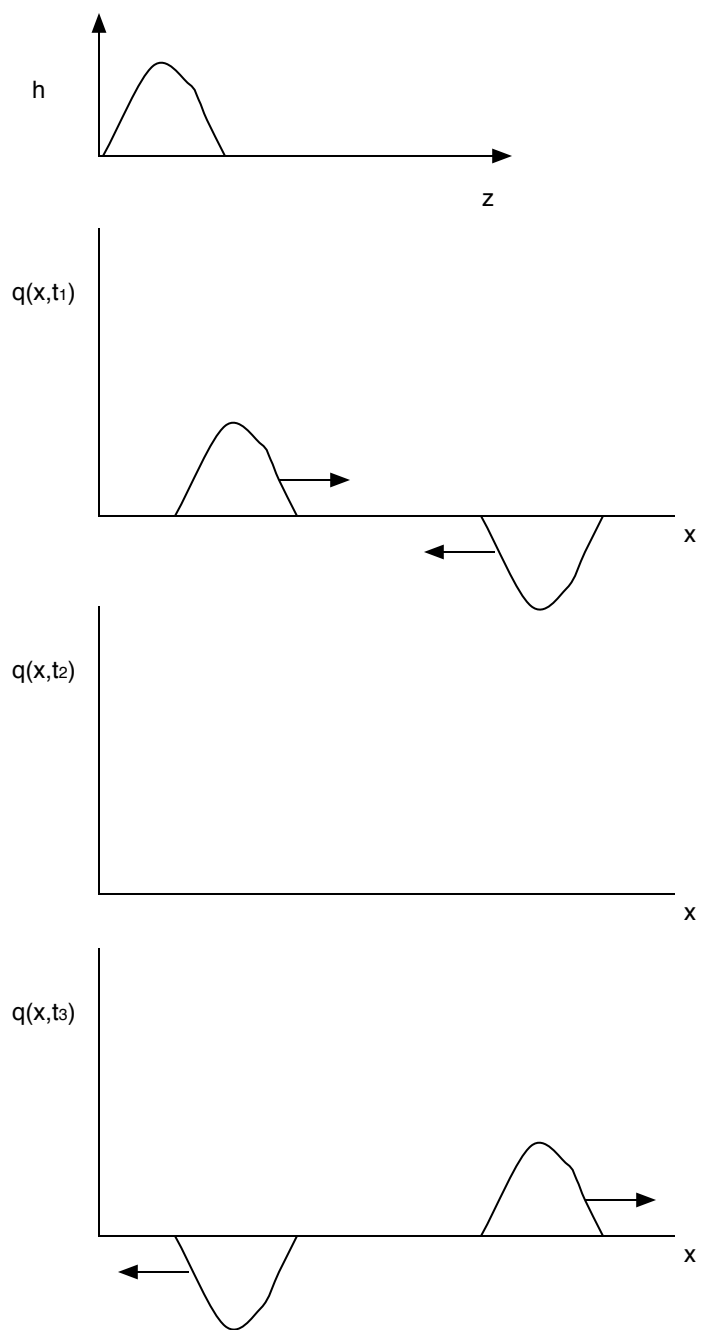
Determine all values of A and k such that this solution satisfies (i) the boundary conditions (7.17), and (ii) the boundary conditions (7.19).

Problem 3.7

Suppose $h(z)$ is a given smooth function such that

$$h(z) = 0 \quad z \leq 0, \text{ or } z \geq 1.$$

You can think of $h(z)$ as defining a “pulse” of finite length. Build a solution to the wave equation which corresponds to two such pulses, one of them inverted, which start out moving toward each other, which then collide and cancel each other out, and then propagate off to arbitrarily large values of $|x|$. (See the following figure.)



8. Fourier analysis.

We now would like to show that one can build up the general solution of the wave equation by superimposing certain elementary solutions. Indeed, the elementary solutions being referred to are those discussed in §6. These elementary solutions will form a very convenient “basis” for the vector space of solutions to the wave equation, just as the normal modes provided a basis for the space of solutions in the case of coupled oscillators. Indeed, as we shall see, the elementary solutions *are* the normal modes for wave propagation. The principal tools needed to understand this are provided by the methods of *Fourier analysis*, which is very useful in analyzing waves in any number of spatial dimensions. To begin, we will take a somewhat superficial tour of the key results of Fourier analysis. Then we’ll see how to use these results to better understand the solutions to the wave equation.

8.1 Fourier Series

Fourier series are a way of representing functions on an interval as a sum of sinusoidal functions, much in the same way as a vector in a vector space can be expanded in a basis. Although it is possible to proceed with more generality, let us restrict our attention to functions f on the interval $x \in [0, L]$ such that

$$f(0) = f(L) = 0. \quad (8.1)$$

(As you might guess, this discussion will be relevant when considering fixed endpoint boundary conditions of the form (5.12)). The main result we need is that, with appropriate technical assumptions about f ,[†] it is possible to express f as

$$f(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi x}{L}\right). \quad (8.2)$$

This is usually called the *Fourier series* of f . (The factor in front of the sum is for later convenience.) Note that each term in the series involves the sine function $\sin\left(\frac{n\pi x}{L}\right)$, which vanishes at $x = 0, L$. This guarantees that the conditions (8.1) are satisfied. Equation (8.2) is to be interpreted as follows. There exists an (infinite) sequence of numbers h_n , $n = 1, 2, 3, \dots$, such that the sequence of partial sums,

$$F_N(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^N h_n \sin\left(\frac{n\pi x}{L}\right) \quad (8.3)$$

[†] The following results only require f to be square (Lebesgue) integrable:

$$\int_0^L dx f^2(x) < \infty.$$

converges to f ,

$$\lim_{N \rightarrow \infty} F_N \longrightarrow f. \quad (8.4)$$

Here “convergence” means that

$$\lim_{N \rightarrow \infty} \int_0^L dx (F_N - f)^2 = 0. \quad (8.5)$$

The constants h_m , $m = 1, 2, 3, \dots$, are determined by the function f via the integrals

$$h_m = \sqrt{\frac{2}{L}} \int_0^L dx f(x) \sin\left(\frac{m\pi x}{L}\right). \quad (8.6)$$

Indeed, as a nice exercise you can verify the identity*

$$\frac{2}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \delta_{nm}. \quad (8.7)$$

In case you haven’t encountered it before, δ_{nm} is the *Kronecker delta*, defined by

$$\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}. \quad (8.8)$$

The identity (8.7) formally produces (8.6) once you integrate both sides of the equation (8.2) against the sine function and interchange the order of integration and summation:

$$\begin{aligned} \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) f(x) &= \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} h_n \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \delta_{nm} h_n \\ &= h_m. \end{aligned} \quad (8.9)$$

(It is not always legitimate to interchange the integration and summation as we do here, but it can be shown that the formulas (8.6) are correct nonetheless.) Evidently, the sequence of numbers h_m , $m = 1, 2, 3, \dots$, contain the same information as the function f itself.† The

* If you are a mathematics fan you can try to prove this result directly. But it’s just as instructive — perhaps more so — for you to check this result using standard resources, *e.g.*, an integral table or using algebraic computing software.

† In this context it is worth noting that the square-integrability of f is equivalent to the square-summability of h_n :

$$\sum_{n=1}^{\infty} h_n^2 < \infty.$$

mapping (8.6) from a function f on an interval to the sequence $\{h_n\}$ is called the *Fourier transform of f* .

This is a good place to work a simple example. Consider a square pulse

$$\eta(x) = \begin{cases} 1 & \text{if } a < x < b, \\ 0 & \text{otherwise,} \end{cases} \quad (8.10)$$

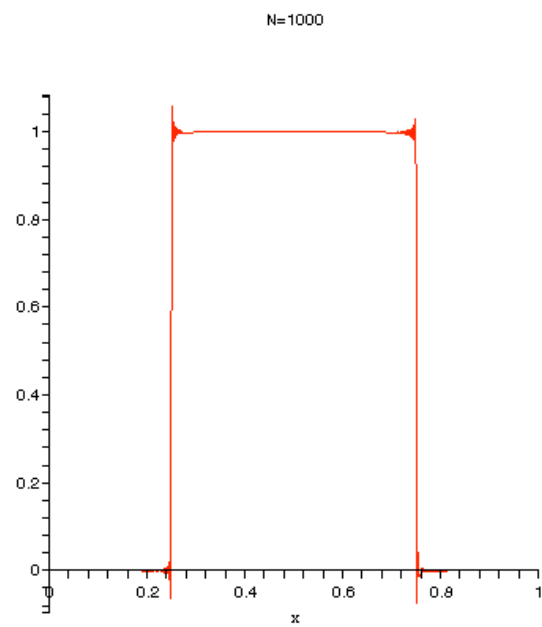
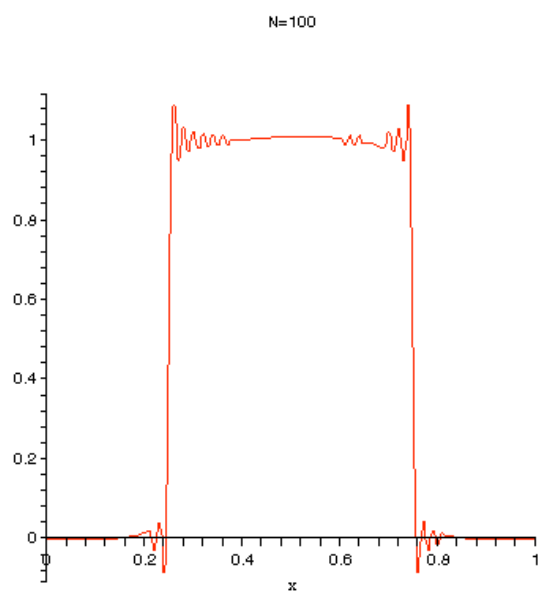
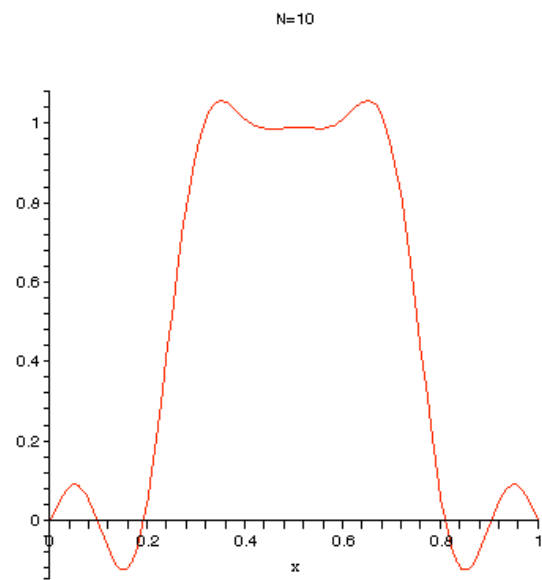
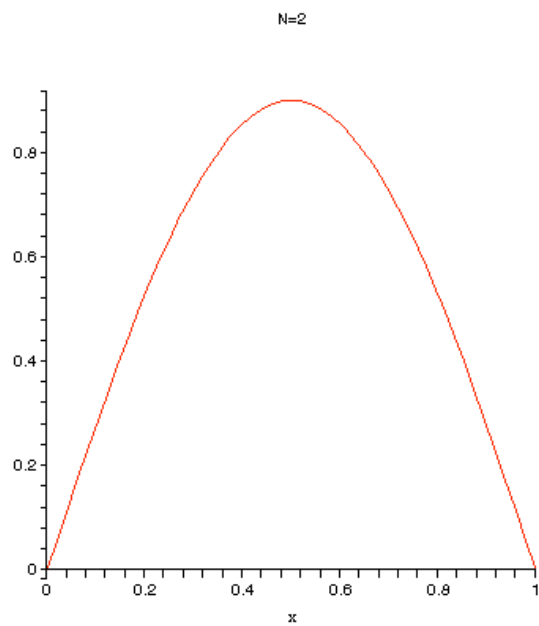
where $0 < a, b < L$. This is a great example because, while the graph of this function looks completely unlike a sinusoidal function, we will see that this function “is” a superposition of sinusoidal functions. To find the superposition, we compute the Fourier transform of η (*exercise*):

$$h_m = \frac{\sqrt{2L}}{m\pi} \left[\cos\left(\frac{m\pi a}{L}\right) - \cos\left(\frac{m\pi b}{L}\right) \right]. \quad (8.11)$$

The claim is, then, that the following series represents the square pulse:

$$\eta(x) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \left[\cos\left(\frac{m\pi a}{L}\right) - \cos\left(\frac{m\pi b}{L}\right) \right] \sin\left(\frac{m\pi x}{L}\right). \quad (8.12)$$

Admittedly, it is hard to tell what is the shape of the function defined in (8.12)! Let us investigate (8.12) graphically. The following sequence of figures depict the partial sums (8.3) for various values of N with $L = 1, a = 1/4, b = 3/4$.



Fourier series approximation to a square pulse. Here $L=1$, $a=1/4$, $b=3/4$.

As you can see, for large enough values of N the superposition of sine functions gives a good approximation to the square pulse. The only place where things don't look quite so nice is at the discontinuities of $\eta(x)$, *i.e.*, at the corners. The relatively larger pointwise error in the Fourier representation of the function at the corners is due to the fact that we are trying to approximate a discontinuous function using a linear combination of smooth functions. While the pointwise error at the corners is getting better with larger N it is *not* true that the pointwise error vanishes as $N \rightarrow \infty$. What is guaranteed to vanish is the limit (8.5), which allows for the function and its Fourier representation at isolated points, *i.e.*, the Fourier series will agree with the function “almost everywhere”. It can be shown that for continuous functions the Fourier series will converge pointwise to $f(x)$, however. For discontinuous functions one can say precisely how the pointwise values of the series behave, but we won't go into that here.

The reason we used sine functions in the Fourier series rather than the obvious alternative — cosine functions — stems from the fact that we chose our interval of interest to begin at $x = 0$ and we demanded that $f(0) = 0$. It is possible to relax these assumptions; here we just state a generalization of our previous results. For any (square-integrable) function $f(x)$ on the finite interval $(-L, L)$ with $f(-L) = f(L)$ there exist (square-summable) numbers α_n and β_n , $n = 0, 1, 2, 3, \dots$, defined as

$$\alpha_0 = \frac{1}{2} \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \quad \alpha_n = \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \cos\left(\frac{n\pi x}{L}\right), \quad n \neq 0 \quad (8.13)$$

$$\beta_n = \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \sin\left(\frac{n\pi x}{L}\right),$$

such that (almost everywhere)

$$f(x) = \sqrt{\frac{1}{L}} \sum_{n=0}^{\infty} \left[\alpha_n \cos\left(\frac{n\pi x}{L}\right) + \beta_n \sin\left(\frac{n\pi x}{L}\right) \right]. \quad (8.14)$$

Note that if we restrict attention to functions that vanish at the endpoints then we recover our previous results (*exercise*). An elegant way to rewrite this series is as follows,

$$f(x) = \sqrt{\frac{1}{2L}} \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}, \quad (8.15)$$

with

$$\gamma_n = \sqrt{\frac{1}{2L}} \int_{-L}^L dx e^{-inx} f(x). \quad (8.16)$$

Here we have combined the information contained in the two sequences of real numbers (α_n and β_n) into a single complex sequence (γ_n). You will work out the precise relationship

between α_n , β_n and γ_n in the Problems. However, one point might confuse you. The real sequences are in some sense only half as big as the complex sequence because $n \geq 0$ in the real sequence case while n is any integer (positive and negative) in the complex sequence case. The resolution of this discrepancy lies in the fact that, since $f(x)$ is real, the constants γ_n must satisfy

$$\gamma_n^* = \gamma_{-n}, \quad (8.17)$$

which you can check (*exercise*) is necessary and sufficient for

$$f^*(x) - f(x) = 0. \quad (8.18)$$

Let us conclude this section by indicating that Fourier analysis and Fourier series have a very natural vector space interpretation. To this end consider the real vector space of functions on the interval $x \in [0, L]$ satisfying the boundary conditions (8.1). We will equip this vector space with the scalar product*

$$(f, g) = \int_0^L dx f g. \quad (8.19)$$

You will prove in the Problems that this “function space” is a vector space and that (8.19) does indeed define a scalar product. The relation (8.2) can be interpreted as the statement that the functions (vectors)

$$e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad (8.20)$$

form a basis for the vector space, *i.e.*, for any vector f :

$$f = \sum_{n=1}^{\infty} h_n e_n. \quad (8.21)$$

Note that this vector space is thus infinite-dimensional. The relation (8.7) can be interpreted as simply the statement that the basis $\{e_n\}$ is orthonormal:

$$(e_n, e_m) = \delta_{nm}. \quad (8.22)$$

And the relation (8.6) is just the usual way of extracting the components of a vector in an orthonormal basis:

$$h_n = (e_n, f). \quad (8.23)$$

Thus, in this context Fourier analysis can be viewed simply as computing relative a particular basis for the vector space! As an *exercise* you can contemplate the vector space interpretation of the more general form of the Fourier series shown in (8.13)–(8.16).

* For simplicity, we do not bother to attach vector notation arrows to the functions/vectors in this discussion.

8.2 Fourier transforms and Fourier integrals

We have seen how to “Fourier analyze” functions on an interval. What happens if a function is non-vanishing on the whole x -axis? Let us have a look at some of the salient features of this situation. So, we now consider a function $f(x)$ of one variable $x \in (-\infty, \infty)$. With appropriate technical assumptions about the behavior of the functions involved (we will at least require that f vanishes as $|x| \rightarrow \infty$ — see below for a little more on this) it can be shown that there is another *complex* function $h(k)$, again called the *Fourier transform* of $f(x)$, such that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ikx} dk. \quad (8.24)$$

The Fourier transform of $f(x)$ is obtained via

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad (8.25)$$

The formulas (8.24) and (8.25) work by virtue of the *Fourier identity*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{ik(x-y)} dk dy. \quad (8.26)$$

Note in particular that the function $f(x) = 0$ corresponds to the Fourier transform $h(k) = 0$ (exercise).

You can view (8.24) and (8.25) as a continuum limit of (8.15), (8.16) obtained by taking $L \rightarrow \infty$. Indeed, for large enough L one can view $\frac{n\pi}{L} \leftrightarrow k$ as varying continuously with n to a better and better approximation since an integer step of n gives a very small change in $\frac{n\pi}{L}$. From this point of view, the Fourier sum over n in (8.15) becomes an integral over k in this limit. The restriction (8.17) becomes

$$h^*(k) = h(-k). \quad (8.27)$$

We will discuss the normalizing factor $1/\sqrt{2\pi}$ a little later.

Some technical details

As before, the restriction (8.27) is needed to keep $f(x)$ a real function. To see this, we demand

$$f(x) = f^*(x). \quad (8.28)$$

In terms of the Fourier representation (8.24) this gives (exercise)

$$\int_{-\infty}^{\infty} h(k) e^{ikx} dk = \int_{-\infty}^{\infty} h^*(k) e^{-ikx} dk. \quad (8.29)$$

In the integral on the right hand side, make the change of variables $k \rightarrow -k$. We then get (exercise)

$$\int_{-\infty}^{\infty} h(k) e^{ikx} dk = \int_{-\infty}^{\infty} h^*(-k) e^{-ikx} dk, \quad (8.30)$$

or

$$\int_{-\infty}^{\infty} [h(k) - h^*(-k)] e^{ikx} dk = 0. \quad (8.31)$$

This means that f is a real function if and only if

$$h(k) = h^*(-k). \quad (8.32)$$

One nice set of technical assumptions one can use to justify the continuous form of the Fourier transform is as follows. Suppose that $f(x)$ satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty, \quad (8.33)$$

which says that f is “square integrable”. Note that this implies that f vanishes as $x \rightarrow \pm\infty$, but is otherwise a fairly weak requirement. For example, f need not be continuous. It can be shown that this guarantees the existence of the Fourier transform h , and that furthermore h is also square-integrable. Indeed, it can be shown that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |h(k)|^2 dk, \quad (8.34)$$

which is known as *Plancherel's identity*. With this set up, the Fourier transform relation is the statement that $f(x)$ and its Fourier integral representation (8.24) will agree “almost everywhere”, which essentially means that they will agree except at isolated points. More precisely, with $h(k)$ defined by (8.25), if we define

$$E_A(x) := f(x) - \frac{1}{\sqrt{2\pi}} \int_{-A}^A h(k) e^{-ikx} dk \quad (8.35)$$

then it can be shown that

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} |E_A(x)|^2 dx = 0. \quad (8.36)$$

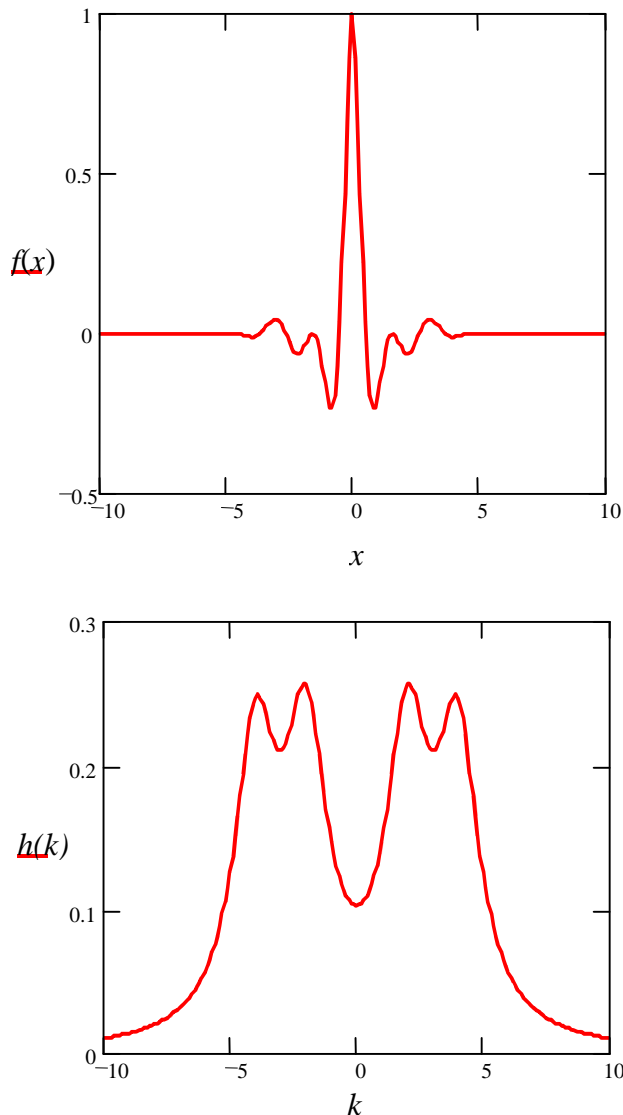


Figure 12. An example function $f(x) = \cos(x)\cos(3x)\exp(-|x|)$ and its Fourier transform $h(k)$.

It is not hard to see that all the preceding results will in fact extend to the case where $f(x)$ is a complex-valued function. Indeed, we could have phrased the discussion in terms of complex-valued functions from the start. One way to see this from the results shown above is to split f into its real and imaginary parts — which are given by real-valued functions — and Fourier analyze each of these parts separately. All of the preceding formulas still hold, except for (8.28)–(8.32), of course.

Once again, we can interpret the formal structure of the preceding discussion in terms of vector spaces. For novelty, let us briefly describe this in the case where we consider

complex-valued functions. So, think of the set of all (square-integrable) complex functions of $x \in (-\infty, \infty)$ as a complex vector space. Indeed, one can multiply complex numbers functions by complex numbers (scalars), add functions to make new functions, *etc.* Thus, in particular, each function is to be viewed as a vector. Using this analogy, we can think of the functions $e_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$ as a family of vectors labeled by k . Then, the Fourier expansion (8.24) can be viewed as an expansion of the vector f in the basis provided by e_k . The sum becomes an integral because of the continuous nature of k . From this heuristic point of view (which can be made quite rigorous), the Fourier transform $h(k)$ can be viewed as providing the components of f in the basis e_k .

8.2 The Delta Function

The Fourier identity (8.26) can be conveniently expressed in terms of another useful mathematical object known as the *delta function*, which was devised by the great physicist Paul Dirac. We introduce the delta function by rewriting equation (8.26) as

$$f(x) = \int_{-\infty}^{\infty} \delta(x-y)f(y) dy. \quad (8.37)$$

Evidently,

$$\delta(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk. \quad (8.38)$$

The “integral” appearing in (8.38) is just a formality since it does not really exist in the conventional sense of a Riemann integral (*exercise*). One should think instead of (8.38) as only being a shorthand expression, to be used inside an integral, such as appears in (8.37) or (8.26). Virtually everything one needs to know about the delta function can be deduced from the formula (8.37), which shows how a number (the number $f(x)$ for a given x) is being associated with a function f . For example, from (8.37) it is not hard to see (*exercises*) that

$$\int_{-\infty}^{\infty} \delta(x)f(x) dx = f(0). \quad (8.39)$$

Despite its name, the delta function is not really a function. Indeed, no ordinary function $\delta(x)$ could satisfy (8.39) for any $f(x)$. The delta “function” is an example of a mathematical object called a *distribution* or *generalized function*. A distribution is a linear operation which associates a number to a (nice enough type of) function. In a systematic development of the idea of a distribution one usually *defines* the delta function as any linear operation that satisfies (8.37) for appropriate functions $f(x)$. One then uses the Fourier identity to give an explicit realization (8.38) of such an operation for an appropriate class of functions.

A convenient mental picture of the delta function arises by revisiting our analogy of functions with vectors in a d -dimensional vector space and then viewing the δ -function

as a continuous version of the *Kronecker delta*, denoted δ_j^i , where $i, j = 1, 2, \dots, d$. The Kronecker delta is defined by

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \quad (8.40)$$

Fourier's identity is then a continuous version of the identity

$$v^i = \sum_{j=1}^d \delta_j^i v^j. \quad (8.41)$$

Continuing with this analogy, the formal definition (8.38) can be viewed as a continuous version of the orthonormality of the basis vectors

$$e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx},$$

where “vector” means complex-valued functions and k labels the basis vectors. The scalar product of two functions is defined as

$$(f, g) = \int_{-\infty}^{\infty} f^*(x) g(x) dx,$$

so that (*exercise*)

$$(e_k, e_l) = \delta(k - l).$$

(Now you can see why the normalization factor is $\frac{1}{\sqrt{2\pi}}$ for the continuous Fourier transform.)

Be careful not to take the analogy between the delta function and the Kronecker delta too literally! For example, it is *not* the case that $\delta(k - l)$ vanishes when $k \neq l$ and is unity when $k = l$. Indeed, since δ is not really a function, it makes no sense to speak of its values.

Some physics/engineering texts will assert that $\delta(x - y)$ is a function which vanishes when $x \neq y$ and is infinite when $x = y$. Of course, strictly speaking, this kind of definition is nonsense. But the idea being promoted here is that one can view the delta function in terms of a limit of well-defined functions. This limit is then used (inside an integral) to define formulas such as (8.37). For example, one can define

$$D_\epsilon(x - y) = \frac{1}{2\pi} \int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} e^{ik(x-y)} dk = \frac{1}{\pi} \frac{\sin(\frac{x-y}{\epsilon})}{x - y}, \quad (8.42)$$

so that

$$\delta(x - y) = \lim_{\epsilon \rightarrow 0} D_\epsilon(x - y). \quad (8.43)$$

You will check in the Problems that, as ϵ becomes small, D_ϵ is narrowly peaked around zero with an increasing maximum. In the limit as $\epsilon \rightarrow 0$ D_ϵ is no longer a function*, but the fundamental property (8.37) still holds in the limit:

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} D_\epsilon(x-y)f(y) dy = f(x). \quad (8.44)$$

8.4 Application to the Wave Equation

Let us now see how to use Fourier analysis to handle the wave equation. To begin, we consider the application of Fourier series to solve the wave equation. Let us consider a string of length L that is free to vibrate except that its ends are held fixed. We model this situation using a displacement $q(x, t)$ satisfying the wave equation along with the *boundary conditions*

$$q(0, t) = q(L, t) = 0. \quad (8.45)$$

This is the continuum limit of the chain of oscillators considered previously. A nicer form of the solution is obtained as follows. We note that at each time t the wave profile $q(x, t)$ is a function of $x \in [0, L]$ satisfying the boundary conditions (8.45). Thus we can express $q(x, t)$ as a Fourier series at each time t :

$$q(x, t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

The wave equation then implies that

$$\sum_{n=1}^{\infty} \left[q_n''(t) + \left(\frac{n\pi v}{L}\right)^2 q_n(t) \right] \sin\left(\frac{n\pi x}{L}\right) = 0,$$

so that

$$q_n''(t) + \left(\frac{vn\pi}{L}\right)^2 q_n(t) = 0.$$

This is just the harmonic oscillator equation! You can interpret this result as indicating the sinusoidal Fourier basis elements as defining a set of “normal modes”. We easily get

$$q_n(t) = \text{Re}(\gamma_n e^{i\omega_n t}),$$

where the coefficients γ_n are complex and

$$\omega_n = \frac{n\pi v}{L}. \quad (8.46)$$

* If it were, it would vanish away from zero and diverge at zero!

Thus we obtain

$$q(x, t) = \text{Re} \left(\sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \gamma_n \sin \left(\frac{n\pi x}{L} \right) e^{i\omega_n t} \right). \quad (8.47)$$

The coefficients γ_n are determined by the initial conditions. Let us suppose that the initial wave profile is of the form

$$q(x, 0) = a(x), \quad \frac{\partial q(x, 0)}{\partial t} = b(x), \quad (8.48)$$

where f and g are some given functions satisfying

$$a(0) = a(L) = 0 = b(0) = b(L). \quad (8.49)$$

We can write

$$a(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \alpha_n \sin \left(\frac{n\pi x}{L} \right), \quad (8.50)$$

$$b(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \beta_n \sin \left(\frac{n\pi x}{L} \right), \quad (8.51)$$

where

$$\alpha_n = \sqrt{\frac{2}{L}} \int_0^L a(x) \sin \left(\frac{n\pi x}{L} \right) dx, \quad (8.52)$$

$$\beta_n = \sqrt{\frac{2}{L}} \int_0^L b(x) \sin \left(\frac{n\pi x}{L} \right) dx. \quad (8.53)$$

It is straightforward to check that (i) $q(x, t)$ solves the wave equation and (ii) matches the initial conditions provided

$$\text{Re}(\gamma_n) = \alpha_n, \quad \text{Im}(\gamma_n) = -\frac{\beta_n}{\omega_n}. \quad (8.54)$$

As a nice exercise you should compare these Fourier series to their discrete counterparts which we found when studying the chain of oscillators.

As an example of the above results, consider an initial wave profile in which

$$\begin{aligned} a(x) &= \begin{cases} 1, & \text{if } L/4 < x < 3L/4; \\ 0, & \text{otherwise,} \end{cases} \\ b(x) &= 0, \end{aligned} \quad (8.55)$$

which is a square pulse, at rest, centered at $x = L/2$.^{*} We have (exercise)

$$\begin{aligned}\alpha_n &= \frac{\sqrt{2L}}{n\pi} \left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{2n\pi}{4}\right) \right) \\ &= \begin{cases} \frac{(-1)^{n+1}\sqrt{8L}}{n\pi} \sin\left(\frac{n\pi}{4}\right), & n \text{ odd} \\ 0 & n \text{ even} \end{cases}\end{aligned}\tag{8.56}$$

and

$$\beta_n = 0.\tag{8.57}$$

We then get (exercise)

$$q(x, t) = \sum_{n=1, \text{ odd}}^{\infty} (-1)^{n+1} \frac{4}{n\pi} \sin\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)\tag{8.58}$$

In the Problems you will be asked to interpret this solution.

Let us now apply results from Fourier analysis to solutions to the wave equation for all $x \in (-\infty, \infty)$, assuming that $q(x, t)$ is square-integrable for all time t . As before, we can Fourier analyze $q(x, t)$ with respect to the x variable and derive a harmonic oscillator type equation for the Fourier transform. This approach will be addressed in the Problems. Here we illustrate a different approach. We consider the function

$$f(u) = f(x + vt),\tag{8.59}$$

which appears in the general solution (7.13) to the one-dimensional wave equation (the other function, $g(s)$, is handled in an identical fashion). We now know that there is a function $h(k)$ such that we can write

$$f(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{iku} dk,\tag{8.60}$$

or, equivalently,

$$f(x + vt) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ik(x+vt)} dk.\tag{8.61}$$

This way of expressing f as an integral can be viewed as a way of writing f as a continuous superposition of the elementary solutions (6.6) with coefficients $h(k)$. Of course,

^{*} Strictly speaking, such initial conditions are beyond the scope of our original view of the wave equation since they correspond to discontinuous solutions — such functions can't be differentiated! You can view the current example as an idealized model of a realistic solution whose initial data are ever so slightly rounded at the corners. Both the general solution to the wave equation (3.10) as well as the Fourier series version (8.47) extend without difficulty to such discontinuous “solutions”, in effect, defining the wave equation for such solutions.

we usually think of a superposition as a sum with constant coefficients, but recall that an integral is really just a continuum limit of a sum, so we can think of the Fourier representation of $f(u)$ as just a continuous generalization of a superposition of elementary solutions. If you like, you can check that this form of $f(u)$ does indeed satisfy the wave equation by direct differentiation, but you already can see that the integral must satisfy the wave equation because it is a function of $x + vt$. A similar line of reasoning expresses $g(s)$ as

$$g(x - vt) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(k) e^{ik(x-vt)} dk. \quad (8.62)$$

Thus the general solution to the 1-dimensional wave equation, (7.13), can be characterized by $h(k)$ and $w(k)$, the Fourier transforms of f and g .

In our discussion we have not yet taken into account the fact that $q(x, t)$ must be real. This is achieved by choosing $f(u)$ and $g(s)$ to be real. As discussed earlier (see (8.32)) to do this we must require

$$h^*(k) = h(-k) \quad \text{and} \quad w^*(k) = w(-k). \quad (8.63)$$

To summarize, we can express the general solution of the wave equation as a superposition of elementary traveling wave solutions via

$$q(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(h(k) e^{ik(x+vt)} + w(k) e^{ik(x-vt)} \right) dk, \quad (8.64)$$

where $h^*(k) = h(-k)$ and $w^*(k) = w(-k)$. We see that to specify a solution we still must choose two functions of one variable ($h(k)$ and $w(k)$). Again, this is equivalent to choosing initial data.

8.5 The Dispersion Relation

In §4 we introduced the idea of a *dispersion relation*, which could be interpreted as a relation between wavelength and wave speed for sinusoidal waves. There is likewise a dispersion relation for solutions of the 1-d wave equation. The (complex form of the) sinusoidal solution of the wave equation (which we just saw in the Fourier expansion) was displayed in (6.6):

$$q(x, t) = A e^{i(kx \pm \omega t)}, \quad (8.65)$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = |k|v$. The speed of this wave is

$$\frac{\omega}{|k|} = v, \quad (8.66)$$

which is obviously independent of wavelength. The dispersion relation

$$\omega = |k|v \quad (8.67)$$

shows that in the superposition (8.64) all the component waves are moving at the same speed. This means that whatever shape the wave displacement has at one time it will retain for all time. If waves with different wavelengths moved at different speeds, the superposition would be, in effect, changing in time and the shape of the wave would change in time (spread out, in fact). Because this *doesn't* happen here, one says that there is no dispersion in the 1-d wave equation.

8.6 Example: Gaussian Wave Packet

As an example, let us return to the Gaussian wave profile displayed in (7.29). Recall that this wave arose by choosing a Gaussian displacement profile (7.27) at time $t = 0$ and a vanishing initial velocity profile (7.28). We have

$$\frac{A}{2} e^{-\frac{(x+vt)^2}{a^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ik(x+vt)} dk, \quad (8.68)$$

and

$$\frac{A}{2} e^{-\frac{(x-vt)^2}{a^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(k) e^{ik(x-vt)} dk. \quad (8.69)$$

To solve for $h(k)$ and $w(k)$ we use the Fourier transform formulas. Set $u = x + vt$. Then we have (exercise)

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A}{2} e^{-\frac{u^2}{a^2}} e^{-iku} du. \quad (8.70)$$

Note that the t dependence has dropped out! This is not too surprising since the formula for $h(k)$ should hold for all t , and we might as well choose $t = 0$ where we again get (8.70) (exercise).

You may not have encountered an integral like (8.70) before. It is an example of a *Gaussian integral*. Note that it is really a combination of the integral of $\exp(-u^2/a^2) \cos(ku)$ and $\exp(-u^2/a^2) \sin(ku)$. The integrals exist because the exponentials die off fast enough at large u so that the “area under the curve” is finite. There are several ways of performing such integrals, which you will probably see in detail in a more advanced course. Here we shall simply state the result. If you have appropriate computer software you should try these integrals out. In any case, it can be shown that

$$\int_{-\infty}^{\infty} e^{-\alpha^2 x^2} \cos(\beta x) dx = \frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right), \quad (8.71)$$

and

$$\int_{-\infty}^{\infty} e^{-\alpha^2 x^2} \sin(\beta x) dx = 0. \quad (8.72)$$

To use these results we identify $\alpha = 1/a$ and $\beta = k$; we then find

$$h(k) = \frac{1}{2\sqrt{2}} Aa \exp\left(-\frac{k^2 a^2}{4}\right). \quad (8.73)$$

Notice that this also a Gaussian, but in “ k -space”. Thus *the Fourier transform of a Gaussian is also a Gaussian*. A similar computation shows that $w(k) = h(k)$ (exercise). Note that both $h(k)$ and $w(k)$ are real functions and satisfy

$$h(-k) = h(k) = w(-k) = w(k), \quad (8.74)$$

as needed for a real solution (exercise).

It is worth noting that while both the initial wave profiles in x -space and k -space are Gaussians, their widths behave in an opposite manner. The width of the x -space Gaussian increases with a while the width of the corresponding k -space Gaussian decreases with a . This is in fact a general “rule of thumb” concerning Fourier transforms. The more “localized” are the non-zero values of a function, the more “de-localized” (or “spread out”) are the values of its Fourier transform. This holds whether or not the original function is in x -space or k -space. You may be familiar with a result of this type from quantum mechanics where it leads to one manifestation of the uncertainty principle.

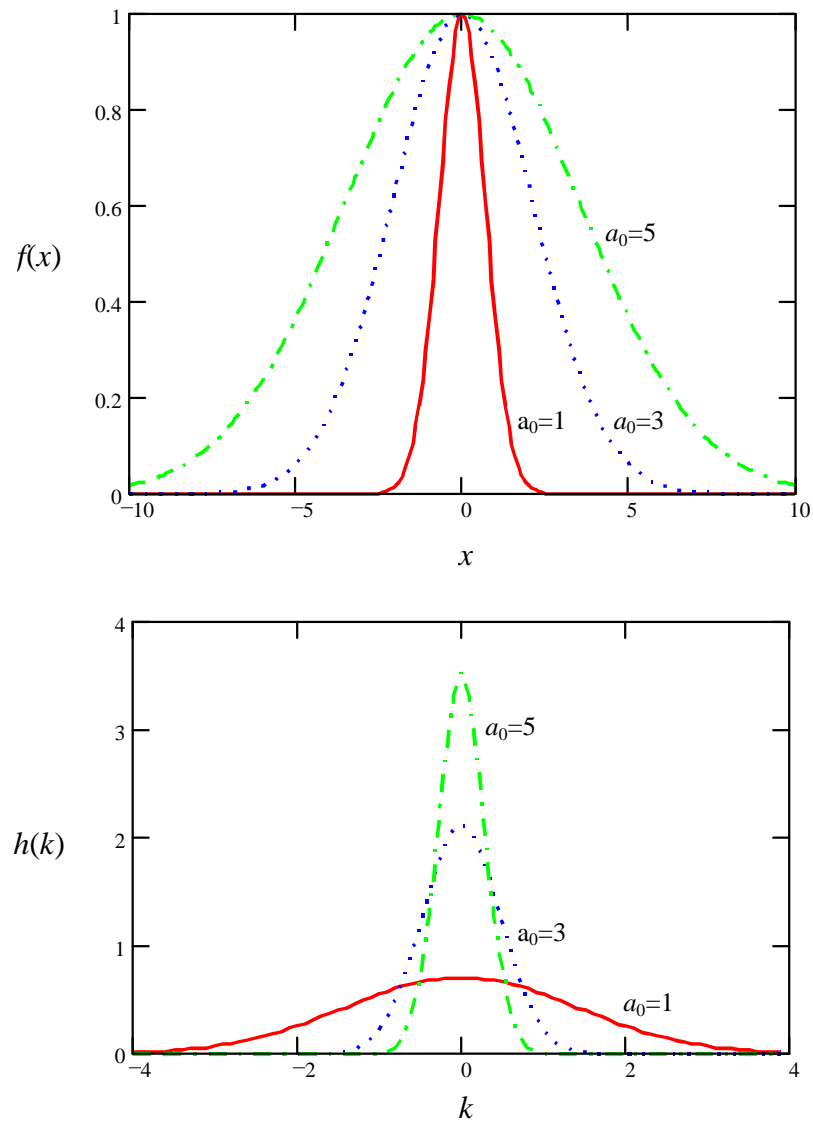


Figure 13. Three Gaussian functions $f(x)$ and their Fourier transforms $h(k)$, which are also Gaussian functions. Notice that as $f(x)$ becomes broader its Fourier transform becomes narrower.

Problem Set 4

Problem 4.1

Make a computer animation which depicts the solution (8.58) of the wave equation with the square pulse initial conditions. Describe the resulting motion.

Problem 4.2

In the text we solved the wave equation with initial data consisting of a Gaussian displacement with vanishing initial velocity profile. Find the solution to the wave equation with initial data

$$q(x, 0) \equiv a(x) = A \exp\left(-\frac{x^2}{a_0^2}\right)$$
$$\frac{\partial q(x, 0)}{\partial t} = -2\frac{AV}{a_0^2}x \exp\left(-\frac{x^2}{a_0^2}\right),$$

where V is a constant with units of speed. What is the physical meaning of this initial data set?

Problem 4.3

Consider a wave propagating along the whole x -axis. Write the wave displacement in terms of its Fourier transform at each time:

$$q(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k, t) e^{ikx} dk.$$

Substitute this expression for $q(x, t)$ into the wave equation and derive an equation for $a(k, t)$. Solve this equation for $a(k, t)$ and show that $q(x, t)$ can be put into the form (8.64).

Problem 4.4

Using a computer, graph the function D_ϵ in (8.42) and show that as ϵ gets smaller D_ϵ becomes a narrowly peaked function with ever-increasing maximum value.

Problem 4.5

Recall the Gaussian wave packet discussed in §8.5. Consider the limit in which the Gaussian width vanishes, $a \rightarrow 0$, and the Gaussian height blows up, $A \rightarrow \infty$, such that the product of the width and height approaches a non-zero constant, $Aa \rightarrow \text{const.} \neq 0$. Show that the solution to the wave equation becomes a delta function displacement at rest at $t = 0$ which then splits into two delta function pulses traveling in opposite directions. *Hint: Consider the limit of the Fourier transforms.*

Problem 4.6

Prove Plancherel's identity (8.34) using the properties of the delta function.

Problem 4.7

Compute the most basic Gaussian integral,

$$I(a) = \int_{-\infty}^{\infty} e^{-a^2 x^2} dx,$$

as follows. First, consider the square of the integral:

$$[I(a)]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{-a^2 y^2} dx dy.$$

Next, notice that this is simply the integral of a function over the whole x - y plane. Change integration variables to polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ (note: $dx dy = r dr d\theta$). You should now be able to easily compute the integral $[I(a)]^2$. Finally, note that $I(a)$ is positive and take the square root to determine the original integral.

Problem 4.8

In §8.5 we considered the Fourier representation of a wave with Gaussian initial displacement and vanishing initial velocity. We computed the Fourier transform and showed that $h(k) = w(k)$. Show that $h(k) = w(k)$ is always true (not just for a Gaussian) whenever we choose vanishing initial velocity profile.

Problem 4.9

In the text we computed the Fourier transform of a square pulse, viewing it as a function on the interval $[0, L]$ which vanishes at the endpoints. We can, of course, view

the square pulse as a (square-integrable) function on the whole x axis. From this point of view, compute the Fourier transform $h(k)$ of the square pulse:

$$f(x) = \begin{cases} 1, & \text{if } a < x < b; \\ 0, & \text{otherwise.} \end{cases}$$

Plot the real part, imaginary part and absolute value of $h(k)$

Problem 4.10

In (8.73) we gave a formula for the Fourier transform $h(k)$ of a Gaussian function $f(x) = e^{-x^2/a^2}$. Use the general formulas we obtained to take the Fourier transform $w(x)$ of that Fourier transform ($h(k)$). Show that you get back the original Gaussian function, *i.e.*, $w(x) = f(x)$.

Problem 4.11

Show that $\int_{-\infty}^{\infty} e^{-ax^2} \sin bx \, dx = 0$. (Hint: this one is really easy.)

Problem 4.12

Find the Fourier series form of the solution $q(x, t)$, $0 < x < L$, to the wave equation with $q(0, t) = q(L, t) = 0$ such that

$$q(x, 0) = x^2 - Lx, \quad \frac{\partial q(x, 0)}{\partial t} = 0.$$

Problem 4.13

Show that, for $a \neq 0$,

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

(*Hint*: this means you must show

$$\int_{-\infty}^{\infty} \delta(ax) f(x) \, dx = \frac{1}{|a|} f(0)$$

for any function $f(x)$.)

Problem 4.14

Verify that $q(x, t)$ defined by (8.47)–(8.54) does indeed satisfy the wave equation and match the initial conditions (8.48).

Problem 4.15

Verify that (7.15) leads to (6.4) as advertised.

Problem 4.16

A function f on an interval $[-L, L]$ is said to be *even* if $f(-x) = f(x)$ for all $x \in [-L, L]$. Similarly, a function is said to be *odd* if $f(-x) = -f(x)$. For even (odd) functions, show that the Fourier series representation (8.15) becomes a series involving only cosines (sines).

Problem 4.17

Show that for any two (real-valued, square-integrable) functions, f and g , on an interval (a, b) the following defines a scalar product on the vector space of such functions:

$$(f, g) = \int_a^b f(x)g(x) dx.$$

(See the Appendix for the definition of a scalar product.)

9. The Wave equation in 3 dimensions.

We now turn to the 3-dimensional version of the wave equation, which can be used to describe a variety of wavelike phenomena, *e.g.*, sound waves and electromagnetic waves. One could derive this version of the wave equation much as we did the one-dimensional version by generalizing our line of coupled oscillators to a 3-dimensional array of oscillators. For many purposes, *e.g.*, modeling propagation of sound, this provides a useful discrete model of a three dimensional solid. We won't be able to go into that here. The point is, though, that if we take the continuum limit as before we end up with the *3-dimensional wave equation* for the displacement $q(\mathbf{r}, t)$ of the oscillator-medium at the point labeled $\mathbf{r} = (x, y, z)$ at time t :

$$\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}. \quad (9.1)$$

The 3-dimensional wave equation is a linear, homogeneous partial differential equation with constant coefficients. Note that if we assume the wave displacement does not depend upon two of the three independent variables, *e.g.*, $q = q(x, t)$ we end up with the one-dimensional wave equation (*exercise*).

9.1 Gradient, Divergence and Laplacian

The right-hand side of (9.1) represents a very important differential operator, known as the *Laplacian*, so let us take a moment to discuss it. The Laplacian itself can be viewed as the composition of two other operators known as the gradient and the divergence, which we will also briefly discuss.

To begin, let $f(\mathbf{r}) = f(x, y, z)$ be a function of three variables. The *gradient* of f is a vector field, *i.e.*, a vector at each position \mathbf{r} , denoted ∇f , defined by

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}, \quad (9.2)$$

where \mathbf{i} , \mathbf{j} , \mathbf{k} are the orthogonal unit vector fields in the x , y , z directions, respectively.* What is the meaning of this vector field? Clearly at any given point, the x , y , z components of ∇f give the rate of change of f at that point in the x , y , z directions, respectively; the other variables being held fixed. More generally, at a given point, say, \mathbf{r}_0 , a unit vector $\hat{\mathbf{n}}$ defines the rate of change of f in the direction of $\hat{\mathbf{n}}$ via the *directional derivative* along $\hat{\mathbf{n}}$, defined by $\hat{\mathbf{n}} \cdot \nabla f(\mathbf{r}_0)$. The direction of the gradient of f at any given point is the direction

* Some texts denote the gradient operation by “grad”:

$$\nabla f \equiv \text{grad} f. \quad (9.3)$$

in which the function f has the greatest rate of change at that point, while the magnitude of the gradient (at the given point) is that rate of change. (Prove these last two facts as a nice exercise.) An important geometric feature of the gradient of a function is that ∇f is always perpendicular to the surfaces $f = \text{constant}$. This you will study in a homework problem. The gradient is an example of a linear differential operator.

The Laplacian is a combination of the gradient with another linear differential operator, called the *divergence*, which makes a function out of a vector field. Let

$$\mathbf{V} = V^x \mathbf{i} + V^y \mathbf{j} + V^z \mathbf{k}$$

be a vector field (each of the three components (V^x, V^y, V^z) is a function of \mathbf{r}). The divergence, denoted $\nabla \cdot \mathbf{V}$, is the function*

$$\nabla \cdot \mathbf{V} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} + \frac{\partial V^z}{\partial z}. \quad (9.5)$$

You may have encountered the divergence in a discussion of the Gauss law of electromagnetism. Often, Gauss law is formulated in terms of the electric flux through a closed surface. But if you take a spherical surface and shrink the surface to infinitesimal size at a point, the divergence of the vector field at that point is what you get in the limit. More precisely, consider a small closed (“Gaussian”) surface enclosing a point. One can compute the flux of the vector field \mathbf{V} through this surface (just as one computes electric flux). Now consider shrinking this surface to the chosen point. The limit as one shrinks the surface to the point of the flux divided by the enclosed volume is precisely the divergence of the vector field at that point. So, you can think of the divergence as a sort of flux per unit volume.

It is now easy to see that the right-hand side of the wave equation (9.1) is the divergence of the gradient of the function $q(\mathbf{r}, t)$ with t held fixed. The resulting differential operator, shown on the right-hand side of (9.1), is denoted by $\triangle f$ or $\nabla^2 f$ and is the *Laplacian*:

$$\nabla^2 f := \nabla \cdot (\nabla f) = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}. \quad (9.6)$$

The Laplacian is a linear differential operator that takes a (twice differentiable) function and produces another function.

The wave equation can thus be compactly written in terms of the Laplacian as

$$\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \nabla^2 q. \quad (9.7)$$

* In some texts one denotes $\nabla \cdot$ by “div”:

$$\nabla \cdot \mathbf{V} \equiv \text{div} \mathbf{V}. \quad (9.4)$$

9.2 Solutions to the Three-Dimensional Wave Equation

Solutions of the 3-dimensional wave equation (9.7) are not any harder to come by than those of the 1-dimensional wave equation. Indeed, if we look for solutions that are independent of y and z , we recover the solutions obtained for the 1-dimensional equation. So, for example, the (complex) wave solution

$$q(\mathbf{r}, t) = Ae^{ik(x-vt)}, \quad (9.8)$$

satisfies the 3-dimensional wave equation (exercise) for any real number k . This is a (complex) wave traveling in the x direction with wavelength $\lambda = 2\pi/|k|$ and speed v . Of course, we can equally well write down solutions corresponding to traveling waves in the y and z directions (exercise). More generally, a (complex) wave with wavelength $\lambda = 2\pi/|k|$ propagating in the direction \mathbf{k} with speed v is given by

$$q(\mathbf{r}, t) = Ae^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \omega = |\mathbf{k}|v. \quad (9.9)$$

To check this you can just plug (9.9) into (9.7) and check that the wave equation is satisfied. Alternatively, you can use the following (slightly tricky) argument, which you should ponder as an exercise. The dot product $\mathbf{k} \cdot \mathbf{r}$ is a scalar that is defined geometrically, that is, independently of the orientation of the (x, y, z) axes. Likewise, the Laplacian is the same no matter the orientation of these axes. If you choose your x axis along the direction of \mathbf{k} , it will be obvious that (9.9) solves (9.7) because of our previous comments.

Solutions to the wave equation of the form (9.9) are called *plane waves* because at any given time they look the same as one moves along any plane $\mathbf{k} \cdot \mathbf{r} = \text{constant}$. We'll have a little more to say on this in §10.

Note the relation between frequency and wavelength associated with the plane wave (9.9):

$$\omega = |\mathbf{k}|v = \frac{2\pi}{\lambda}v. \quad (9.10)$$

This is the dispersion relation for plane waves in three dimensions. We can view this restriction as fixing the frequency ω in terms of the magnitude $|\mathbf{k}|$ but leaving the *wave vector* \mathbf{k} itself free. In effect, the dispersion relation (8.67), $\omega = |k|v$, still holds in the three-dimensional setting, provided we interpret $|k|$ as the magnitude of the wave vector. This is because each plane wave appearing in the Fourier decomposition of the general solution is, mathematically, indistinguishable from a complex sinusoidal solution of the one-dimensional wave equation.

Unfortunately, it is not quite as easy to write a simple expression for the *general* solution to the 3-dimensional wave equation as it was in the 1-dimensional case.* In particular,

* There is a formula analogous to (7.26), but it is a little too complicated to be worth going into here.

our trick of changing variables to $x \pm vt$ will not help here. However, Fourier analysis *does* generalize to any number of dimensions. The idea is that, given a function $f(x, y, z)$, we can take its Fourier transform one variable at a time. Let us briefly see how this works. For simplicity, we will only consider the case of waves on all of 3-dimensional space, *i.e.*, we will use the continuous version of the Fourier transform.

The Fourier transform $h(\mathbf{k}) = h(k_x, k_y, k_z)$ of $f(\mathbf{r}) = f(x, y, z)$ is defined by

$$h(\mathbf{k}) = (2\pi)^{-3/2} \int_{\text{all space}} e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) d^3x, \quad (9.11)$$

where $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$, and the 2π factor is a conventional—and convenient—normalization. Every (*e.g.*, square-integrable) function $f(\mathbf{r})$ can be expressed as

$$f(\mathbf{r}) = (2\pi)^{-3/2} \int_{\text{all k space}} e^{i\mathbf{k}\cdot\mathbf{r}} h(\mathbf{k}) d^3k \quad (9.12)$$

for some (*e.g.*, square-integrable) function $h(\mathbf{k})$. The integration region in each of these formulas is denoted by “all space”, which means that each of (x, y, z) in (9.11) and each of (k^x, k^y, k^z) in (9.12) run from $-\infty$ to ∞ .

Note that $e^{i\mathbf{k}\cdot\mathbf{r}}$ can be viewed as a (complex) plane wave profile at a fixed time; the wave is traveling in the direction \mathbf{k} with wavelength $\lambda = 2\pi/k$, where

$$k = |\mathbf{k}| = \sqrt{(k^x)^2 + (k^y)^2 + (k^z)^2} \quad (9.13)$$

is the magnitude (length) of the vector \mathbf{k} . Thus, the essence of Fourier analysis is that every function can be expressed as a superposition of waves with (i) varying amplitudes (specified by $h(\mathbf{k})$), (ii) varying directions (specified by \mathbf{k}/k), and (iii) varying wavelengths (specified by k). The precise contributions from the ingredients (i)–(iii) depend upon the particular function being Fourier analyzed. We now use this basic fact from Fourier analysis to get a handle on the general solution of the wave equation; this will generalize our one-dimensional result.

Suppose $q(\mathbf{r}, t)$ is a solution to the wave equation. Let us define

$$p(\mathbf{k}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k}\cdot\mathbf{r}} q(\mathbf{r}, t) d^3x. \quad (9.14)$$

This is just the Fourier transform of q at each time t . It is easy to see that the wave equation implies

$$\frac{\partial^2 p}{\partial t^2} + k^2 v^2 p = 0. \quad (9.15)$$

As also occurred in the one-dimensional case, for each value of \mathbf{k} this is just the harmonic oscillator equation. The wave vector, in effect, labels degrees of freedom that oscillate

with a frequency $\omega = |k|v$. This is not an accident, of course, given the relation between the wave equation and a collection of harmonic oscillators which we discussed earlier. The solution to (9.15) is then of the form

$$p(\mathbf{k}, t) = A(\mathbf{k}) \cos(\omega t) + B(\mathbf{k}) \sin(\omega t). \quad (9.16)$$

To fix the coefficients A and B we need to consider initial conditions. Suppose that at, say, $t = 0$ the wave has displacement and velocity profiles given by

$$q(\mathbf{r}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } \mathbf{k} \text{ space}} e^{i\mathbf{k} \cdot \mathbf{r}} a(\mathbf{k}) d^3k \quad (9.17)$$

and

$$\frac{\partial q(\mathbf{r}, 0)}{\partial t} = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } \mathbf{k} \text{ space}} e^{i\mathbf{k} \cdot \mathbf{r}} b(\mathbf{k}) d^3k. \quad (9.18)$$

Then it is easy to see that*

$$A(\mathbf{k}) = a(\mathbf{k}) \quad B(\mathbf{k}) = \frac{1}{\omega} b(\mathbf{k}). \quad (9.19)$$

Evidently,

$$q(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } \mathbf{k} \text{ space}} e^{i\mathbf{k} \cdot \mathbf{r}} \left\{ a(\mathbf{k}) \cos(\omega t) + b(\mathbf{k}) \frac{\sin(\omega t)}{\omega} \right\} d^3k \quad (9.20)$$

satisfies the wave equation where

$$a(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k} \cdot \mathbf{r}} q(\mathbf{r}, 0) d^3x, \quad (9.21)$$

and

$$b(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k} \cdot \mathbf{r}} \frac{\partial q(\mathbf{r}, 0)}{\partial t} d^3x. \quad (9.22)$$

The general solution (9.20) to the 3-d wave equation can be viewed as a superposition of the elementary (complex) plane wave solutions that we studied earlier. To see this, just note that

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}), \quad \sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}), \quad (9.23)$$

and then rearrange the terms in (9.20) to get the solution into the form

$$q(\mathbf{r}, t) = (2\pi)^{-3/2} \int_{\text{all } \mathbf{k}\text{-space}} \left[c(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - kvt)} + c^*(\mathbf{k}) e^{-i(\mathbf{k} \cdot \mathbf{r} - kvt)} \right] d^3k, \quad (9.24)$$

* Note that the coefficient $B(\mathbf{k})$ is not defined as $k \rightarrow 0$ (exercise), but the product $B(\mathbf{k}) \sin(\omega t)$ is well-defined at $k = 0$.

where $c(\mathbf{k})$ are determined by the initial data. Physically, you can think of this integral formula as representing a (continuous) superposition of plane waves over their possible physical attributes. To see this, consider a plane wave of the form

$$q(\mathbf{r}, t) = \text{Re} \left[c e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \quad (9.25)$$

where c is a complex number. You can check that this is a wave traveling in the direction of \mathbf{k} , with wavelength $\frac{2\pi}{k}$, and with amplitude $|c|$. The argument of c , namely $(\frac{c}{|c|})$ adds a constant to the phase of the wave (*exercise*). The integral in (9.20) is then a superposition of waves in which one varies the amplitudes ($|c|$), relative phases ($c/|c|$), wavelengths ($2\pi/k$), and directions of propagation (\mathbf{k}/k) from one wave to the next.

Exactly as we did for the space of solutions to the one-dimensional wave equation, we can view the space of solutions of the three-dimensional wave equation as a vector space (*exercise*). From this point of view, the plane waves form a basis for the vector space of solutions. Equivalently, every solution to the wave equation can be obtained by superimposing real plane wave solutions of the form

$$q(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) \quad (9.26)$$

by varying the amplitude A , the wave vector \mathbf{k} and the phase ϕ (*exercise*).

10. Why “plane” waves?

Let us now pause to explain in more detail why we called the elementary solutions (9.9) and (9.26) *plane waves*. The reason is that the displacement $q(\mathbf{r}, t)$ has the symmetry of a plane. To see this, fix a time t (take a “snapshot” of the wave) and pick a location \mathbf{r} . Examine the wave displacement q (at the fixed time) at all points in a plane that is (i) perpendicular to \mathbf{k} , and (2) passes through \mathbf{r} . The wave displacement will be the same at each point of this plane. To understand this most easily, simply choose, say, the x -axis to be along the vector \mathbf{k} . The planes perpendicular to \mathbf{k} are parallel to the y - z plane. In these new coordinates the wave (9.26) takes the simple form (*exercise*)

$$q(\mathbf{r}, t) = A \cos(kx - \omega t + \phi). \quad (10.1)$$

Clearly, at a fixed t and x , $q(\mathbf{r}, t)$ is the same anywhere on the plane obtained by varying y and z .

A more formal — and perhaps more instructive — way to see the plane wave symmetry of (9.26) is to fix a time t and ask for the locus of points upon which the wave displacement is constant. At a fixed time, the wave displacement q is a function of 3 variables. As you know, the locus of points where a function takes the same values defines a surface. Well,

with $t = \text{constant}$, the surfaces of fixed q arise when $\mathbf{k} \cdot \mathbf{r} = \text{constant}$ (*exercise*). But the equation $k_x x + k_y y + k_z z = \text{constant}$ (with each of k_x, k_y, k_z a constant) is the equation for a plane (*exercise*)! This plane is everywhere orthogonal to the wave vector \mathbf{k} , which can be viewed as a constant vector field (*i.e.*, a vector field whose Cartesian components are the same everywhere). To see this, we recall from our discussion in §9 that the gradient of a function is always perpendicular to the surfaces upon which the function is constant. We just saw that the plane of symmetry (where q doesn't change its value) arises when the function $\mathbf{k} \cdot \mathbf{r}$ is constant. Thus, the plane of symmetry for the plane wave is orthogonal to the (constant) vector field

$$\nabla(\mathbf{k} \cdot \mathbf{r}) = \mathbf{k}. \quad (10.2)$$

The wave vector is thus normal to the planes of symmetry of a plane wave. As time evolves, the displacement profile on a given plane of symmetry moves along \mathbf{k} . In this way \mathbf{k} determines the propagation direction of the plane wave. The wave vector thus determines the wavelength and direction of motion of a plane wave.

PROBLEM SET 5

Problem 5.1

Let \mathbf{A} be a given constant vector field vector. Show that the equation $\mathbf{A} \cdot \mathbf{r} = c$, where c is a constant, defines a plane. If $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the $c = 0$, where is the plane?

Problem 5.2

Prove that the gradient of a function $f(\mathbf{r})$ is always orthogonal to the surfaces $f(\mathbf{r}) = \text{constant}$. (Hint: This one is easy; think about the directional derivative of f along any direction tangent to the surface.)

Problem 5.3

Consider the sphere defined by $x^2 + y^2 + z^2 = 1$. Compute the gradient of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

and check that it is everywhere orthogonal to the sphere. Consider a linear function

$$f(\mathbf{r}) = \mathbf{a} \cdot \mathbf{r},$$

where \mathbf{a} is a fixed, constant vector field. Compute the gradient of f and check that the resulting vector field is perpendicular to the plane $f(\mathbf{r}) = 0$.

Problem 5.4

Compute the divergence of the following two vector fields:

- (a) $\mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}$, $r = \sqrt{x^2 + y^2 + z^2} > 0$, (Coulomb electric field)
- (b) $\mathbf{B}(\mathbf{r}) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$, $x^2 + y^2 > 0$, (magnetic field outside a long wire)
- (c) $\mathbf{D}(\mathbf{r}) = \mathbf{r}$ (electric field inside a uniform ball of charge).

Problem 5.5

Derive (9.15) and (9.19).

Problem 5.6

Consider a spherically symmetric function $f = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$. Show that its Fourier transform takes the following form:

$$\frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k}\cdot\mathbf{r}} f(r) d^3x = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty r f(r) \sin(kr) dr.$$

(*Hint:* Use spherical polar coordinates, choosing your z axis along \mathbf{k} .) Note that the transform is spherically symmetric also in \mathbf{k} space. Use this formula to compute the Fourier transform of a 3-dimensional Gaussian

$$f = e^{-a^{-2}(x^2+y^2+z^2)}.$$

Problem 5.7

Derive (9.24) from (9.20). In particular, express $c(\mathbf{k})$ in terms of the initial data.

Problem 5.8

Let f and g be two functions. We can take the gradient of g to get a vector field, which we denote by ∇g . We can multiply this vector field by the function f to get another vector field, $f\nabla g$. As with any vector field, we can form a function by taking the divergence of $f\nabla g$; using the definitions of gradient and divergence show that

$$\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g. \quad (10.3)$$

11. Separation of variables.

There is yet another way to find the general solution to the wave equation which is valid in 1, 2, or 3 (or more!) dimensions. This method is quite important and, as we shall see, can often be used for other linear homogeneous differential equations. This technique for solving the wave equation is called the method of *separation of variables*.

We begin by assuming that the solution of the wave equation is a superposition of elementary solutions $q(\mathbf{r}, t)$ each of which can be expressed as a product:

$$q(\mathbf{r}, t) = T(t)X(x)Y(y)Z(z). \quad (11.1)$$

We shall show that each of these 4 functions satisfies a relatively simple ordinary differential equation. To this end, we insert the trial solution (11.1) into the wave equation to obtain (*exercise*):

$$\frac{1}{v^2}T''XYZ - TX''YZ - TXY''Z - TXYZ'' = 0, \quad (11.2)$$

where the prime on a function denotes a derivative with respect to the argument of that function, *e.g.*, $X' = \frac{dX(x)}{dx}$. Away from points where $q = 0$, we can divide both sides of this equation by q to get

$$\frac{1}{v^2} \frac{T''}{T} - \frac{X''}{X} - \frac{Y''}{Y} - \frac{Z''}{Z} = 0. \quad (11.3)$$

Note that each term in (11.3) is a function, respectively, of the *different* variables (t, x, y, z) . Because of this property, we say that the wave equation is “separable”, or “separates” in Cartesian coordinates. As we shall see, the property of being separable really refers not just to the equation, but also to the coordinate system employed. Typically, one can only expect linear, homogeneous partial differential equations to separate.

Because the variables are separated in (11.3), this equation represents a very stringent requirement; it amounts to saying that each of the functions T''/T , X''/X , Y''/Y , Z''/Z is in fact a constant! To see this, let us isolate the T''/T term on one side of the equation. We then get an equation of the form

$$f(t) = g(x, y, z), \quad (11.4)$$

where we are denoting

$$f(t) = \frac{1}{v^2} \frac{T''}{T} \quad (11.5)$$

and

$$g(x, y, z) = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \quad (11.6)$$

Because we can vary x, y, z independently of t , (11.4) holds if and only if $f(t) = a = g(x, y, z)$, where a is a constant. To see this directly, simply take the partial derivative of (11.4) with respect to t to find $f' = 0$ (*exercise*).

So, at this point we know that for some constant a

$$\frac{1}{v^2} \frac{T''}{T} = a, \quad (11.7)$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = a. \quad (11.8)$$

But we can play the same game again! We write

$$\frac{X''}{X} = a - \frac{Y''}{Y} - \frac{Z''}{Z}, \quad (11.9)$$

and we see that both sides must equal a constant, say, b :

$$\frac{X''}{X} = b. \quad (11.10)$$

Applying this logic repeatedly, we conclude that (*exercise*)

$$\begin{aligned} \frac{1}{v^2} \frac{T''}{T} &= a, \\ \frac{X''}{X} &= b, \\ \frac{Y''}{Y} &= c, \\ \frac{Z''}{Z} &= d, \end{aligned} \quad (11.11)$$

where a, b, c, d are constants (“separation constants”) and

$$a - b - c - d = 0, \quad i.e., \quad a = (b + c + d). \quad (11.12)$$

The single *partial* differential equation (9.7) has thus been turned into 4 *ordinary* differential equations (11.11) involving 3 arbitrary separation constants.

It is now a simple matter to solve the quartet (11.11) of linear, homogeneous ordinary differential equations for T, X, Y, Z . They are all of the form

$$f'' = -\alpha^2 f, \quad (11.13)$$

which is the harmonic oscillator equation again! This equation has (complex) solutions (*exercise*)

$$f(u) = \beta e^{\pm i\alpha u},$$

where α and β are constants. To recover our plane wave “basis” for the general solution to the wave equation we set

$$b = -k_x^2, \quad c = -k_y^2, \quad d = -k_z^2, \quad (11.14)$$

and

$$a = -k_x^2 - k_y^2 - k_z^2 = -k^2. \quad (11.15)$$

Thus our separation of variables (complex) solution is a plane wave of the form

$$q(\mathbf{r}, t) = Ae^{ik_x x} e^{ik_y y} e^{ik_z z} e^{i\omega t} = Ae^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad (11.16)$$

where

$$\omega = \pm kv \quad (11.17)$$

as before. Thus the “separation constants” make up the frequency of the wave and the components of the wave vector.

We can build up the general solution to the wave equation by superposition of these elementary separation of variables solutions and arrive again at the Fourier expansions of §9 (*exercise*).

12. Cylindrical Coordinates.

We have seen how to build solutions to the wave equation by superimposing plane waves with various choices for amplitude, phase and wave vector \mathbf{k} . In this way we can build up solutions which need not have the plane symmetry (*exercise*), or any symmetry whatsoever. Still, as you know by now, many problems in physics are fruitfully analyzed when they are modeled as having various symmetries, such as cylindrical symmetry or spherical symmetry. For example, the magnetic field of a long, straight wire carrying a steady current can be modeled as having cylindrical symmetry. Likewise, the sound waves emitted by a pointlike source are nicely approximated as spherically symmetric. Now, using the Fourier expansion in plane waves we can construct such symmetric solutions — indeed, we can construct any solution to the wave equation. But, as you also know, we have coordinate systems that are adapted to a variety of symmetries, *e.g.*, cylindrical coordinates, spherical polar coordinates, *etc.* When looking for waves with some chosen symmetry it is advantageous to get at the solutions to the wave equation directly in these coordinates, without having to express them as a superposition of plane waves. Our task now is to see how to express solutions of the wave equations in a useful fashion in terms of such *curvilinear* coordinate systems.

12.1 The Wave Equation in Cylindrical Coordinates

Our first example will involve solutions to the wave equation in cylindrical coordinates (ρ, ϕ, z) , which we shall now define. To begin, we point out that the purpose of coordinates is to provide a unique label for every point in some region of space. A common method of defining a new system of coordinates is to take some existing coordinate system and

define the new system relative to it. To this end, fix some Cartesian (*i.e.*, (x, y, z)) coordinates, which provide unique labels for all of space in a very well-known way. Cylindrical coordinates are denoted by (ρ, ϕ, z) and are related to Cartesian coordinates as follows. A point in space is labeled by (i) its perpendicular distance ρ from the z -axis; (ii) the angle ϕ between a line obtained by projecting into the x - y plane a line drawn perpendicularly from the z -axis to the point and the positive x -axis; and (iii) the z coordinate of the point. If this verbal description leaves you a little dazed, an analytic statement of this is somewhat simpler. A point with Cartesian coordinates (x, y, z) is labeled with cylindrical coordinates (ρ, ϕ, z) , where

$$\rho = \sqrt{x^2 + y^2} \quad (12.1)$$

$$\phi = \tan^{-1} \left(\frac{y}{x} \right). \quad (12.2)$$

Note that ϕ is only defined *modulo* integral multiples of 2π . For example, given a fixed value of ρ and z , $\phi = \frac{\pi}{4}$ and $\phi = -\frac{7\pi}{4}$ define the same point in space.

You can think of cylindrical coordinates as a simple 3-dimensional extension of the usual two-dimensional polar coordinates in the sense that on each plane $z = \text{constant}$, ρ and ϕ are polar coordinates. Given cylindrical coordinates (ρ, ϕ, z) of a point, we can recover the Cartesian coordinates (x, y, z) of that point from (*exercise*)

$$\begin{aligned} x &= \rho \cos \phi, \\ y &= \rho \sin \phi \end{aligned} \quad (12.3)$$

and the z value, of course. Note that $\rho \geq 0$ and, without loss of generality, $0 \leq \phi \leq 2\pi$.

Warning: cylindrical coordinates are not well behaved on the z -axis. For example, what is ϕ for a point on the z -axis?

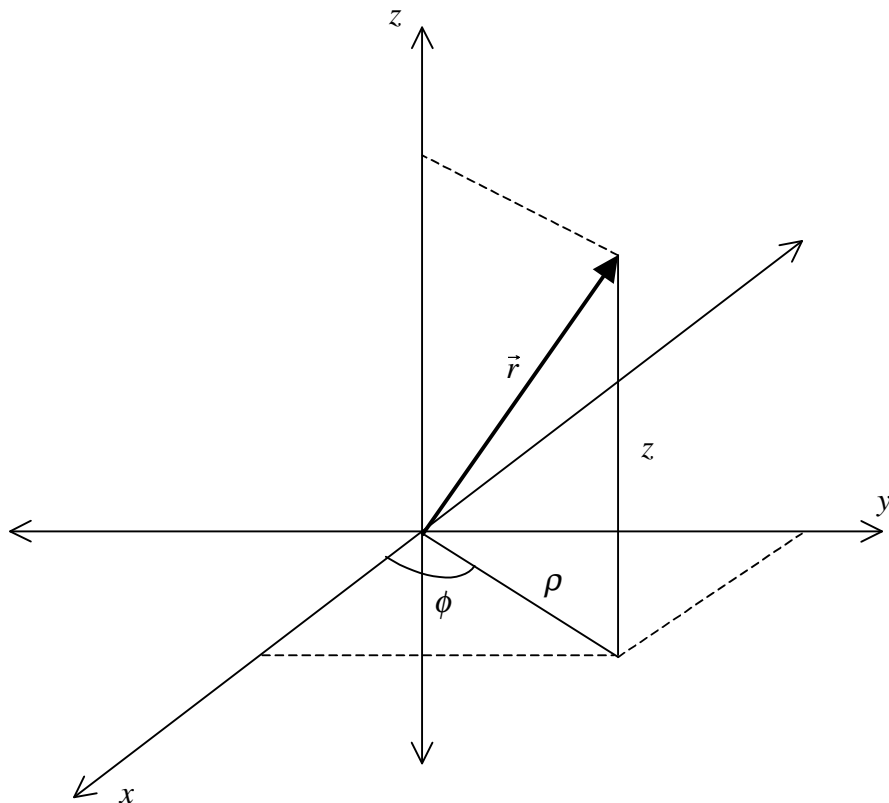


Figure 14. Illustration of cylindrical coordinates ρ , ϕ , and z .

As we shall explore a little later, cylindrical coordinates are particularly useful when one is studying waves with cylindrical symmetry because the surface $\rho = \text{constant}$ is a right cylinder of radius ρ . Of course, this is why these coordinates are called cylindrical! Note that z and ϕ provide coordinates for points on the cylinder $\rho = \text{constant}$ (*exercise*).

One way to get solutions of the wave equation in terms of cylindrical coordinates we can simply translate our Fourier expansions in terms of this coordinate system. In particular, the plane wave solution takes the form

$$\begin{aligned} q(\rho, \phi, z, t) &= \text{Re} \left(A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right) \\ &= \text{Re} \left(A e^{i[k_x \rho \cos \phi + k_y \rho \sin \phi + k_z z - \omega t]} \right). \end{aligned} \quad (12.4)$$

Because the cylindrical coordinates are not adapted to the plane symmetry (as are the Cartesian coordinates), the plane symmetry of the plane wave is hidden in (12.4). General

solutions in cylindrical coordinates can be obtained by taking superpositions of such solutions. But as mentioned above, we want to be able to get solutions which have cylindrical symmetry without having to perform complicated integrals. To do this, we return to the wave equation, express it in terms of cylindrical coordinates and try to solve it directly.

To translate the wave equation into cylindrical coordinates amounts to finding the cylindrical coordinate expression of the Laplacian. There are a number of tricks for getting this expression, and you can find the result in any text on mathematical methods for physicists or on electromagnetism. It is a nice exercise in multivariable calculus — mainly the chain rule — to get the result, so let us sketch the computation. Recall the Laplacian in Cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (12.5)$$

Given any function $g(x, y, z)$ we can express it as a function, denoted f , of (ρ, ϕ, z) by simply substituting for (x, y, z) in terms of (ρ, ϕ, z) :

$$f(\rho, \phi, z) = g(\rho \cos \phi, \rho \sin \phi, z). \quad (7.25)$$

An example of this appears in (12.4). Our goal is to express $\nabla^2 g$ in terms of f and its derivatives with respect to (ρ, ϕ, z) .*

Let us begin with the term $\frac{\partial^2}{\partial x^2}$ in $\nabla^2 g$. The partial derivative of g with respect to x is related to the partial derivatives of f with respect to (ρ, ϕ, z) by

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}. \quad (12.6)$$

To get an expression in cylindrical coordinates for $\frac{\partial g}{\partial x}$ in terms of partial derivatives of f with respect to (ρ, ϕ, z) we take the expression (12.6) and express (x, y, z) in terms of (ρ, ϕ, z) . We get (exercise)

$$\frac{\partial g}{\partial x} = \cos \phi \frac{\partial f}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial f}{\partial \phi}. \quad (12.7)$$

Here we used the fact that if $u = u(x)$, then (exercise)

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx}, \quad (12.8)$$

also

$$\frac{\partial z}{\partial x} = 0, \quad (12.9)$$

and

$$\frac{\partial \rho}{\partial x} = \cos \phi, \quad \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{\rho}. \quad (12.10)$$

* Note that we using our picky – but clear – notation which distinguishes the form of a function in different coordinate systems.

Make sure you understand the basic equations (12.8)–(12.10).

We now need to take another derivative—what a mess! Computers equipped with algebraic computing software, *e.g.*, *MathCad*, *Mathematica* or *Maple*, are very good at these calculations. But it is instructive to do it ourselves. The chain rule says:

$$\frac{\partial^2 g}{\partial x^2} = \left(\frac{\partial^2 f}{\partial \rho^2} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \phi}{\partial x} \right) \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial x^2} + \left(\frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \phi^2} \frac{\partial \phi}{\partial x} \right) \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial x^2}. \quad (12.11)$$

Similarly,

$$\frac{\partial^2 g}{\partial y^2} = \left(\frac{\partial^2 f}{\partial \rho^2} \frac{\partial \rho}{\partial y} + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \phi}{\partial y} \right) \frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial y^2} + \left(\frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial y} + \frac{\partial^2 f}{\partial \phi^2} \frac{\partial \phi}{\partial y} \right) \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial y^2}. \quad (12.12)$$

We now combine these two expressions with $\frac{\partial^2 f}{\partial z^2}$, and use the facts (exercises)

$$\frac{\partial \rho}{\partial y} = \sin \phi, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho}, \quad (12.13)$$

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{\sin^2 \phi}{\rho}, \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{2 \cos \phi \sin \phi}{\rho^2}, \quad (12.14)$$

$$\frac{\partial^2 \rho}{\partial y^2} = \frac{\cos^2 \phi}{\rho}, \quad \frac{\partial^2 \phi}{\partial y^2} = -\frac{2 \cos \phi \sin \phi}{\rho^2}. \quad (12.15)$$

to get the remarkably simple result

$$\begin{aligned} \nabla^2 g &\equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \\ &= \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \quad (12.16)$$

We can thus express the wave equation for $q(\rho, \phi, z, t)$ in cylindrical coordinates as

$$\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial q}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 q}{\partial \phi^2} + \frac{\partial^2 q}{\partial z^2}. \quad (12.17)$$

Note that here and henceforth we are using the same symbol q for the wave displacement irrespective of coordinate system.

12.2 Separation of Variables in Cylindrical Coordinates

How are we to solve equation (12.17)? The only method we have exhibited for Cartesian coordinates that has a chance of generalizing to cylindrical coordinates is the method of separation of variables. Let's try it. Set

$$q(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t), \quad (12.18)$$

substitute this trial solution into the wave equation, and then divide the resulting equation by q to find (exercise)

$$\frac{1}{v^2} \frac{T''}{T} = \frac{1}{\rho} \frac{(\rho R')'}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z}. \quad (12.19)$$

Again we see a separation of variables: on the left hand side we have a function of t only, while on the right hand side we have only a function of (ρ, ϕ, z) . Thus both sides must equal a constant; let us call it $-k^2$. We can solve the resulting equation for T ,

$$T'' = -v^2 k^2 T, \quad (12.20)$$

via

$$T = A e^{\pm i v k t}, \quad (12.21)$$

where we are using the complex form for the solution and, as usual, we must remember to take the real part of the solution at the end.

We have reduced (12.19) to an equation of the form

$$\frac{1}{\rho} \frac{(\rho R')'}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} = -k^2 - \frac{Z''}{Z}, \quad (12.22)$$

which equates a function of ρ and ϕ with a function of z . The left and right hand sides must therefore equal a constant, call it $-a^2$:

$$\frac{Z''}{Z} = a^2 - k^2. \quad (12.23)$$

The solution to (12.23) is of the form

$$Z = B e^{\pm i \sqrt{k^2 - a^2} z}. \quad (12.24)$$

We are now left with

$$\frac{1}{\rho} \frac{(\rho R')'}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} = -a^2. \quad (12.25)$$

which can be written as

$$\rho \frac{(\rho R')'}{R} + a^2 \rho^2 = -\frac{\Phi''}{\Phi}. \quad (12.26)$$

The separation of variables continues! Both sides of this equation must equal a constant. Let us call it b^2 . We have

$$\Phi'' = -b^2 \Phi, \quad (12.27)$$

which has the (complex) solution

$$\Phi = C e^{\pm i b \phi}, \quad (12.28)$$

where C is a complex constant.

Here we encounter a slightly novel feature arising from our use of cylindrical coordinates. If $q(t, \rho, \phi, z)$ is to be a function associating a physically measurable quantity to each point of space at each instant of time we should demand that

$$q(\rho, 0, z, t) = q(\rho, 2\pi, z, t). \quad (12.29)$$

Equation (12.29) forces b in (12.28) to be an integer $n = 0, 1, 2, 3, \dots$ (*exercise*).

With this restriction in mind, we can finally try to solve for R . The equation we must solve is (*exercise*)

$$\rho \frac{(\rho R')'}{R} + a^2 \rho^2 - n^2 = 0, \quad (12.30)$$

or

$$R'' + \frac{1}{\rho} R' + (a^2 - \frac{n^2}{\rho^2}) R = 0. \quad (12.31)$$

Let us make a simple change of variables to put this equation into a standard form. Let us set $x := a\rho$.^{*} Of course, this x has nothing to do with the x of Cartesian coordinates. Note that x is a dimensionless quantity (*exercise*). With $\mathcal{R}(x) := R(x/a)$, the radial equation is (*exercise*)

$$\frac{d^2 \mathcal{R}}{dx^2} + \frac{1}{x} \frac{d\mathcal{R}}{dx} + (1 - \frac{n^2}{x^2}) \mathcal{R} = 0. \quad (12.32)$$

This linear, homogeneous, ordinary differential equation is known as *Bessel's equation*. The solutions are called *Bessel functions* and *Neumann functions*. These solutions will, of course, depend on which integer (n) is used in the equation. The Bessel and Neumann functions are examples of “special functions”. There is a menagerie of special functions which are useful in analyzing the basic equations of mathematical physics (wave equation, Laplace equation, *etc.*). You may not have seen the Bessel or Neumann functions before, but they are really no more sophisticated than, say, the very familiar sine, cosine, or exponential functions. Indeed, we could define the sine and cosine functions as the two “independent” real solutions of

$$f'' = -\omega^2 f, \quad (12.33)$$

where ω is a given constant (analogous to n in Bessel's equation (12.32)). This is a familiar pattern in mathematical physics: special functions can be defined as solutions to certain (ordinary) differential equations. Just as in the case of sine and cosine, for a given n there are actually 2 independent solutions of Bessel's equation — the Bessel and Neumann functions. But in the case at hand, only the Bessel functions, are well-behaved at $\rho = 0$, and we will limit our attention to this type of solution, which is normally denoted $J_n(x)$. The singular Neumann solutions are useful when one is interested in solving the wave equation in regions not including the axis $\rho = 0$.

* Recall that the symbol “ $:=$ ” means “is defined to be”.

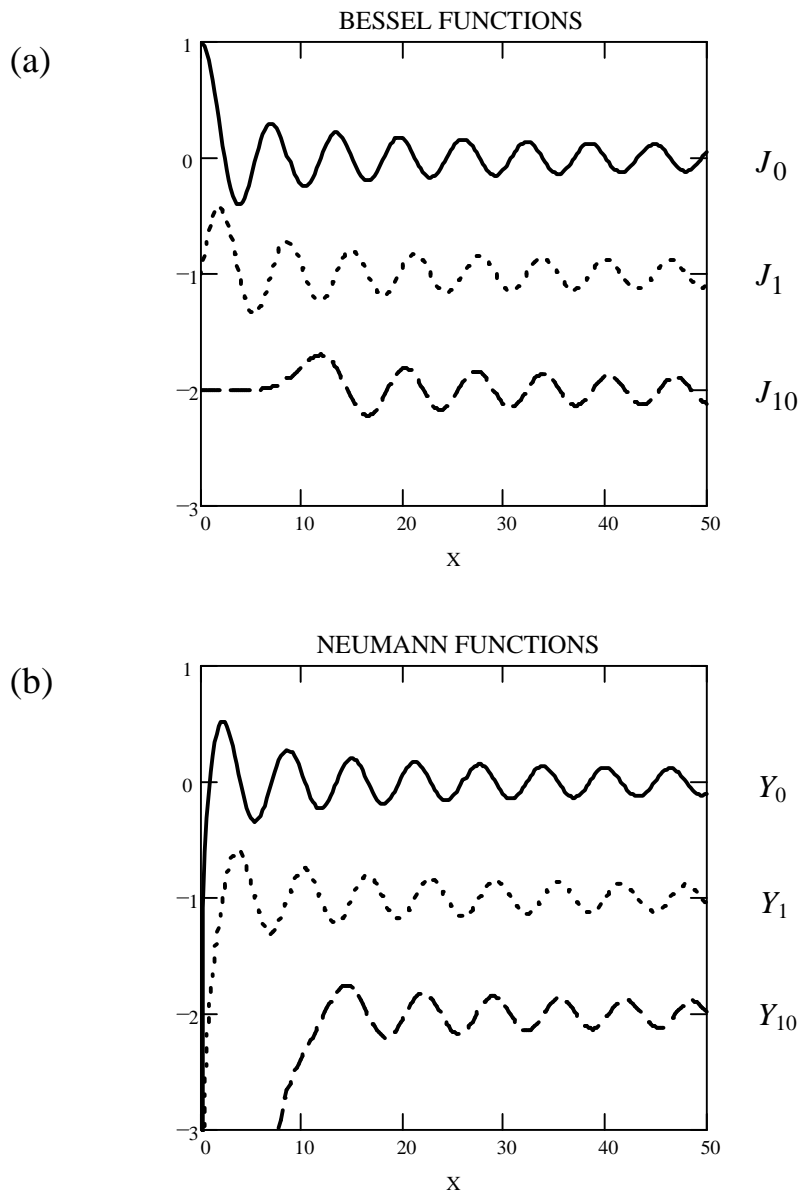


Figure 15. (a) Bessel functions $J_0(x)$, $J_1(x)$, and $J_{10}(x)$. For clarity $J_1(x)$ and $J_{10}(x)$ have been displaced in the vertical direction by -1 and -2 , respectively. (b) Neumann functions $Y_0(x)$, $Y_1(x)$, and $Y_{10}(x)$. For clarity $Y_1(x)$ and $Y_{10}(x)$ have been displaced in the vertical direction by -1 and -2 , respectively.

Remark:

It is worth mentioning that this sort of situation is quite common when solving differential equations that have singularities in their coefficients — the $1/x$ in (12.32). Since the

equation is, strictly speaking, only defined when $x \neq 0$, the mathematics is quite happy to supply solutions that are only defined on the domain $x \neq 0$. Moreover, differential equations are by their nature *local* – they really only constrain the behavior of a function in the neighborhood of any given point. Global information, such as boundary conditions, domains of validity, *etc.* must be imposed as additional conditions on the solutions to the differential equations.

Needless to say, the Bessel and Neumann functions have been exhaustively investigated, and one can look up their values and basic properties in any set of math tables. Alternatively, any decent computer mathematics package will know about Bessel and Neumann functions. We will not need any details about these functions, but for the purposes of the following discussion it is worth noting the following qualitative features possessed by the Bessel functions. The Bessel functions $J_n(x)$ are real functions and oscillatory in x with decreasing amplitude for increasing x and, as mentioned above, they take on finite values at $x = 0$ (corresponding to points on the z -axis). In fact, it turns out that only J_0 is non-zero on the at $x = 0$; $J_n(0) = 0$ for $n > 0$. It is clear that Bessel's equation (12.32) does not distinguish between n and $-n$ (*exercise*) so we do not expect J_n and J_{-n} to represent different solutions. This is true, but it is customary to define J_n such that

$$J_{-n} = (-1)^n J_n. \quad (12.34)$$

12.3 Solutions to the Wave Equation in Cylindrical Coordinates

Let us summarize the results of the last section. Using separation of variables we can get a complex solution to the wave equation in cylindrical coordinates via

$$q(\rho, \phi, z, t) = C e^{\pm i k v t} J_n(a \rho) e^{\pm i n \phi} e^{\pm i \sqrt{k^2 - a^2} z}. \quad (12.35)$$

Sometimes this elementary solution is expressed as

$$q(\rho, \phi, z, t) = C e^{\pm i \omega t} J_n(a \rho) e^{\pm i n \phi} e^{\pm i \kappa z}, \quad (12.36)$$

where

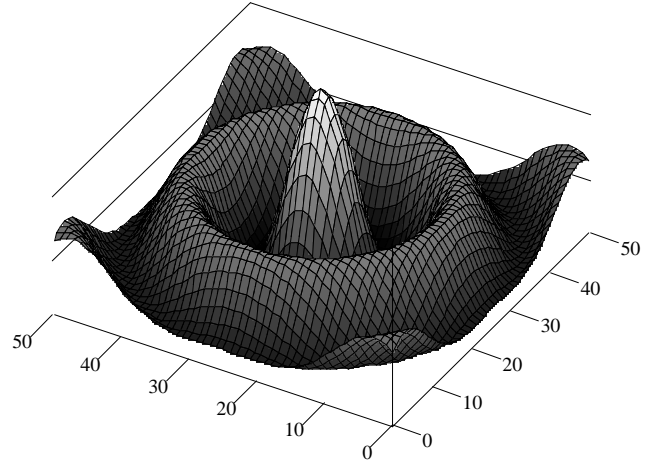
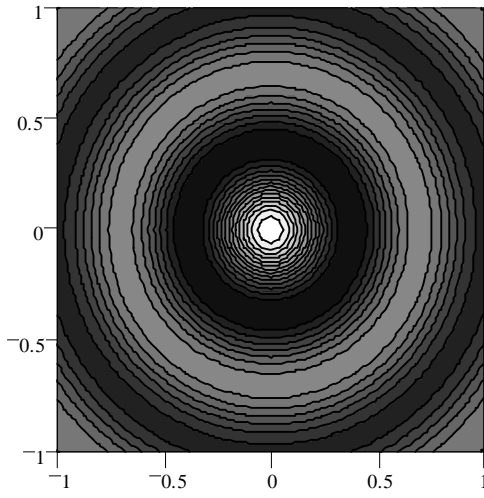
$$\omega = k v, \quad \kappa = \sqrt{\frac{\omega^2}{v^2} - a^2}.$$

Of course, we must take the real (or imaginary) parts of (12.35) to get a real solution; for example, we have a real solution of the form (*exercise*)

$$q(\rho, \phi, z, t) = b \cos(k v t + n \phi + \kappa z + \alpha) J_n(a \rho), \quad (12.37)$$

where all parameters (b, k, a, α) are real numbers.

(a)



(b)

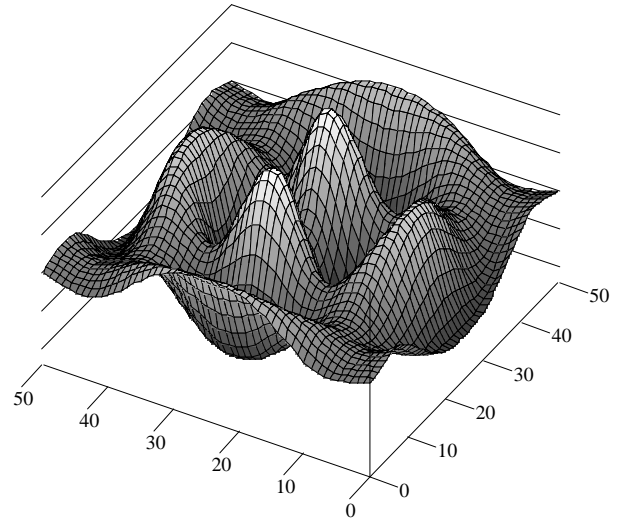
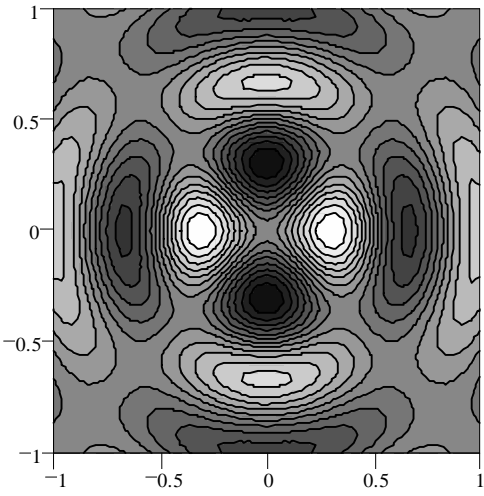


Figure 16. Spatial part of the separation-of-variables solutions in cylindrical coordinates. The solutions shown have no z dependence, i.e., $k = a$. (a) $J_0(\rho)$. (b) $\cos(2\phi)J_2(\rho)$.

From (12.37) we see that, at a given point of space, the amplitude of the displacement will oscillate sinusoidally in time. If we take a snapshot of the displacement profile at an instant of time we see a rather complicated oscillatory behavior in ρ , ϕ , and z . To get a feel for this, let us suppose that $\kappa = 0$ and $n = 0$. Then the solution (12.37) is of the form (*exercise*)

$$q(\rho, \phi, z, t) = b \cos(\omega t + \alpha) J_0(a\rho). \quad (12.38)$$

At any fixed location the displacement oscillates harmonically with frequency ω . At any given instant of time the displacement is the same at all points on the cylinder $\rho = \text{constant}$. Equivalently, the solution does not depend on the coordinates ϕ and z . We say that the solution exhibits *cylindrical symmetry*. As ρ increases from zero — still at a fixed time — the displacement oscillates with decreasing amplitude.

A simple physical picture of such a cylindrically symmetric wave can be obtained as follows. Consider the (two-dimensional) surface of a lake. Since the cylindrically symmetric solution doesn't depend upon z , we simply ignore z . Let q denote the displacement of the water in the vertical direction at a given time. The displacement $q = 0$ means that the lake is calm, that is, its surface is flat. Now drop a stone in the lake and choose the origin of our coordinates on the lake at the point where the stone is dropped. You know what happens physically: concentric circular waves are produced which spread from the point where the stone was dropped with amplitude decreasing as the distance from the disturbance increases. But this is precisely the kind of situation our cylindrically symmetric solution represents! Indeed, have a look at the cylindrically symmetric solution (12.38), which represents traveling waves propagating radially with decreasing amplitude as one moves away from the center. That the cylindrically symmetric solution to the wave equation matches up with qualitative behavior of water waves is no accident: for small displacements the restoring force on small elements of the water can be approximated as a Hooke's law force, and all our work shows that this leads to the wave equation. You might complain that a more realistic wave on the lake is not as symmetric as our simplest solution (12.38) above. But more complicated solutions can be obtained by considering superpositions of solutions of the form (12.35).

This last point deserves some elaboration. In Cartesian coordinates we saw how one could build the general solution to the wave equation using Fourier integrals, which could be viewed as a superposition of elementary plane wave solutions. The plane wave solutions could be obtained using separation of variables in Cartesian coordinates. A similar result can be established (though we won't do it here) when solving the wave equation in cylindrical coordinates. Essentially, any solution of the wave equation in cylindrical coordinates can be obtained by a suitable superposition of the elementary solutions (12.35) we have found via separation of variables.

13. Spherical Coordinates.

We can play similar games with another popular coordinate system: *spherical coordinates* (also called “spherical polar coordinates”). These coordinates are denoted (r, θ, ϕ) and are defined by

$$r = \sqrt{x^2 + y^2 + z^2} \quad (13.1)$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) \quad (13.2)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \quad (13.3)$$

Note that $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. Careful! Spherical polar coordinates are not well-behaved on the z -axis (*exercise*). A point labeled (r, θ, ϕ) has Cartesian coordinates (*exercise*)

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta. \end{aligned} \quad (13.4)$$

The spherical coordinates of a point p can be obtained by the following geometric construction. The value of r represents the distance from the point p to the origin (which you can put wherever you like). The value of θ is the angle between the positive z -axis and a line l drawn from the origin to p . The value of ϕ is the angle made with the x -axis by the projection of l into the x - y plane ($z = 0$). Note: for points in the x - y plane, r and ϕ (not θ) are polar coordinates. The coordinates (r, θ, ϕ) are called the *radius*, *polar angle*, and *azimuthal angle* of the point p , respectively. It should be clear why these coordinates are called spherical. The points $r = a$, with $a = \text{constant}$, lie on a sphere of radius a about the origin. Note that the angular coordinates can thus be viewed as coordinates on a sphere. Indeed, they label latitude and longitude (*exercise*).

You should take note of the fact that many mathematics texts use a different labeling scheme for spherical coordinates in which the roles of θ and ϕ are interchanged. The convention being used here is found in most of the physics literature. To be fair, the mathematicians’ convention is a little more logical since the normal notation for polar coordinates in the x - y plane is (r, θ) .

Spherical coordinates are, of course, particularly useful when studying wave phenomena exhibiting spherical symmetry. For example, the sound waves emitted by an exploding firework shell can be modeled as spherically symmetric with respect to an origin at the explosion’s location. In other words, the compression/rarefaction of air at a point only depends on the distance from the point to the explosion, not on the angular location of the point relative to the explosion (ignoring obstacles, of course). Similarly, the light emitted from “point sources” (*i.e.*, sources which are sufficiently small compared to the distance

to the points of interest) is best studied using spherical polar coordinates centered on the light source.

13.1 The Wave Equation in Spherical Coordinates

How do we find solutions to the wave equation in spherical coordinates? You might be able to guess how we are going to proceed: express the wave equation in spherical coordinates for a function $q(r, \theta, \phi, t)$ and solve by separation of variables. We will not go through

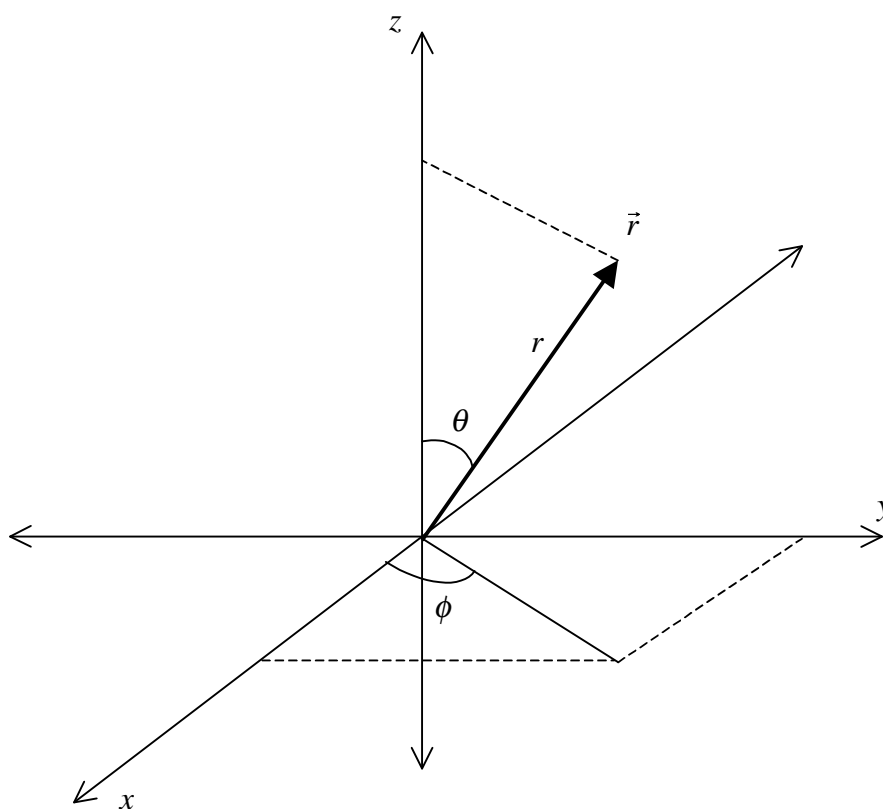


Figure 17. Illustration of spherical coordinates r , θ , and ϕ .

the chain rule derivation of the wave equation in spherical coordinates. It is the same kind of calculation we did for cylindrical coordinates. Here are the results. The Laplacian in

spherical coordinates is

$$\nabla^2 f(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}, \quad (13.5)$$

so the wave equation in spherical coordinates takes the form

$$\frac{1}{v^2} \frac{\partial^2}{\partial t^2} q(r, \theta, \phi, t) = \nabla^2 q(r, \theta, \phi, t), \quad (13.6)$$

where the Laplacian is given by (13.5) for each time t . Given the form of the Laplacian, this equation certainly looks formidable, but it can be converted into 4 tractable ordinary differential equations by separation of variables.

13.2 Separation of Variables in Spherical Coordinates

To solve (13.6) using the method of separation of variables we (i) assume the solution is of the form

$$q(r, \theta, \phi, t) = R(r)\Theta(\theta)\Phi(\phi)T(t), \quad (13.7)$$

(ii) substitute (13.7) into the wave equation, and (iii) divide the resulting equation by q . We obtain (*exercise*)

$$\frac{1}{v^2} \frac{T''}{T} = \frac{1}{r^2 R} \left(r^2 R' \right)' + \frac{1}{r^2 \sin \theta} \frac{1}{\Theta} \left(\sin \theta \Theta' \right)' + \frac{1}{r^2 \sin^2 \theta} \frac{\Phi''}{\Phi}. \quad (13.8)$$

We now perform the familiar separation of variables analysis. Right away we see that there is a constant k such that (*exercise*)

$$T'' = -v^2 k^2 T \quad (13.9)$$

and hence the complex form of the solution for T is of the form (*exercise*)

$$T = Ae^{\pm ivkt}, \quad (13.10)$$

as before. This is not too surprising; wave phenomena are always characterized by harmonically varying displacement in time—this cannot be affected by a choice of spatial coordinates.

We now continue the analysis to see how to characterize the spatial dependence of the waves. If we use (13.9) in (13.8) and multiply both sides by $r^2 \sin^2 \theta$, the usual logic implies that there is a constant, denoted by a , such that (*exercise*)

$$\Phi'' = -(a^2 + k^2)\Phi, \quad (13.11)$$

and we solve this via

$$\Phi = Be^{\pm im\phi}, \quad m = 0, 1, 2, \dots, \quad (13.12)$$

where

$$m^2 = a^2 + k^2. \quad (13.13)$$

Note that we have restricted m to integer values so that q is well-defined. Now we are left with

$$\frac{1}{r^2 \sin \theta} \frac{1}{\Theta} (\sin \theta \Theta')' - \frac{m^2}{\sin^2 \theta} = -\frac{1}{R} (r^2 R')' - k^2 r^2. \quad (13.14)$$

Again, each side must equal a constant, which is conventionally taken to be $-l(l+1)$. We thus get an equation for Θ :

$$\frac{1}{\sin \theta} (\sin \theta \Theta')' + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0, \quad (13.15)$$

and an equation for R :

$$(r^2 R')' + [k^2 r^2 - l(l+1)] R = 0. \quad (13.16)$$

Notice that the equation (13.15) for Θ depends on l and m , but not k , while the equation (13.16) for R depends on l and k but not m .

Once again we have reduced the wave equation to 4 ordinary differential equations. We could easily solve two of the equations (for T and Φ), but the equations for Θ and R are a little more complicated.

The equation for Θ defines a special function analogous to the cosine, sine, or Bessel function. This new special function is called the *associated Legendre function*, denoted by $P_{lm}(\theta)$. While we won't go into the details here, an important result of analyzing the equation for Θ is that (i) the constant l must be a non-negative integer:

$$l = 0, 1, 2, \dots \quad (13.17)$$

and (ii) for a given l , the allowed values of m are

$$m = -l, -l+1, \dots, -1, l-1, 0, 1, \dots, l. \quad (13.18)$$

The restrictions on l and m are needed for the solution to be well-defined at $\theta = 0, \pi$.^{*} This result is the motivation for defining the angular coordinate separation constant in the

^{*} As you can easily see, because spherical polar coordinates are not defined on the z axis the Laplacian in those coordinates is not defined there either. Strictly speaking, then, the equation can only be solved in the open interval $\theta \in (0, \pi)$ and the mathematics has no way of “knowing” that the solutions should exist at the endpoints of this interval. When we insist upon this we get the restriction on l and m . Requiring regularity at $\theta = 0, \pi$ also rejects another set of linearly independent solutions from consideration. (Generally speaking, there will be two linearly independent solutions to a second order ordinary differential equation.)

form $l(l+1)$. We shall not try to prove the results (13.17) and (13.18); you will probably see a proof in a more advanced course.

Some examples of associated Legendre functions (with convenient normalizations) are

$$P_{00} = 1, \quad P_{10} = \cos \theta, \quad P_{11} = \sin \theta, \quad P_{20} = \frac{1}{2}(3 \cos^2 \theta - 1), \quad P_{2,1} = 3 \sin \theta \cos \theta. \quad (13.19)$$

As a nice exercise you should verify that these functions do solve the equation for Θ with the indicated values of l and m . A general formula is obtained from

$$P_{lm} = \frac{1}{2^l l!} (1-x^2)^{|m|/2} \left(\frac{d}{dx} \right)^{|m|+l} (x^2-1) \quad (13.20)$$

where $x = \cos \theta$.

The equation for R — the “radial equation” — has solutions called *spherical Bessel functions* $R = j_l$ and *spherical Neumann functions* $R = n_l$. Note that these solutions depend upon the choice of k and l . All of these fancy special functions are well-studied and have well-understood properties. You can find a derivation of these solutions and discussion of their features in an upper level text. We will content ourselves with exhibiting a few of the spherical Bessel solutions and a general formula. The spherical Neumann functions are not defined at the origin (where they “become infinite”) so we will not bother with them here.† The first 3 spherical Bessel functions are

$$j_0 = \frac{\sin kr}{kr}, \quad j_1 = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}, \quad j_2 = \left(\frac{3}{(kr)^2} - \frac{1}{kr} \right) \sin kr - \frac{3}{(kr)^2} \cos kr. \quad (13.21)$$

You should check that these are solutions to the radial equation as a nice exercise. With $x = kr$ a general formula is

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x}. \quad (13.22)$$

13.3 Some Simple Solutions to the Wave Equation in Spherical Coordinates

The simplest solution to the wave equation in spherical coordinates is obtained by setting $l = 0$ in the separation of variables solution. When $l = 0$ it follows that $m = 0$ (exercise) and it is easy to verify that in this case both Θ and

† As in the cylindrically symmetric case, the spherical Neumann functions are useful when considering solutions to the wave equation which do not include the origin.

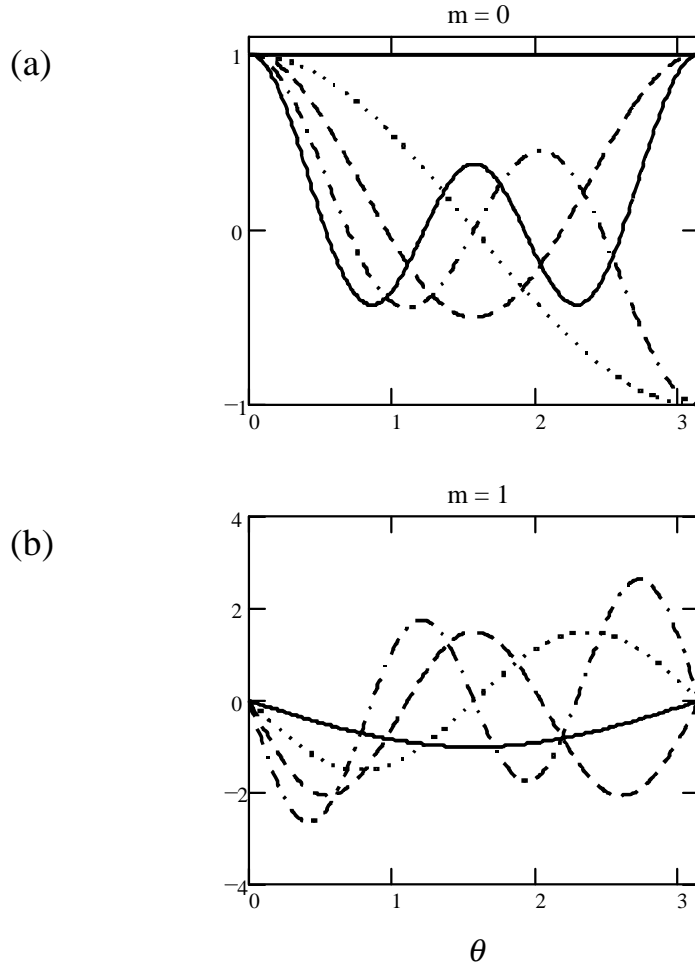


Figure 18. Some generalized Legendre functions. (a) five lowest order functions (in l) for $m = 0$. (b) four lowest order functions (in l) for $m = 1$.

Φ are constants (*exercise*). The radial function is given by the spherical Bessel function

$$R(r) = \frac{\sin kr}{kr}. \quad (13.23)$$

The other independent solution to the radial equation is the $l = 0$ spherical Neumann function, which is given by

$$R(r) = \frac{\cos kr}{kr}. \quad (13.24)$$

You can check that both of these radial functions solve the radial equation when $l = 0$. You can easily check as an *exercise* that the spherical Bessel function is well-behaved as $r \rightarrow 0$ (indeed, $R(0) = 1$) while the spherical Neumann function is not defined at $r = 0$,

as we mentioned earlier. Since we are considering solutions valid everywhere in space, we shall not consider the spherical Neumann function solution any further.*

The function (13.23) is sinusoidal in r with decreasing amplitude as r grows (much like the ordinary Bessel function). Note that $R(0) = 1$ by l'Hospital's rule (*exercise*). The spherically symmetric solution to the wave equation thus takes the form of a sinusoidal profile in r with a harmonic time dependence. We have (*exercise*)

$$q(r, \theta, \phi, t) = A \cos(\omega t + \alpha) \frac{\sin kr}{kr}, \quad (13.25)$$

where

$$\omega = kv. \quad (13.26)$$

The solution (13.25) is *spherically symmetric*, which means that at each time t the solution is the same everywhere on any sphere $r = \text{const.}$, as you can easily see from the fact that the solution is independent of θ and ϕ . Physically, the solution (13.25) can be viewed as representing the displacement (compression) of air due to a source at $r = 0$ emitting sound at a fixed frequency $\omega/2\pi$. The decrease of the amplitude at large distances represents the decreasing intensity of the emitted sound, which is required by conservation of energy (see §14).

As another example, let us suppose that $l = 1$, $m = 0$ so that

$$q(r, \theta, \phi, t) = A \cos(\omega t + \alpha) \cos \theta \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr} \right]. \quad (13.27)$$

In this case the solution depends upon r and θ but not ϕ . Such solutions are called *azimuthally symmetric* (or *axially symmetric*). As an *exercise* you should verify that this solution is well-defined at $r = 0$.

Exercise: Devise an example of a source of sound which would yield (at least ideally) azimuthally symmetric waves.

* If we consider solutions in a region not including the origin, then the solution to the wave equation with $l = 0$ is a superposition of the spherical Bessel and Neumann functions.

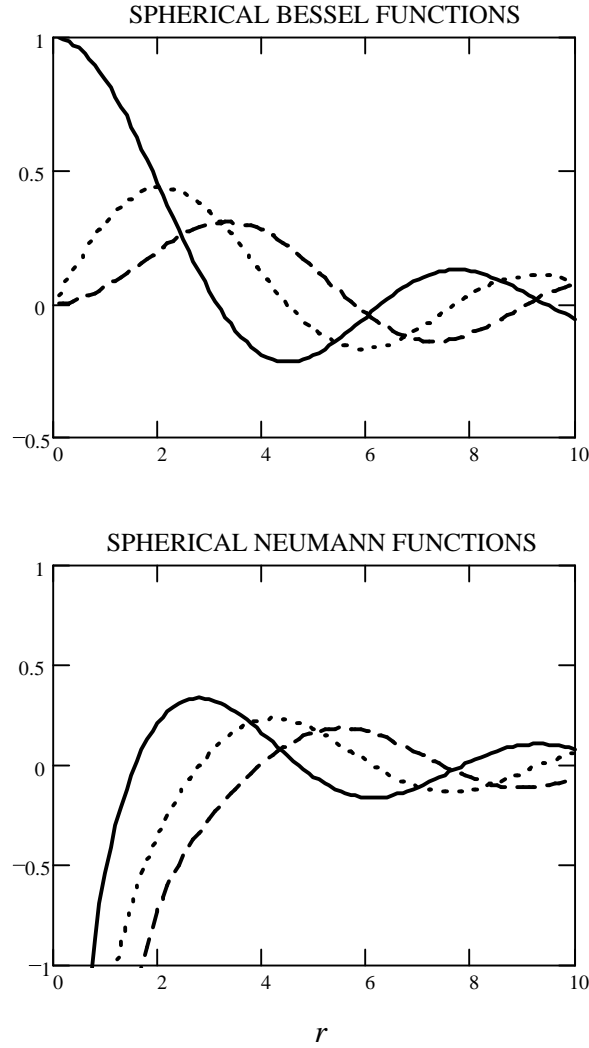


Figure 19. Spherical Bessel and spherical Neumann functions. In each graph the functions for $l = 0, 1$, and 2 are shown. Note that the spherical Neumann functions are all singular at the origin while the spherical Bessel functions remain finite.

We will not elaborate this point, but once again it can be shown that the general solution of the wave equation can be obtained by superpositions of the separation of variables solutions over all values of k, l, m . Note that this involves (i) an integral over k from 0 to ∞ , (ii) a sum over l from 0 to ∞ , and (iii) for each l , a sum over m from $-l$ to l . Thus the separation of variables solutions form a basis for the vector space of solutions to the wave equation.

Finally, we mention that various kinds of boundary conditions can be handled using

separation of variables. For boundary conditions which are imposed on surfaces where one of the coordinates is constant, *e.g.*, on a sphere centered at the origin, one simply imposes the boundary conditions on the solutions to the relevant ordinary differential equation (see the Problems for an example). This will, in general, limit the range of the separation constants and/or the allowed superpositions.

PROBLEM SET 6

Problem 6.1

Use the method of separation of variables to find a nonzero solution of the 3-dimensional wave equation that vanishes on the faces of a cube. (You can think of this as a mathematical model of sound waves in a room.)

Problem 6.2

Suppose that a function only depends upon the distance from the z -axis:

$$F(x, y, z) = f(\sqrt{x^2 + y^2}).$$

Consider the expression of F in cylindrical coordinates. Show that

$$\frac{\partial F}{\partial \phi} = 0,$$

- (i) directly in cylindrical coordinates (easy!)
- (ii) using the chain rule starting in Cartesian coordinates.

Problem 6.3

Show that spherical polar coordinates are identical to cylindrical coordinates when labeling points in the x - y plane.

Problem 6.4

Suppose that we are considering cylindrically symmetric solutions to the wave equation, $q = q(\rho, t)$. Starting from the wave equation in Cartesian coordinates, use the chain rule to derive the wave equation satisfied by $q(\rho, t)$.

Problem 6.5

If one looks for solutions to the wave equation that do not depend upon time, $q = q(\mathbf{r})$, then one must solve the Laplace equation

$$\nabla^2 q(\mathbf{r}) = 0.$$

Find the general form of the solution to this equation if one assumes

(a) cylindrical symmetry: $q = q(\rho)$

(b) spherical symmetry: $q = q(r)$.

Problem 6.6

Verify that the spherically symmetric functions

$$q_1(r, t) = \cos(kvt) \frac{\sin(kr)}{kr}, \quad q_2(r, t) = \cos(kvt) \frac{\cos(kr)}{kr}$$

solve the (three-dimensional) wave equation. Show that q_1 is well-defined at the origin while q_2 is not.

Problem 6.7

Find a non-zero solution to the wave equation that is spherically symmetric and vanishes on the surface of a sphere (centered at the origin, radius R).

Problem 6.8

Using your favorite mathematical software, write a program to compute the spherical Bessel functions from the formula (13.22). Verify the results shown in (13.21).

Problem 6.9

Using your favorite mathematical software, write a program to display an animation depicting the cylindrically symmetric solution (12.38).

14. Conservation of Energy.

After all of these developments it is nice to keep in mind the idea that the wave equation describes (a continuum limit of) a network of coupled oscillators. This raises an interesting question. Certainly you have seen by now how important energy and momentum — and their conservation — are for understanding the behavior of dynamical systems such as an oscillator. If a wave is essentially the collective motion of many oscillators, might not there be a notion of conserved energy and momentum for waves? If you've ever been to the beach and swam in the ocean you know that waves do indeed carry energy and momentum which can be transferred to other systems. How to see energy and momentum and their conservation laws emerge from the wave equation? One way to answer this question would be to go back to the system of coupled oscillators and try to add up the energy and momentum of each oscillator at a given time and take the continuum limit to get the total energy and momentum of the wave. Of course, the energy and momentum of each individual oscillator is not conserved (*exercise*). Indeed, the propagation of a wave depends upon the fact that the oscillators are coupled, *i.e.*, can exchange energy and momentum. What we want to do here, however, is to show how to keep track of this energy flow in a wave, directly from the continuum description we have been developing. This will allow us to define the energy (and momentum) densities of the wave as well as the total energy contained in a region.

14.1 The Continuity Equation

To begin, let us limit attention to one spatial dimension for simplicity. You can think of the system under study as a rope or string under tension. The transverse displacement of the rope from equilibrium at the position x at time t is denoted by $q(x, t)$ as before. We understand how to analyze wave phenomena in this sort of system and we now seek some kind of energy which is conserved.

As it happens, conservation of energy* in the dynamics of a continuous medium mathematically manifests itself in the existence of a *continuity equation*. With one spatial dimension, this is an equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0, \quad (14.1)$$

where $\rho = \rho(x, t)$ and $j = j(x, t)$ are some functions built from $q(x, t)$ and its derivatives. We shall give some formulas for ρ and j in terms of q in a moment. Do not confuse ρ with the radial coordinate of the cylindrical coordinate system. The continuity equation (14.1) is to be satisfied whenever $q(x, t)$ satisfies the wave equation. The physical meaning of ρ and j are as follows. The function $\rho(x, t)$ represents the *energy density* (actually, energy

* Or for that matter, conservation of any other extensive quantity, *e.g.*, mass, charge, *etc.*

per unit length in our one-dimensional model) of the wave at the point x and time t . The function $j(x, t)$ represents the x -component of the *energy current density* of the wave.* The energy current density is a vector field which represents the flow of energy from point to point (*i.e.*, reflects the exchange of energy among the oscillators).

The continuity equation is an “energy balance” equation. To see this, view $\rho(x, t)dx$ as the energy contained in an infinitesimal region, $[x, x + dx]$ at time t . Eq. (14.1) says that the time rate of change of energy in an infinitesimal region ($\frac{\partial \rho}{\partial t}dx$) is given by (*exercise*)

$$-\frac{\partial j(x, t)}{\partial x}dx = j(x)dx - j(x + dx), \quad (14.2)$$

which should be viewed as the net flow of energy *into* the region per unit time.† This is precisely the familiar statement of the principle of conservation of energy: “energy cannot be created or destroyed”.

To see this conservation of energy business from a non-infinitesimal point of view, consider the energy $E(t)$ contained in a region $a \leq x \leq b$ at time t (a and/or b can be infinite). We get this by adding up the energy per unit length with an integral:

$$E(t) = \int_a^b \rho(x, t) dx. \quad (14.3)$$

From the continuity equation, the time rate of change of energy in this region is given by

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_a^b \frac{\partial \rho(x, t)}{\partial t} dx \\ &= - \int_a^b \frac{\partial j(x, t)}{\partial x} dx \\ &= j(a, t) - j(b, t). \end{aligned} \quad (14.4)$$

Physically, the expression $j(a, t) - j(b, t)$ is the energy entering the region per unit time. (Keep in mind that if j is positive it indicates a flow of energy in the positive x direction.) Mathematically, $j(a, t) - j(b, t)$ is *negative* the flux‡ of the energy current vector field through the closed region $[a, b]$ (*exercise!*). In particular, a positive (negative) flux indicates an energy flow out (into) the region (*exercise*). If the net flux is zero, then the energy in the region is conserved. This can happen because the flux vanishes at the boundaries of the region, or because the inward flux in at one endpoint of the region is balanced by the

* Of course, in one spatial dimension the x component is the only component of a vector field.

† If $j(x)$ is positive (negative) then the energy current density vector points into (out of) the region at x . Similarly, if $j(x + dx)$ is positive (negative) the energy current density vector points out of (into) the region at $x + dx$.

‡ Here “flux” is defined just as in the context of the Gauss law of electrostatics. We shall define it formally a little later.

outward flux out at the other. If the net flux is non-zero, the system in the region $[a, b]$ is exchanging energy with its “environment” outside of $[a, b]$.[†]

Let us amplify these last remarks a bit by considering boundary conditions in a little more detail. If no boundary conditions are imposed than the waves are free to transmit energy to and from the “environment”, that is, the region outside of $[a, b]$. The energy in the interval is not constant and one says that one has an “open system”. If the wave displacement vanishes at the endpoint of the interval,

$$q(a, t) = 0 = q(b, t) \quad (14.5)$$

then the energy in the interval is constant. This would occur, for example, if one considers a vibrating string under tension with fixed ends (think: guitar string).^{*} One can also let the interval become infinite, in which case the conditions (14.5) are asymptotic conditions. If we consider periodic boundary conditions,

$$q(a, t) = q(b, t) \quad (14.6)$$

then the energy in the interval is constant, basically because energy that flows “out” at $x = b$ is also flowing back “in” at $x = a$. In both of these cases one can view the system as “closed”, that is, not interacting with its environment.

By the way, while we have been using the word “energy” in the discussion above, it is clear that we have not really used any fact except that ρ and j satisfy a continuity equation. The conservation law (14.4) will arise no matter what is the physical interpretation of ρ and j . Indeed, we will see later how a continuity equation is used to describe electric charge conservation in electrodynamics. The connection with energy comes when we specify exactly how ρ and j are to be constructed from the wave displacement q . For the energy density we define

$$\rho = \frac{1}{2} \left[\left(\frac{\partial q}{\partial t} \right)^2 + v^2 \left(\frac{\partial q}{\partial x} \right)^2 \right]. \quad (14.7)$$

You can think of the first term in (14.7) as a kinetic energy (density) and the second term as a potential energy (density). Indeed, the form of the energy per unit length can be

[†] You may wonder how this way of thinking about conservation laws jibes with the conservation laws you have seen for mechanical systems, *e.g.*, coupled oscillators. Roughly speaking, the conserved total energy involves a sum of the individual energies (including any interaction energy) over the degrees of freedom of the system. In the continuum, such as we have here, the degrees of freedom are – in effect – being labeled by x and the sum over degrees of freedom is the integral over x .

^{*} Of course, real vibrating strings eventually stop vibrating because the energy of the strings is transferred elsewhere by dissipative processes. We are obviously not including these effects here.

understood via the continuum limit of the total energy of the chain of oscillators. For the energy current density we have

$$j = -v^2 \frac{\partial q}{\partial t} \frac{\partial q}{\partial x}. \quad (14.8)$$

You can think of j as being proportional to a momentum density for the wave.

Remarks:

- (1) Given a (system of) partial differential equation(s), there are systematic mathematical ways to find functions ρ and j satisfying a continuity equation (if such an equation exists). When such techniques are applied to the wave equation one can derive the formulas (14.7) and (14.8).
- (2) Our use of the words “energy” and “momentum” is slightly misleading because of dimensional considerations. Depending upon the units of the wave displacement $q(x, t)$, the quantities ρ and j/v^2 need not have dimensions of energy per unit length, or momentum per unit length. You can check as a nice *exercise* that if q has dimensions of square root of mass times square root of distance, then ρ has dimensions of energy per unit length and j/v^2 has dimensions of momentum per unit length. Of course, q need not have these units since it can represent variety of physical displacements (position, current, temperature, *etc.*). Thus in applications you will often see other dimensionful constants in the definition ρ and j . For example, suppose that q represents a spatial displacement, *e.g.*, in a vibrating string, or in a fluid or in an elastic medium (such as a continuum limit of a chain of oscillators), so that q has units of length. Then, we need a constant μ , with dimensions of mass per unit length, to appear in the energy density:

$$\rho = \frac{\mu}{2} \left[\left(\frac{\partial q}{\partial t} \right)^2 + v^2 \left(\frac{\partial q}{\partial x} \right)^2 \right].$$

Similarly, in this example we would define

$$j = -\mu v^2 \frac{\partial q}{\partial t} \frac{\partial q}{\partial x}.$$

The constant μ would represent, say, the mass density (really mass per unit length) of the one-dimensional medium. You can easily see that we could absorb μ into the definition of q : $q \rightarrow q/\sqrt{\mu}$, thus giving q units of square root of mass times square root of distance. To keep things looking reasonably simple, we assume that such a redefinition has been performed in what follows.

Now we show that, when q satisfies the wave equation, ρ and j satisfy the continuity

equation. The time derivative of ρ is

$$\frac{\partial \rho}{\partial t} = \frac{\partial q}{\partial t} \frac{\partial^2 q}{\partial t^2} + v^2 \frac{\partial q}{\partial x} \frac{\partial^2 q}{\partial x \partial t}. \quad (14.9)$$

The space derivative of j is

$$\frac{\partial j}{\partial x} = -v^2 \left[\frac{\partial q}{\partial t} \frac{\partial^2 q}{\partial x^2} + \frac{\partial q}{\partial x} \frac{\partial^2 q}{\partial t \partial x} \right]. \quad (14.10)$$

The continuity equation (14.1) is now easily seen to be satisfied (*exercise*). In detail, the first terms in (14.9) and (14.10) cancel because q satisfies the wave equation, and the second terms in (14.9) and (14.10) cancel thanks to the commutativity of partial derivatives.

We emphasize that the continuity equation, and the conservation of energy it represents, only holds for solutions to the wave equation. This is analogous to the situation in Newtonian mechanics where the energy of a particle,

$$E = \frac{1}{2} m \dot{\mathbf{r}}^2(t) + V(\mathbf{r}(t)), \quad (14.11)$$

is conserved,

$$\frac{dE}{dt} = 0, \quad (14.12)$$

provided the particle obeys its equations of motion, *i.e.*, Newton's second law:

$$\frac{d^2 \mathbf{r}(t)}{dt^2} = -\frac{1}{m} \nabla V(\mathbf{r}(t)). \quad (14.13)$$

But the energy will in general not be conserved for motions of the system $\mathbf{r}(t)$ not obeying Newton's second law (*exercise*).

14.2 Some simple examples

Let us look at a couple of examples. First, we consider a sinusoidal wave in one dimension:

$$q(x, t) = A \cos(kx - \omega t), \quad (14.14)$$

where $\omega/k = v$. The energy density is (*exercise*)

$$\rho(x, t) = (A\omega)^2 \sin^2(kx - \omega t). \quad (14.15)$$

The energy current density is (*exercise*)

$$j(x, t) = A^2 \omega k v^2 \sin^2(kx - \omega t). \quad (14.16)$$

Note that for $k > 0$ the wave moves to the right, and the energy current is positive. This reflects a net transport of energy to the right. If $k < 0$, then the wave moves to the left and the current is correspondingly negative.

The total energy in a region $-L \leq x \leq L$ is (see Problem Set 6)

$$E(t) = (A\omega)^2 \left[L - \frac{1}{4k} \sin\{2(kL - \omega t)\} - \frac{1}{4k} \sin\{2(kL + \omega t)\} \right]. \quad (14.17)$$

As an exercise you can check that E in (14.17) is positive as it should be. (*Exercise: Why would you expect $E(t)$ to be positive?*) The net flux of energy into the region is

$$j(-L, t) - j(L, t) = A^2 \frac{\omega^3}{k} \left[\sin^2(kL + \omega t) - \sin^2(kL - \omega t) \right] = A^2 \frac{\omega^3}{k} \sin(2kL) \sin(2\omega t). \quad (14.18)$$

It is interesting to examine the special case in which the size of the region is an integral number of half-wavelengths: $2L = \frac{n}{2}\lambda$, $n = 1, 2, 3, \dots$, i.e., $kL = \frac{1}{2}\pi n$. In this case the energy (14.17) is time-independent (*exercise*):

$$E = \frac{n\lambda}{8} (A\omega)^2. \quad (14.19)$$

As you should expect, the net energy flux vanishes:

$$j(-L, t) - j(L, t) = 0, \quad (14.20)$$

although the energy current is not identically zero at either boundary point. This result can be interpreted as saying that when $2L = \frac{1}{2}n\lambda$, the energy leaving the box from the right is exactly matched by the energy entering the box from the left.

A similar phenomenon occurs with a standing wave (*exercise*). Let us consider a solution of the form

$$q(x, t) = \begin{cases} A \sin\left(\frac{n\pi}{L}x\right) \cos \omega t, & -L < x < L; \\ 0 & |x| \geq L, \end{cases} \quad (14.21)$$

where $n = 1, 2, \dots$. You can easily check that this is a solution of the wave equation provided $\omega = \frac{n\pi}{L}v$ (*exercise*). Note in particular that $q(\pm L, t) = 0$. Equation (14.21) represents a standing wave. It can be obtained by superposing a left and right moving traveling wave in the region $-L < x < L$ (check this as an *exercise*). You can visualize it as a bit of taut string of length $2L$ held fixed at $x = \pm L$, e.g., in a guitar. The different values of n label the different harmonics you can get when you pluck the string. By pinning down the string at $x = \pm L$ we prohibit energy from being transferred outside of that region. Instead, the energy of the wave is reflected. Physically we would expect that

the total energy in the region $-L < x < L$ is conserved (up to dissipative effects that are not modeled by the wave equation). Let us check this. The energy density is (exercise)

$$\rho(x, t) = \frac{1}{2} A^2 \omega^2 \left[\sin^2\left(\frac{n\pi}{L}x\right) \sin^2 \omega t + \cos^2\left(\frac{n\pi}{L}x\right) \cos^2 \omega t \right], \quad -L < x < L, \quad (14.22)$$

and zero elsewhere. The total energy is

$$\begin{aligned} E(t) &= \int_{-\infty}^{\infty} \rho(x, t) dx, \\ &= \int_{-L}^L \rho(x, t) dx \\ &= \frac{1}{2} A^2 \omega^2 L, \end{aligned} \quad (14.23)$$

which is indeed a constant in time. Note that the energy varies as the square of the amplitude and frequency and is proportional to the size of the region where the wave is allowed to exist. Because the energy is time independent, from the continuity equation we expect that the flux of energy at $x = \pm L$ should vanish. We have

$$j(x, t) = \frac{1}{2} v^2 \omega \frac{n\pi}{L} \sin\left(\frac{2n\pi}{L}x\right) \cos(2\omega t), \quad (14.24)$$

and from this expression you can see that, for each value of n ,

$$j(\pm L, t) = 0. \quad (14.25)$$

We have seen that if the wave is confined to a finite region (with appropriate boundary conditions), the total energy in that region will be conserved. We can generalize this to arbitrarily large regions. In particular, suppose that $q(x, t)$ vanishes as $x \rightarrow \pm\infty$. Our Gaussian wave (7.29) is a good example of such a solution to the wave equation (exercise). In this case you can check that even though the energy in any finite region need not be constant in time, the energy contained in all of space is a constant in time. This is because j will vanish as $x \rightarrow \pm\infty$. So, while different parts of the medium (string, air, etc.) can exchange energy, the total energy cannot be lost (provided no waves are “at infinity” after finite time).

14.3 The Continuity Equation in 3 Dimensions

Let us now see how our preceding discussion generalizes to waves which are 3-dimensional in nature. To begin, we need to generalize the continuity equation. We note that the energy density $\rho = \rho(\mathbf{r}, t)$ is a *scalar field*, as is its time derivative: they both assign a number (a scalar) to each point of space (and instant of time). The energy current density is a *vector field*: it has a magnitude and direction at each point of space (and instant of time).

We denote this vector field by $\mathbf{j} = \mathbf{j}(\mathbf{r}, t)$. Evidently, the continuity equation equates space derivatives of \mathbf{j} with time derivatives of ρ , which means we need to make a scalar from the derivatives of \mathbf{j} . This is done using the divergence (see §9.1); the continuity equation in 3 dimensions takes the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (14.26)$$

It is instructive to see how (14.26) leads to a notion of conservation of energy in this 3-dimensional setting. We begin with a simple example. Consider a cubical region defined by $0 \leq x \leq L$, $0 \leq y \leq L$, and $0 \leq z \leq L$. Now integrate the continuity equation over the volume of the box. Defining

$$E(t) = \int_0^L \int_0^L \int_0^L \rho(\mathbf{r}, t) dx dy dz, \quad (14.27)$$

we find (good exercise)

$$\begin{aligned} \frac{dE(t)}{dt} = - \int_0^L \int_0^L \left\{ [j^x(L, y, z, t) - j^x(0, y, z, t)] dy dz \right. \\ + [j^y(x, L, z, t) - j^y(x, 0, z, t)] dx dz \\ \left. + [j^z(x, y, L, t) - j^z(x, y, 0, t)] dx dy \right\}. \end{aligned} \quad (14.28)$$

To get this result we wrote out the divergence and used the fundamental theorem of integral calculus. Compare this result with (14.4).

Again we interpret the formula for dE/dt as equating the time rate of change of energy contained in the volume of the box with the net flux of energy into the box through the walls. This is what we mean by conservation of energy. The first term on the right hand side of (14.28) is the flux through the faces at $x = 0$ and $x = L$. In particular, we see that the x component of \mathbf{j} controls the flux in the x direction. You can easily see that we have corresponding statements for the y and z components of the energy current density and the flux. If the net flux through the walls in the box vanish, then the energy in the cube will be constant in time.

We now provide the generalizations to three dimensions of formulas for the energy density and energy current density in terms of a solution $q(\mathbf{r}, t)$ to the wave equation. The energy density is

$$\rho = \frac{1}{2} \left[\left(\frac{\partial q}{\partial t} \right)^2 + v^2 \nabla q \cdot \nabla q \right], \quad (14.29)$$

and the current density is

$$\mathbf{j}(\mathbf{r}, t) = -v^2 \frac{\partial q}{\partial t} \nabla q. \quad (14.30)$$

You should compare these formulas with their one-dimensional counterparts (14.7) and (14.8). The function $\rho(\mathbf{r}, t)$ represents the energy per unit volume at the point \mathbf{r} at the time t . The vector field $\mathbf{j}(\mathbf{r}, t)$ represents the net energy transported across a surface, per unit area per unit time, at the point \mathbf{r} at the time t . As an exercise you can verify, just as we did in 1 dimension, that ρ and \mathbf{j} satisfy the 3-d continuity equation provided q satisfies the wave equation.

14.4 Divergence theorem

We did not have to use a rectangular box to compute the energy — any region would do. The continuity equation still guarantees conservation of energy. It is worth spelling this out in detail since it brings into play a very important integral theorem known as *the divergence theorem*, which you may have encountered in electrostatics (via Gauss's law).

To begin, here is the geometric setting. Let V be a three-dimensional region (volume) enclosed by a surface S . At each point of the surface we have the unit normal \mathbf{n} , which by definition points out of V . For example, V could be the volume contained in a sphere of radius R . In this case the unit normal, at the point with coordinates (x, y, z) on the sphere $x^2 + y^2 + z^2 = R^2$, can be written as (*exercise*)

$$\mathbf{n} = \frac{x}{R}\mathbf{i} + \frac{y}{R}\mathbf{j} + \frac{z}{R}\mathbf{k}, \quad (14.31)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the usual x, y, z unit vectors. More generally, if the surface S is expressed as the level surface of a function,

$$f(x, y, z) = \text{constant}, \quad (14.32)$$

then the unit normal can be obtained by (i) taking the gradient ∇f , (ii) evaluating the vector field on the surface (14.32), and then (iii) normalizing the result so that you have a unit vector at each point of S . You can check this recipe for the sphere example above, where $f = x^2 + y^2 + z^2$ (*exercise*).

The *divergence theorem* states that the integral of the divergence of a vector field \mathbf{A} over a volume V is the same as the *flux* of the vector field through the bounding surface S . In formulas:

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot \mathbf{n} dS, \quad (14.33)$$

where dV and dS are the volume and surface elements respectively. The little circle on the surface integral sign is there to remind us that the surface is *closed*, that is, it has no boundary.* The divergence theorem is essentially an elaborate use of the fundamental

* For example, a sphere, *i.e.*, the surface of a ball in 3-d, is a closed surface. A hemisphere is not a closed surface. The boundary surface of a volume is always a closed surface.

theorem of integral calculus (in the guise of repeated integration by parts — re-examine our cube example). Indeed, you can easily check that (14.33), gives the result we obtained by direct integration in (14.28) for dE/dt when the volume is a cube of length L (see the homework problems).

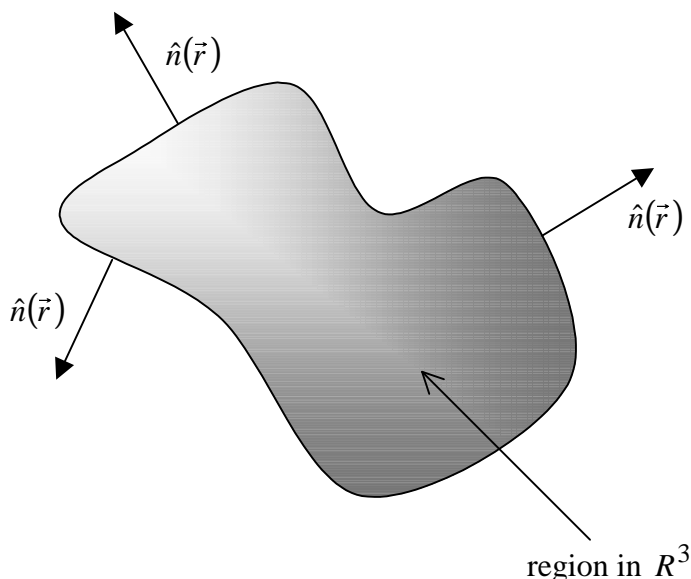


Figure 20. An enclosed region in R^3 illustrating several outwardly directed normal vectors associated with the surface enclosing the region.

With the divergence theorem in hand we can consider the integrated version of the continuity equation — *i.e.*, the energy conservation law. It reads (*exercise*)

$$\frac{d}{dt} \int_V \rho dV = - \oint_S \mathbf{j} \cdot \mathbf{n} dS, \quad (14.34)$$

showing that the change of energy in the region is *minus* the energy flux through the boundary, which is the net energy flow *into* the region.

Incidentally, by using the identity (14.33) we can recover our previous interpretation of the divergence, presented in §9. As we stated in that section, the idea is simply that in a limit in which the volume V becomes arbitrarily small (“infinitesimal”), the integral on the left side of (14.33) becomes — to an arbitrarily good approximation — $(\nabla \cdot \mathbf{A}) V$. Dividing both sides by V we see that, the divergence of \mathbf{A} is the flux per unit volume through the boundary of the small volume V .

14.5 Applications of the 3-Dimensional Continuity Equation

Let us look at an illustrative example. Consider a spherical wave (see §13.3):

$$q(\mathbf{r}, t) = A \frac{\sin(kr)}{kr} \cos \omega t, \quad (14.35)$$

where $\omega = kv$. The energy density associated with (14.35) is (exercise)

$$\rho = \frac{1}{2} A^2 \left(\omega^2 \sin^2 \omega t \frac{\sin^2 kr}{(kr)^2} + v^2 \cos^2 \omega t \left[\frac{\cos kr}{r} - \frac{\sin kr}{kr^2} \right]^2 \right). \quad (14.36)$$

Let us consider the time rate of change of energy $E(t)$ contained in a sphere of radius R centered at the origin:

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_0^{2\pi} \int_0^\pi \int_0^R r^2 \sin \theta \frac{\partial \rho(r, t)}{\partial t} d\phi d\theta dr \\ &= 4\pi \int_0^R r^2 \frac{\partial \rho(r, t)}{\partial t} dr \\ &= -4\pi A^2 v^3 \sin \omega t \cos \omega t \sin kR \left(\cos kR - \frac{\sin kR}{kR} \right). \end{aligned} \quad (14.37)$$

The result of the integral appearing in (14.37) is probably not obvious to you, but can be obtained using the usual tricks. You might try checking the result using your favorite symbolic mathematics software. Note that if $R = n\pi/k$, *i.e.*, R is a multiple of half a wavelength, then the energy in the sphere is conserved, this can be viewed as a spherically symmetric version of the standing wave phenomenon (exercise).

According to our interpretation of the continuity equation via the divergence theorem, $\frac{dE}{dt}$ should be equal to minus the flux of the energy current through the sphere of radius R . This is easily checked because of spherical symmetry. In detail, the energy current is (exercise)

$$\mathbf{j} = A^2 \omega v^2 \frac{\sin kr}{kr} \sin \omega t \cos \omega t \left[\frac{\cos kr}{r} - \frac{\sin kr}{kr^2} \right] \frac{\mathbf{r}}{r}. \quad (14.38)$$

The normal to the sphere of radius r is $\mathbf{n} = \frac{\mathbf{r}}{r}$, so that, with S being the sphere of radius R

$$\begin{aligned} \oint_S \mathbf{j} \cdot \mathbf{n} d^2 S &= \int_0^{2\pi} \int_0^\pi R^2 \sin \theta A^2 \omega v^2 \frac{\sin kR}{kR} \sin \omega t \cos \omega t \left[\frac{\cos kR}{R} - \frac{\sin kR}{kR^2} \right] d\theta d\phi \\ &= 4\pi A^2 v^3 \sin kR \sin \omega t \cos \omega t \left[\cos kR - \frac{\sin kR}{kR} \right]. \end{aligned} \quad (14.39)$$

where the integral was easily evaluated because the energy current was spherically symmetric! Comparing with the result (14.37) we obtained earlier for the time rate of change of energy in the sphere we see that the conservation law is satisfied.

Finally, let us consider the flux of energy at large distances, *i.e.*, we consider the limit of (14.39) as $R \rightarrow \infty$. More precisely, we consider the leading order behavior of the flux as the boundary of the region of interest is taken to be at a radius which is much larger than the wavelength of the wave. By inspecting this formula you will see that at large distances we have (*exercise*)

$$\oint_S \mathbf{j} \cdot \mathbf{n} d^2S \approx \pi A^2 v^3 \sin(2kR) \cos(2\omega t), \quad kR \gg 1. \quad (14.40)$$

You see that despite the fact that the wave amplitude decreases with increasing r , the energy flux through a sphere does not in fact decrease. This is because the spherical wave has an energy current density that decreases slowly with radius, *i.e.*, like $1/r^2$. This is the prototypical behavior of a *radiation field*, that is a wave disturbance produced by a bounded source (a point source at the origin in this example) that carries energy off “to infinity”, *i.e.*, to arbitrarily large distances from the source.

PROBLEM SET 7

Problem 7.1

Recall the mechanical system consisting of two coupled oscillators. The kinetic energy T for the system is defined as usual ($T = \frac{1}{2}m(v_1^2 + v_2^2)$). The potential energy is denoted by $V(x_1, x_2)$ and is defined so that the force, F_i , on the i^{th} particle ($i = 1, 2$) is given by

$$F_i = -\frac{\partial V}{\partial x_i}.$$

Find the form of V , and prove that the total energy $E = T + V$ is conserved, that is, $\frac{dE}{dt} = 0$ for solutions of the equations of motion.

Problem 7.2

Solutions to the wave equation have a conserved momentum. The momentum density for a wave $q(x, t)$ is defined by

$$\rho = \frac{\partial q}{\partial t} \frac{\partial q}{\partial x}.$$

Find the corresponding momentum current density j for the wave. (*Hint:* Use the continuity equations.)

Problem 7.3

Recall the Gaussian wave

$$q(x, t) = A \left[e^{-(x-vt)^2} + e^{-(x+vt)^2} \right].$$

Compute the total energy contained in this wave by integrating the energy density $\rho(x, t)$ over all x and show that the result does not depend upon the time t .

Problem 7.4

In the previous problem, it is shown that the total energy of the Gaussian wave is time independent. Explain this result by showing that the energy current density j vanishes as $x \rightarrow \infty$.

Problem 7.5

Verify that

$$\rho(\mathbf{r}, t) = \frac{1}{2} \left[\left(\frac{\partial q}{\partial t} \right)^2 + v^2 (\nabla q)^2 \right],$$

and

$$\mathbf{j}(\mathbf{r}, t) = -v^2 \frac{\partial q}{\partial t} \nabla q.$$

satisfy the continuity equation when q satisfies the (3-d) wave equation.

Problem 7.6

If $q(\mathbf{r}, t)$ depends only upon x and t (*i.e.*, q is independent of y and z) show that the 3-dimensional forms for the energy density, energy current density, and continuity equation reduce to the 1-dimensional results.

Problem 7.7

Use the divergence theorem (14.33) to verify (14.28).

Problem 7.8

Verify (14.17). Show that the time rate of change of energy in the region is the net flux of energy into the region (14.18).

Problem 7.9

Derive the approximate formula (14.40).

Problem 7.10

Show that the quantity

$$\Pi(t) = \int_{-\infty}^{\infty} dx \frac{\partial q(x, t)}{\partial t}$$

is independent of t (*i.e.*, is a conserved quantity) for all solutions q of the one-dimensional wave equation whose first derivatives vanish at infinity,

$$\lim_{x \rightarrow \pm\infty} \frac{\partial q(x, t)}{\partial x} = 0.$$

15. The Schrödinger Equation.

An important feature of the wave equation is that its solutions $q(\mathbf{r}, t)$ are uniquely specified once the initial values $q(\mathbf{r}, 0)$ and $\partial q(\mathbf{r}, 0)/\partial t$ are specified. As was mentioned before, if we view the wave equation as describing a continuum limit of a network of coupled oscillators, then this result is very reasonable since one must specify the initial position and velocity of an oscillator to uniquely determine its motion. It is possible to write down other “equations of motion” that exhibit wave phenomena but which only require the initial values of the dynamical variable — not its time derivative — to specify a solution. This is physically appropriate in a number of situations, the most significant of which is in quantum mechanics where the wave equation is called the *Schrödinger equation*. This equation describes the time development of the observable attributes of a particle via the *wave function* (or probability amplitude) ψ . In quantum mechanics, the complete specification of the initial conditions of the particle’s motion is embodied in the initial value of ψ . The price paid for this change in the allowed initial data while asking for a linear wave equation is the introduction of complex numbers into the equation for the wave. Indeed, the values taken by ψ are complex numbers. In what follows we shall explore some of the elementary features of the wave phenomena associated with the Schrödinger equation.

15.1 One-Dimensional Schrödinger equation

Let us begin again in one spatial dimension, labeled by x . We consider a *complex-valued function* ψ . This is a function that associates a complex number $\psi(x, t)$ to each point x of space and instant t of time. In other words, at each (x, t) , $\psi(x, t)$, is a complex number. Consequently, we can — if desired — break ψ into its real and imaginary parts:

$$\psi(x, t) = f(x, t) + ig(x, t), \quad (15.1)$$

where f and g are real functions. We can also use a polar representation:

$$\psi(x, t) = R(x, t)e^{i\Theta(x, t)}, \quad R \geq 0. \quad (15.2)$$

See §1.3 for a review of complex variable notation.

The complex-valued function ψ is called the *wave function* — you’ll see why shortly. The wave function ψ is required to satisfy the *Schrödinger equation*:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}. \quad (15.3)$$

Here $V = V(x, t)$ is some given real-valued function of space and time representing the potential energy function of the particle, \hbar is Planck’s constant (h) divided by 2π , and

m is a parameter representing the mass of the particle. The Schrödinger equation is a complex, linear, homogeneous, partial differential equation with variable coefficients (thanks to $V(x, t)$). It is equivalent to a pair of real, coupled, linear differential equations for the real and imaginary parts of ψ as you can see by using the fact that equality of complex numbers means separate equality of their real and imaginary parts (*exercise*).

The Schrödinger equation specifies the time evolution of a quantum mechanical particle,* thus it plays a role in quantum mechanics roughly akin to the role played by the famous $\vec{F} = m\vec{a}$ in Newtonian mechanics. While we often speak of *the* Schrödinger equation, strictly speaking there is no single differential equation valid for all situations. Rather, each potential energy function defines a Schrödinger equation appropriate to the physical system. This is also true with $\vec{F} = m\vec{a}$ in Newtonian mechanics; one uses different forms for \vec{F} depending upon the physical situation. We also note that in some applications of the Schrödinger equation it is useful to allow the potential energy function V to be complex valued. Such potentials can be used to model processes involving particle decay. For simplicity we shall assume that the potential energy function is real. (See the Problems and also §15.4 for hints as to what happens when we let V be complex-valued.)

While we certainly won't be offering a course in quantum mechanics in this text, it is worth commenting on the physical meaning of solutions to (15.3). The simplest use of the wave function ψ is via the rule that $\psi^*(x, t)\psi(x, t) dx$ is the probability that a measurement of the particle's position at time t will find the particle in the region between x and $x + dx$. More complicated expressions involving ψ are used to give the probability distributions for other particle observables. You will have a chance to get used to such ideas in a later course in quantum mechanics. Fortunately, we do not really need to understand much of quantum mechanics in order to see the basic wave phenomena embodied in the Schrödinger equation. Still, from time to time it will be appropriate to make a few remarks concerning the physical interpretation of some of our results.

15.2 Free Particle Solution of the Schrödinger Equation

Let us now try to understand the sense in which (15.3) is a wave equation. This is most easily done by considering the special case $V(x, t) = 0$, which physically corresponds to the motion of a free particle. Although you can probably guess solutions to this equation, let us apply some of the techniques we have developed.

We begin with separation of variables; we try a solution of the form

$$\psi(x, t) = X(x)T(t), \quad (15.4)$$

* There is also a Schrödinger equation for systems of particles, not to mention even more exotic dynamical systems. But we will stick to the simplest case of a single particle.

and substitute to find (*exercise*):

$$-\frac{\hbar}{2m} \frac{X''}{X} = i \frac{T'}{T}. \quad (15.5)$$

As usual, we conclude that

$$-\frac{\hbar}{2m} X'' = \alpha X, \quad (15.6)$$

and

$$iT' = \alpha T,$$

where α is some constant. These equations are easily solved:

$$X(x) = Ae^{ikx}, \quad T(t) = Be^{-i\omega(k)t}, \quad (15.7)$$

where k is any constant and

$$\omega(k) = \frac{\hbar k^2}{2m}. \quad (15.8)$$

Note that we could have written $X(x) = Ae^{\pm ikx}$, but we can get both choices of sign by choosing k positive or negative, so for simplicity we drop the \pm . Keep in mind, though, that for a given ω there are two independent solutions for $X(x)$, namely $e^{\pm i|k|x}$.

Since ψ is complex-valued, there is no obvious *a priori* restriction on whether k is real or not. As it happens, physical considerations in conjunction with the principles of quantum mechanics end up requiring k to be real in this example, so we'll only consider that case. The solution

$$\psi(x, t) = Ce^{i(kx - \omega(k)t)} \quad (15.9)$$

is then a complex form of a traveling wave (*i.e.*, its real and imaginary parts are traveling waves). We do not need to take the real part of ψ , however, since the wave function is allowed to be complex. Like the wave equation, the Schrödinger equation is linear and homogeneous. This means that one can take linear combinations of solutions (with complex coefficients) to get new solutions — a fact that has far-reaching physical consequences in quantum mechanics. The *general solution* of the free particle ($V(x, t) = 0$) Schrödinger equation is a familiar superposition of traveling waves:

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(k) e^{i(kx - \omega(k)t)} dk. \quad (15.10)$$

As a homework problem you will derive this same form of the solution using Fourier transform methods.

Let us make a few comments regarding the physical meaning of (15.9) and (15.10). Physically, the wave function (15.9) represents a free particle with momentum $p = \hbar k$ and energy $E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$. Recall that $\psi^*(x)\psi(x)dx$ is interpreted as the probability

for finding the particle between x and $x + dx$. This probability is the same throughout all space for a particle described by (15.9) because $\psi^*\psi = \text{constant}$, independent of x (*exercise*). Thus the particle in a state described by (15.9) has an equal chance to be found anywhere in space. By contrast, the momentum of the particle in this state is known with certainty to have the value $\hbar k$. This state of affairs is an extreme manifestation of the position-momentum *uncertainty principle*: the statistical spread in the position is inversely proportional to that of the momentum. Thus, in particular, if the momentum is known precisely (vanishing statistical uncertainty) then the position takes all values with equal probability. The general solution (15.10) of the free particle Schrödinger equation, being a superposition over plane waves, corresponds to a superposition of momenta and energies. Because of this superposition, neither the energy or momentum of a free particle described by (15.10) has a precise value in the sense that there is a probability distribution describing the range of possible outcomes of a measurement of these observables.

Exercises: What choice of $C(k)$ in (15.10) corresponds to (15.9)? What is the wavefunction at $t = 0$ if we set $C(k) = 1$?

Equation (15.8) defines the relation between frequency and wave number (equivalently, wavelength) for solutions of the free particle Schrödinger equation. It is the dispersion relation for this Schrödinger equation. Compare the dispersion relation for the Schrödinger equation with the dispersion relation (8.67) for the wave equation in one dimension. The latter exhibits a linear relation between frequency and wave number while the former exhibits a quadratic relation. To understand the implications of these different dispersion relations let us recall that, in general, sinusoidal waves of the form $A \sin(kx - \omega t)$ travel with speed given by ω/k . For the wave equation, (8.67) tells us that $\omega/k = v$, *i.e.*, the waves travel with speed v irrespective of the frequency (or wavelength) of the wave. For the Schrödinger equation, (15.8) tells us that

$$\frac{\omega}{k} = \frac{\hbar k}{2m}, \quad (15.11)$$

which implies that the speed of the sinusoidal wave depends upon the wavelength.* Thus waves with different wavelengths travel at different speeds, the shorter wavelengths having the higher speeds (*exercise*). At any given time we can Fourier analyze any solution of the free particle Schrödinger equation into a superposition of sinusoidal waves with varying wavelengths (see (15.10)). Since each of these waves travels with a different speed, the shape of the wave will not be preserved in time as it is in the case of the wave equation

* Note that this result says the sinusoidal wave speed is *one half* the momentum $\hbar k$ of the particle divided by the mass! So one cannot interpret the free particle wave motion as particle motion; the slogan “particles are waves” has to be handled with care.

— the Schrödinger wave will in fact “disperse” as the shorter wavelengths “outrun” the longer wavelengths (see fig. 21 below). This is the origin of the term “dispersion relation” for formulas such as (8.67) and (15.8).

In contrast to the wave equation, the general solution to the Schrödinger equation involves only one undetermined complex function of one variable. We see this explicitly in the free particle case (15.10), where the undetermined function is represented by $C(k)$. This reflects the fact that only the initial value of the wave function $\psi(x, 0)$ is needed to uniquely fix the solution. Thus suppose $\psi(x, 0) = f(x)$, where $f(x)$ is some given function. Then $C(k)$ is the Fourier transform of $f(x)$ (*exercise*). According to the rules of quantum mechanics, $C(k)$ defines the probability amplitude for momentum, that is, $C^*(k)C(k)dk$ is the probability for finding momentum between $\hbar k$ and $\hbar k + \hbar dk$. Let us illustrate this with an example which we have already explored mathematically.

Consider the initial condition

$$\psi(x, 0) = Ae^{-\frac{x^2}{a^2}}. \quad (15.12)$$

Physically, this corresponds to a particle which is most likely found at the origin, but has a non-vanishing probability to be found anywhere on the x -axis. The likelihood for finding the particle away from the origin grows as the parameter a is increased, *i.e.*, as the width of the Gaussian increases. From our previous work with the Gaussian profile, you can check that its Fourier transform is (*exercise*)

$$C(k) = \frac{1}{2\sqrt{2}}Aae^{-\frac{k^2 a^2}{4}}. \quad (15.13)$$

$C(k)$ (and hence $|C(k)|^2$) is also a Gaussian. Evidently, the momentum is most likely to be zero in this state, but the likelihood for finding a non-zero momentum increases as the parameter a decreases. The probability distribution in position has its width increasing with increasing a , while the probability distribution in momentum, has its width decreasing with increasing a . This is a good example of the uncertainty principle for position and momentum: as the probability distribution in position (momentum) becomes more tightly localized around a given value the probability distribution in momentum (position) becomes more de-localized. Speaking more loosely, as the position of the particle becomes more (less) uncertain the momentum of the particle becomes less (more) uncertain.

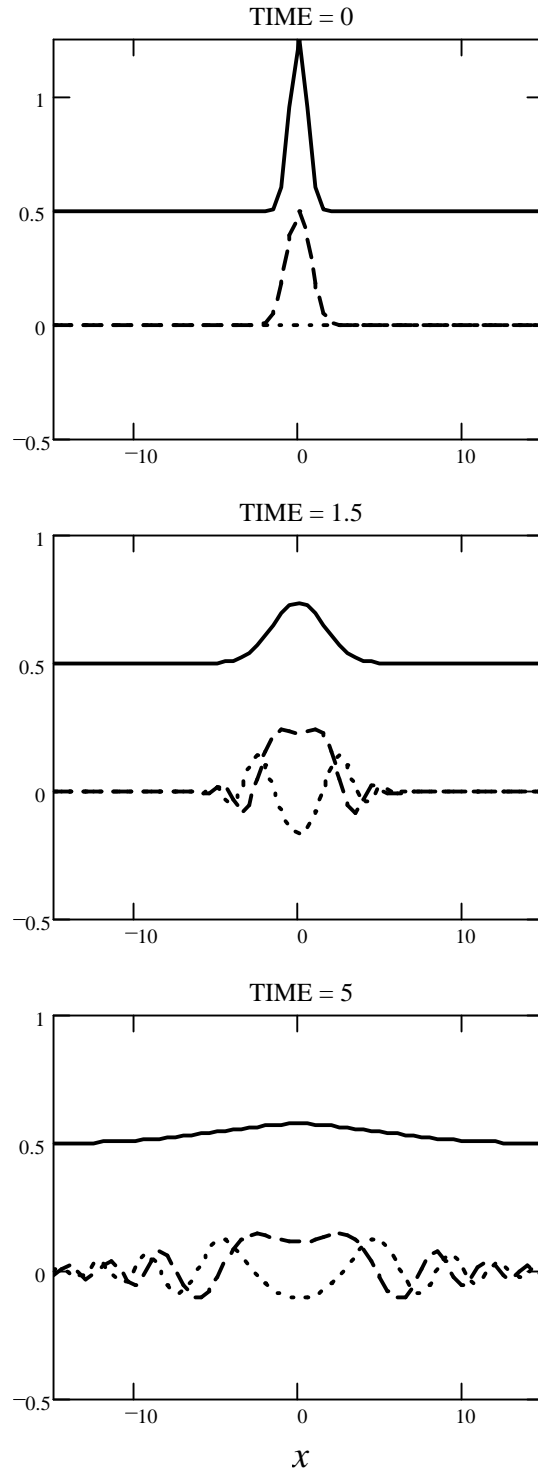


Figure 21. Time dependence of the Gaussian wave-packet solution to the Schrödinger equation. In each graph $\text{Re}(\psi)$ is the dashed line, $\text{Im}(\psi)$ is the dotted line, and $\psi^* \psi$ is the solid line. Note that the time dependence is different than for the wave equation.

15.3 The 3-Dimensional Schrödinger Equation

The generalization of the Schrödinger equation for a particle to three dimensions involves the Laplacian again:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r},t)\psi = i\hbar\frac{\partial\psi}{\partial t}. \quad (15.14)$$

Now, of course, the wave function depends on the position in three-dimensional space, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the time t , $\psi = \psi(\mathbf{r}, t)$. You can easily see that this equation reduces to (15.3) if the y and z dependence of ψ and V are eliminated. The meaning of the wave function is a simple extension of the 1-d result: $|\psi(\mathbf{r}, t)|^2 d^3x$ is the probability that the particle is found in a volume element d^3x at the point \mathbf{r} at time t . The free particle case ($V = 0$) is easily treated by separation of variables and/or Fourier methods, as you will explore in a homework problem.

15.4 Conservation of Probability, Normalization

The Schrödinger equation admits a very important conservation law, which provides a nice example of the continuity equation formalism we discussed earlier. To derive the conservation law, we need both the Schrödinger equation (15.14) and its complex conjugate

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + V(\mathbf{r},t)\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}. \quad (15.15)$$

(*Note:* Here we have used the assumption that the potential energy is a real function.) We can construct a continuity equation as follows. Multiply the Schrödinger equation (15.14) by ψ^* and multiply the complex conjugate equation (15.15) by ψ . Take the difference of the two resulting equations to get (*exercise*)

$$-i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = 0. \quad (15.16)$$

Thus, if ψ satisfies (15.14) (with V real), then it also satisfies (15.16). Next, we recall the identity (10.3). Let us apply (10.3) to the vector field $\psi\nabla\psi^*$:

$$\nabla \cdot (\psi\nabla\psi^*) = \nabla\psi \cdot \nabla\psi^* + \psi\nabla^2\psi^*. \quad (15.17)$$

Similarly

$$\nabla \cdot (\psi^*\nabla\psi) = \nabla\psi^* \cdot \nabla\psi + \psi^*\nabla^2\psi. \quad (15.18)$$

Subtracting these two results and using the fact that the dot product is commutative ($A \cdot B = B \cdot A$) we get (*exercise*)

$$\nabla \cdot [\psi\nabla\psi^* - \psi^*\nabla\psi] = \psi\nabla^2\psi^* - \psi^*\nabla^2\psi. \quad (15.19)$$

Thus the second term in parenthesis in (15.16), involving the Laplacian, can be expressed as the divergence of a vector field. It is straightforward to check that the first term in parenthesis in (15.16), involving the time derivative, can be expressed as a time derivative (*exercise*):

$$\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi). \quad (15.20)$$

From these manipulations we see that the result (15.16) can be expressed as a continuity equation (*exercise*)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (15.21)$$

where

$$\rho = \psi^* \psi \quad (15.22)$$

$$\mathbf{j} = \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi). \quad (15.23)$$

Note that the reality of the function V was crucial for this result. If we used a complex potential energy this continuity equation would not arise (see problems).

We can now use our previous experience with continuity equations to derive a conservation law. Recall that a continuity equation such as (15.21) implies that the time rate of change of the volume integral of ρ over a volume R will be controlled by the flux of \mathbf{j} through the boundary S of R . Thus,

$$N(t) = \int_R \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) d^3x \quad (15.24)$$

satisfies

$$\frac{dN(t)}{dt} = \int_S \mathbf{j} \cdot \mathbf{n} dS, \quad (15.25)$$

provided, of course, that ψ satisfies the Schrödinger equation (15.14). In particular, *with boundary conditions chosen so that the flux of \mathbf{j} through S vanishes*, the probability for finding the particle in the region R will be time-independent. One says that “probability is conserved”.

This conservation law allows us to *normalize* the solutions to the Schrödinger equation. Recall that $\rho(\mathbf{r}, t) d^3x$ is the probability that the particle is located in an infinitesimal neighborhood d^3x of \mathbf{r} at time t . Suppose that the particle is restricted to a region R of space (which may in fact be all of space). The total probability for finding the particle anywhere in R at any given time should be unity. Thus we should demand that at any time t

$$\int_R \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) d^3x = 1. \quad (15.26)$$

One says that the wave function is “normalized” (to unity); this normalization is crucial for the physical interpretation of the wave function in terms of probabilities. In particular, the initial (say, $t = 0$) wave function should be normalized:

$$\int_R \psi^*(\mathbf{r}, 0) \psi(\mathbf{r}, 0) d^3x = 1. \quad (15.27)$$

In fact, it is enough specify initial/boundary conditions such that (15.27) holds and the wave function is guaranteed to be normalized for all time if it satisfies the Schrödinger equation. Indeed, if (15.27) is satisfied, then with boundary conditions chosen such that the flux of \mathbf{j} through the boundary of R vanishes, (15.25) guarantees that (15.26) is satisfied (*exercise*). This result is quite important since the solutions to the Schrödinger equation are uniquely determined by the initial (normalized) wave function. If the wave function at later times were not normalized, then the probability interpretation of quantum mechanics would not work.

15.5 Boundary Conditions, Particle in a Box

Our argument that took us from the Schrödinger equation to the continuity equation and to the conservation of probability relied upon using appropriate boundary conditions. Appropriate boundary conditions for the normalization are such that the flux of \mathbf{j} through the boundary of the region of interest should vanish. If this region is all of space, this is accomplished by using solutions to the Schrödinger equation such that $\psi \rightarrow 0$ (at a fast enough rate) as $r \rightarrow \infty$. Physically, this corresponds to requiring that the particle never escapes to infinity (at any finite time).

It is often physically appropriate to limit the spatial domain of the particle. A common model system used in quantum mechanics is a “particle in a box”. This is a model of a particle which is confined to some finite region in space. For example, a spherical box would be the points $r < a$, $a = \text{constant}$, and we would demand that $\psi = 0$ at $r \geq a$. This means that the particle is never outside the spherical box. Let us explore a simplified model of a particle in a box in a little more detail.

We again restrict our attention to one spatial dimension for simplicity. We consider a free particle moving in a box in which $-L < x < L$. We look for a solution of the free particle Schrödinger equation that is non-zero in the box, but is zero outside the box. Since the zero function always satisfies the Schrödinger equation, we have already solved the Schrödinger equation outside the box (*exercise*). We will restrict our attention to solutions which continuously “join” this exterior solution, *i.e.*, the solutions must continuously vanish at the walls of the box:

$$\psi(-L, t) = \psi(L, t) = 0. \quad (15.28)$$

For example, a simple function which vanishes at the boundaries $x = \pm L$ is

$$f(x) = N \sin\left(\frac{n\pi}{L}x\right), \quad (15.29)$$

where N is a constant (determined by normalization) and n is an integer. We require the particle to be *somewhere*, so it won't do to let $\psi = 0$ everywhere. Thus we restrict attention to $n \neq 0$. Let us choose this sine function to represent the initial wave function for the particle in the box:

$$\psi(x, 0) = \begin{cases} N \sin\left(\frac{n\pi}{L}x\right), & \text{if } -L \leq x \leq L; \\ 0, & \text{if } |x| \geq L. \end{cases} \quad (15.30)$$

The constant N is determined by normalization:

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1.$$

As an *exercise* you should do this integral in and show that, up to a phase factor of the form $e^{i\alpha}$, with α real,

$$N = \sqrt{\frac{1}{L}}. \quad (15.31)$$

We suppose that (15.30) is the initial wave function. To find the solution at time t with this initial condition, we can use Fourier analysis, but let us take the following shortcut. We already have a very simple class of solutions obtained using the separation of variables technique (see (15.7)). The solutions shown there do not satisfy the boundary conditions (15.28) because the function $X(x)$ shown there does not satisfy the boundary conditions. However both the Schrödinger equation as well as the ordinary differential equations satisfied by X and T are linear so we can build new solutions by taking linear combinations. And it is easy to take appropriate linear combinations to get solutions which *do* satisfy (15.28) (*exercise*). You can easily check (*exercise*) that with

$$\psi(x, t) = X(x)T(t), \quad (15.32)$$

where

$$X(x) = \sqrt{\frac{1}{L}} \sin kx, \quad T(t) = e^{-i\omega(k)t}, \quad (15.33)$$

and

$$k = \frac{n\pi}{L}, \quad \omega(k) = \frac{\hbar k^2}{2m} \quad (15.34)$$

we get a solution to the free particle Schrödinger equation that is always normalized, satisfies the boundary conditions $\psi(\pm L, t) = 0$, and agrees with the initial condition (15.30).

Physically, the wave function (15.32)—(15.34) represents a particle with energy

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2,$$

moving in the region $-L \leq x \leq L$.[†]

[†] “Moving” is a bit misleading because the probability distribution for position (and mo-

PROBLEM SET 8

Problem 8.1

Solve the free particle, three-dimensional Schrodinger equation using separation of variables.

Problem 8.2

Show that

$$\psi(x, t) = \int_{-\infty}^{\infty} C(k) e^{i[kx - \omega(k)t]} dk, \quad \omega(k) = \frac{\hbar k^2}{2m}.$$

solves the free particle, one-dimensional Schrodinger equation for any choice of the complex-valued function $C(k)$.

Problem 8.3

Suppose that the potential energy function is complex-valued. Repeat the steps of the derivation of (15.21) and show how the continuity equation changes if the imaginary part of V is non-zero.

Problem 8.4

Verify the normalization (15.31) for the wave function (15.30).

Problem 8.5

Solve the Schrödinger equation for a particle moving in one dimension in the potential $V = V_0$, where V_0 is a constant. Compare the probability distributions for position compare in the cases $V_0 = 0$ and $V_0 \neq 0$.

Problem 8.6

Using the representations (15.1) and (15.2), respectively, rewrite the Schrödinger equation as a pair of real equations for real functions.

mentum, too) does not depend upon time (*exercise*). To see time-dependent phenomena one needs to consider superpositions of wave functions with different energies.

16. The Curl.

In §17–§20 we will study the mathematical basics behind the propagation of light waves, radio waves, microwaves, *etc.* All of these are, of course, examples of electromagnetic waves, that is, they are all the same (electromagnetic) phenomena just differing in their wavelength. The (non-quantum) description of all electromagnetic phenomena is provided by the *Maxwell equations*. These equations are normally presented as differential equations for the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$. You may have been first introduced to them in an equivalent integral form. In differential form, the Maxwell equations involve the divergence operation, which we mentioned before, and another vector differential operator, known as the *curl*. In preparation for our discussion of electromagnetic waves, we explore this vector differential operator in a little detail.

16.1 Vector Fields

Like the divergence, the curl operates on a *vector field*. To begin, recall that a vector field is different from what one usually thinks of as simply a “vector”. A vector is an arrow. A vector has magnitude and direction. A vector is an ordered set of 3 numbers, *etc.* . More abstractly, a vector is an element of a vector space. A vector *field* is an assignment of a vector to each point of space (and instant time). You pick the point, and the vector field gives you an arrow. So a vector field is really an infinite collection of vectors. The electric and magnetic fields are examples of vector fields, although one sometimes gets lazy and simply calls them vectors.

Let \mathbf{V} be a vector field in 3-dimensional space. Thus, at each point \mathbf{r} we have assigned a vector, denoted by $\mathbf{V}(\mathbf{r})$. We can break the vector field into its Cartesian components

$$\mathbf{V}(\mathbf{r}) = V^x(\mathbf{r})\mathbf{i} + V^y(\mathbf{r})\mathbf{j} + V^z(\mathbf{r})\mathbf{k} \quad (16.1)$$

at \mathbf{r} . Because these components will, in general, vary with the choice of \mathbf{r} the *components* of \mathbf{V} , *i.e.*, $V^x(\mathbf{r})$, $V^y(\mathbf{r})$, $V^z(\mathbf{r})$, are functions. This is a signal that we are dealing with a vector field rather than a vector.* Perhaps you should keep in mind the analogous case with a function (also called a “scalar field”). A function f assigns a number $f(\mathbf{r})$ (or scalar) to each point \mathbf{r} , just as a vector field V assigns a vector $V(\mathbf{r})$ to each point \mathbf{r} .

Since the components of a vector field are functions, the possibility arises for using operations of differentiation and integration when dealing with vector fields. We have

* It is possible to have a *constant vector field*, in which the Cartesian components happen to be constants. A constant vector field is completely determined by a single vector, namely, the value of the constant vector field at any one point. This allows the (somewhat confusing) custom in which one treats a constant vector field as an ordinary vector. The familiar unit vectors \mathbf{i} , \mathbf{j} , \mathbf{k} are examples of vectors which are also used as (constant) vector fields.

already seen some examples: the divergence and the divergence theorem. Let us now look at another important differential operation that can be applied to vector fields.

16.2 The Curl

The curl is a linear differential operator that produces a new vector field from the first derivatives of a given vector field. It is defined as follows. Let \mathbf{V} be a vector field, as given in (16.1). The curl of \mathbf{V} , denoted by $\nabla \times \mathbf{V}$, is defined as that vector field whose Cartesian components are given by

$$\nabla \times \mathbf{V} = \left(\frac{\partial V^z}{\partial y} - \frac{\partial V^y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial V^x}{\partial z} - \frac{\partial V^z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial V^y}{\partial x} - \frac{\partial V^x}{\partial y} \right) \mathbf{k}. \quad (16.2)$$

Just as with divergence and gradient (see (9.3) and (9.4)) some texts denote the curl using “curl”:

$$\nabla \times \mathbf{V} \equiv \text{curl} \mathbf{V}. \quad (16.3)$$

16.3 Why is the Curl Formula So Complicated?

At this point, there are (at least) two questions you may have. First, why is this rather complicated formula (16.2) useful? And second, how can you remember a formula like this? The second question is easiest, so we start with it. You will note that the formula is very much like the cross product, where

$$\mathbf{A} \times \mathbf{B} = (A^y B^z - A^z B^y) \mathbf{i} + (A^z B^x - A^x B^z) \mathbf{j} + (A^x B^y - A^y B^x) \mathbf{k}. \quad (16.4)$$

The pattern of terms is the same in the cross product and the curl: we have *cyclic permutations* of xyz , that is

$$xyz \longrightarrow yzx \longrightarrow zxy.$$

Have a look; you really only have to remember one component of the formula, say, the x -component and then use the cyclic permutation rule (*exercise*).

Of all the possible things one could write down, why the curl? Well, you could also have asked this about the gradient or the divergence. The answer comes in several layers. First of all, consider the options: we can of course define any differential operator we want, *e.g.*, we could make up an operator called $\$$ which is defined via

$$\$ \mathbf{A} = \frac{\partial A^y}{\partial z} \mathbf{i}.$$

(We won’t dignify this silly formula with an equation number.) This formula certainly defines a vector field by differentiating the vector field \mathbf{A} , but the result carries no particularly interesting information; indeed, it carries a lot of useless information. For example,

this vector \mathbf{A} always points along the x -axis, but the orientation of the x -axis in space is arbitrary. Thus the output of our fictitious operator $\$$ is rather arbitrary and hence not particularly useful. Differential operators like gradient, divergence, and curl are singled out because they capture intrinsic, coordinate independent information about the objects they differentiate. Thus, the gradient of a function is a vector field that can be used to compute the rate of change of a function in any direction (via the directional derivative). In addition, the gradient has the nice geometrical interpretation as being orthogonal to the level surfaces of the function. The gradient is essentially the only linear differential operator one can apply to a function that yields a vector field and does not depend on extraneous information — just the function. Similarly, the divergence and curl capture useful information about the vector field they act on — and nothing else. Again, they are essentially unique as linear differential operators that act on a vector field and give a coordinate independent result. But what is the meaning of that result?

16.4 Geometric Meaning of the Curl

It is far from obvious from (16.2) what the curl actually tells you about the vector field. We can give (without proof) an interpretation of the curl which is analogous to that which we gave for the divergence. To do this we need to revisit something you should have seen before: the line integral. Let us pause to define it.

Given a vector field \mathbf{V} , fix a closed curve C . We can now define the line integral or “circulation” of \mathbf{V} around C . This integral is obtained by taking the dot product of \mathbf{V} with the unit tangent to C at each point on the curve and then adding up (well, integrating) the results around the curve. If you prefer formulas, you can compute the circulation by writing the curve parametrically as

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in [a, b]. \quad (16.5)$$

The unit tangent vector \mathbf{T} has components

$$T^x = \frac{1}{N} \frac{df}{dt}, \quad T^y = \frac{1}{N} \frac{dg}{dt}, \quad T^z = \frac{1}{N} \frac{dh}{dt}, \quad (16.6)$$

where

$$N = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}. \quad (16.7)$$

The line element for the curve is

$$d\mathbf{l} = \mathbf{T} N dt. \quad (16.8)$$

The circulation is then the integral

$$\oint_C \mathbf{V} \cdot d\mathbf{l} = \int_a^b dt \left(V^x \frac{df}{dt} + V^y \frac{dg}{dt} + V^z \frac{dh}{dt} \right). \quad (16.9)$$

Evidently, the circulation measures how much the vector field is, well, circulating — or “curling” — in the direction of the curve. Now consider filling in the closed curve with a surface S , *i.e.*, consider a surface S which has C as its boundary.[†] (You have certainly used this idea when studying Faraday’s and Ampere’s law.) Each point of the surface will have a normal vector. Consider shrinking C and S to infinitesimal size. The component of the curl along the normal at a given point is the circulation per unit area in the limit as the area is made to vanish. It turns out that the circulation doesn’t depend upon how you fill in the surface. So, if you like, you can interpret any desired component of the curl of \mathbf{V} as the circulation per unit area of \mathbf{V} around the boundary of a very small circular disk whose normal is in the direction of the desired component.

There is an important theorem which captures this interpretation of the curl even when the curve is not becoming small — it is known as *Stokes’ Theorem*. Once again consider a closed curve C bounding a surface S with normal \mathbf{n} . Given a vector field \mathbf{V} , its curl is another vector field $\nabla \times \mathbf{V}$. Stokes’ Theorem equates the flux of $\nabla \times \mathbf{V}$ through S to the circulation of \mathbf{V} around C :

$$\int_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS = \oint_C \mathbf{V} \cdot d\mathbf{l}. \quad (16.10)$$

If you consider the limit in which the surface shrinks to zero area you can more or less see from this formula how to recover the interpretation of the curl we gave in the previous paragraph.

16.5 Some Important Identities for Div, Grad, Curl

There are some very important identities relating the gradient, divergence and curl that we shall need, some of which you will verify in a homework problem. In the following \mathbf{V} is an arbitrary vector field and f is an arbitrary function. We have

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0, \quad (16.11)$$

$$\nabla \times \nabla f = 0, \quad (16.12)$$

We see from (16.11) that if a vector field arises as the curl of another vector field, then its divergence is zero. Note, though, that not every vector field can be written as a curl, otherwise every vector field would have vanishing divergence — a counterexample being the vector field with Cartesian components (x, y, z) (*exercise*).

We see from (16.12) that if a vector arises as the gradient of a function, then its curl is zero. Note though, not every vector field can be expressed as a gradient of a function,

[†] There is more than one way to do this — think of how you might fill in a circle. As it happens, the circulation and the resulting interpretation of the curl are independent of how you fill in the surface.

otherwise every vector field would have vanishing curl — a counterexample being a vector field with Cartesian components $(-y, x, 0)$ (*exercise*).

Normally, one can assume the converse to the results above: if a vector has vanishing divergence, then it must be a curl. So, if a vector field \mathbf{v} satisfies

$$\nabla \cdot \mathbf{v} = 0, \quad (16.13)$$

then there exists a vector field \mathbf{w} such that

$$\mathbf{v} = \nabla \times \mathbf{w}. \quad (16.14)$$

Because of (16.12) the vector field \mathbf{w} cannot be unique. Indeed we can take any function f and redefine \mathbf{w} via

$$\mathbf{w}' = \mathbf{w} + \nabla f \quad (16.15)$$

and still get (*exercise*)

$$\mathbf{v} = \nabla \times \mathbf{w}'. \quad (16.16)$$

Likewise, if a vector field \mathbf{w} has vanishing curl,

$$\nabla \times \mathbf{w} = 0, \quad (16.17)$$

then it must be a gradient, *i.e.*, there is a function f such that

$$\mathbf{w} = \nabla f. \quad (16.18)$$

The function f is unique up to an additive constant (*exercise*).

Note: Equations (16.11) and (16.12) always hold — they're identities. The converse results just described are only guaranteed to hold in regions of space free of any “holes”. Moreover, these results are not guaranteed to be compatible with boundary conditions imposed on the vector fields.

There is one more identity that we shall need. It involves the double-curl of a vector field:

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}. \quad (16.19)$$

To use this formula you should use Cartesian components (V^x, V^y, V^z) , then the Laplacian is computed component-wise. If a vector field has vanishing divergence, then any given component (*e.g.*, the x component) of its double-curl is just the Laplacian on that component of \mathbf{V} (*e.g.*, V^x).

17. Maxwell Equations.

With our brief review of vector analysis out of the way, we can now discuss the Maxwell equations. We use the Gaussian system of electromagnetic units and let c denote the speed of light in vacuum. The Maxwell equations are differential equations for the electric field $\mathbf{E}(\mathbf{r}, t)$, and the magnetic field $\mathbf{B}(\mathbf{r}, t)$, which are defined by the force they exert on a test charge q at the point \mathbf{r} at time t . This force is defined by the *Lorentz force law*:

$$\mathbf{F}(\mathbf{r}, t) = q \left(\mathbf{E}(\mathbf{r}, t) + \frac{\mathbf{v}(t)}{c} \times \mathbf{B}(\mathbf{r}, t) \right), \quad (17.1)$$

where $\mathbf{v}(t)$ is the particle's velocity at time t . Equation (17.1) is used to determine the motion of a charged particle in a given electromagnetic field assuming the effect of the particle on the field can be neglected. Equation (17.1) can also be used to measure a given electromagnetic field by observing the motion of charged particles.

The Maxwell equations tell how the electromagnetic field arises from “sources”, which are again charged particles. In macroscopic applications it is usually convenient to model the source of the electromagnetic field as a continuous electric charge density $\rho(\mathbf{r}, t)$ and electric current density $\mathbf{j}(\mathbf{r}, t)$. (You may now note that we are anticipating with our notation that ρ and \mathbf{j} will satisfy a continuity equation corresponding to conservation of electric charge.) The Maxwell equations are

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (17.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (17.3)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j}, \quad (17.4)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (17.5)$$

Our goal is to see how on earth it is possible to find wavelike phenomena coming from these equations. But first it is worth pausing to get a feel for the basic features of these equations.

17.1 The Basic Structure of Maxwell Equations

First of all, the equations (17.2)–(17.5) are 8, coupled, first-order, partial differential equations (with constant coefficients in Cartesian coordinates) for the 6 unknown functions contained in the components of \mathbf{E} and \mathbf{B} . Viewing the Maxwell equations as equations for \mathbf{E} and \mathbf{B} given ρ and \mathbf{j} , the equations are “linear-inhomogeneous” thanks to the “source terms” defined by ρ and \mathbf{j} . Because the equations are inhomogeneous, it is not possible to superimpose solutions $(\mathbf{E}_1, \mathbf{B}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2)$ to get new solutions without altering

the charge and current densities (*exercise*). On the other hand, given any solution to the Maxwell equations (for a given ρ and \mathbf{j}) one can add any solution of the homogeneous Maxwell equations (where $\rho = 0 = \mathbf{j}$) to get a new solution of the inhomogeneous equations (*Exercise: Prove this.*) As a special case of this last property, if one is solving the Maxwell equations in a region of space where $\rho = 0$ and $\mathbf{j} = 0$, then the equations are homogeneous and one *can* superimpose solutions.

The equations (17.2) and (17.3) represent 2 “scalar” equations, while equations (17.4) and (17.5) are “vector equations”. A vector equation equates the components of two vectors. Thus the equations (17.4) and (17.5) each represent 3 (coupled) equations in which the x component of the left-hand side is equated to the x component of the right hand side, and so on.

Usually, the Maxwell equations, as presented above, are meant to be solved for \mathbf{E} and \mathbf{B} once the charge density and its motion (the current density) are specified. For example, one can let the charge density be that of a uniform ball of positive charge held fixed in space so that the current density vanishes. As you might guess, the solution of these equations has vanishing magnetic field and a Coulomb-type electrostatic field outside the ball. (Do you remember what happens inside the ball?) Note that this way of using the Maxwell equations assumes that the motion of the sources is completely known (or else, how could we specify ρ and \mathbf{j} ?). For many purposes this is a reasonable physical assumption. But, strictly speaking, this way of describing electrodynamics is at best an approximate description. As you can imagine, many applications (*e.g.*, the electrodynamics of the ionosphere) will require us to also figure out how the sources are moving. This is a complicated problem and quite non-linear: the sources generate the electromagnetic field according to the Maxwell equations (17.2)–(17.5), the electromagnetic field affects the sources according to the Lorentz force law (17.1), but the motion of the charges determines the fields, *etc.* Needless to say, we will be content to study the case where the motion of the sources is prescribed, that is, explicitly given.

Note that only 4 of the 8 Maxwell equations, (17.2) and (17.4), involve the sources. These are often called the *inhomogeneous equations* because they are linear inhomogeneous in the unknowns. The other 4 which do not involve the sources, (17.3) and (17.5), are likewise known as the *homogeneous equations*. The inhomogeneous equation involving ρ shows that the charge density gives a divergence to the electric field. This is reasonable: electric charges create electric fields; this Maxwell equation tells what part of the vector field is affected by charges. The last Maxwell equation (17.3), on the other hand, shows that the magnetic field never has a divergence. By analogy with the electric field, this equation can be viewed as stating that there is no such thing as “magnetic charge”. The magnetic field can have a curl however, (17.4), and this arises from either a time varying electric field or from a current density (moving charges). Thus a moving charge creates

a magnetic field and, as Maxwell first postulated, so does a time varying electric field. Finally, from (17.5), the electric field can also have a curl, but only if there is a time varying magnetic field—a phenomenon characterized by Faraday.

Note also that only 6 of the 8 equations, (17.4) and (17.5) involve a time derivative, that is, only 6 equations concern themselves with how the fields change in time. For this reason these equations are called the *evolution equations*. The remaining two divergence equations are called *constraint equations* since they restrict the fields at any given time. It can be shown that the constraint equations only need to be satisfied initially, *i.e.*, at a single instant of time; the evolution equations will guarantee they will be satisfied at later times. This is important because otherwise we would be in danger of having too many equations (8) and not enough unknowns (6).

17.2 Continuity Equation and Conservation of Charge

We now consider an important consistency condition that must be satisfied by the sources if the Maxwell equations are to admit any solution at all. Besides being an important feature of the equations, this condition follows from a nice manipulation of vector differentiation. Take the time derivative of (17.2) and interchange time and space derivatives to get (*exercise*)

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}. \quad (17.6)$$

Compare this result with the divergence of (17.4) (*exercise*):

$$-\frac{1}{c} \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \nabla \cdot \mathbf{j}, \quad (17.7)$$

to find (*exercise*)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (17.8)$$

This is our old friend the continuity equation. What this computation means is that the Maxwell equations have solutions for \mathbf{E} and \mathbf{B} only if the 4 functions $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ are chosen to satisfy the continuity equation (17.8) given above. Recall that this equation is a differential version of a conservation law; the conserved quantity in this case being the electric charge. More precisely, the total charge Q contained in a volume V at time t , defined by

$$Q = \int_V \rho(\mathbf{r}, t) d^3x, \quad (17.9)$$

changes in time according to the net flux of the current density \mathbf{j} through the boundary S of V :

$$\frac{dQ}{dt} = - \oint_S \mathbf{j} \cdot \mathbf{n} dS. \quad (17.10)$$

If the net flux of charge through the boundary (which may be “at infinity”) vanishes, then the charge contained in V is conserved. When we use the Maxwell equations to solve for the electromagnetic field due to a given charge distribution, that distribution must be specified so that charge is conserved in the sense of (17.8) or else the equations cannot have a solution.*

Given the continuity equation, we can now consider the status of the constraint equations (17.2) and (17.3). It is straightforward to show that if they are satisfied at one time, say $t = 0$ by the initial \mathbf{E} and \mathbf{B} , then they are automatically solved at later times provided (i) the electromagnetic field at later times satisfies the evolution equations, and (ii) (17.8) is satisfied by the sources. See the Problems for details.

18. The Electromagnetic Wave Equation.

Let us now see how the Maxwell equations (17.2)–(17.5) predict the existence of electromagnetic waves. For simplicity we will consider a region of space and time in which there are no sources (*i.e.*, we consider the propagation of electromagnetic waves in vacuum). Thus we set $\rho = 0 = \mathbf{j}$ in our space-time region of interest. Now all the Maxwell equations are linear, homogeneous.

Begin by taking the curl of (17.4) to find (*exercise*)

$$\nabla \times (\nabla \times \mathbf{B}) - \frac{1}{c} \nabla \times \frac{\partial \mathbf{E}}{\partial t} = 0. \quad (18.1)$$

Of course we assume that the components of the electromagnetic field are at least twice continuously differentiable functions of space and time. We are therefore allowed to assume that partial derivatives commute and we have, using the identity (16.19), (*exercise*)

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{E} = 0. \quad (18.2)$$

Next we use the fact that (i) the magnetic field is divergence-free (see (17.3)) and (ii) the equation (17.5) to find (*exercise*)

$$-\nabla^2 \mathbf{B} + \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \quad (18.3)$$

Thus (each component of) \mathbf{B} satisfies the wave equation with wave velocity c ! As a homework problem you will play an analogous game with the electric field to show

$$-\nabla^2 \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \quad (18.4)$$

* It is no accident that the Maxwell equations, in effect, force the conservation of electric charge. Indeed, our current field theoretic description of all fundamental interactions (electromagnetic, weak, strong, and gravitational) is geared to force such conservation laws through the use of variational principles and the “principle of local gauge invariance”. Unfortunately, a discussion of such ideas would be beyond the scope of this course.

So, each component of \mathbf{E} and \mathbf{B} satisfies a wave equation, *i.e.*, we can have traveling wave disturbances in the electromagnetic field. As you know, such disturbances correspond to light waves, radio waves, microwaves, *etc.*

From our computations above it is tempting to think that electromagnetic fields satisfying the Maxwell equations are really just 6 copies of the waves we have been studying all along, one for each component of \mathbf{E} and \mathbf{B} . In particular it appears that each component of the electric and magnetic fields goes its own way, independently of the others. This is not true. What we *have* shown is that, given an \mathbf{E} and \mathbf{B} solving the (source-free) Maxwell equations, the electric and magnetic fields necessarily satisfy uncoupled wave equations. What we have *not* shown (and is not true) is that, given electric and magnetic fields solving uncoupled wave equations (18.3) and (18.4), we obtain solutions to the Maxwell equations. Put differently, that \mathbf{E} and \mathbf{B} solve the wave equation is *necessary but not sufficient* for them to solve the source-free Maxwell equations. We will now show how to use the solutions to (18.3) and (18.4) to build *bona fide* solutions to the Maxwell equations in the form of electromagnetic waves. For simplicity, we will restrict our attention to electromagnetic plane waves. (More general waves can be obtained by superposition in the usual way.) The strategy is to take a plane wave solution to (18.4), restrict it to satisfy (17.2) (with $\rho = 0$) and then define the magnetic field so that the remaining Maxwell equations are satisfied. Here are the details.

Because each component of \mathbf{E} satisfies the wave equations, we can consider a plane wave electric field of the form:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi). \quad (18.5)$$

Here the constant vector field \mathbf{E}_0 determines the amplitude and (constant) direction of the electric field, and the phase ϕ is a constant. You can check that (18.5) is a solution to (18.4) if and only if the dispersion relation is (*exercise*)

$$\omega = kc. \quad (18.6)$$

Of course we could also use a sine function, or take the real part of an exponential, or take linear combinations (the source-free Maxwell equations are linear homogeneous), but this is good enough for our discussion. We emphasize that (18.5) does not yet necessarily define a solution to the Maxwell equations — they still need to be imposed. To begin with, let us impose $\nabla \cdot \mathbf{E} = 0$. You can easily check that this equation means we have to choose the wave vector and the amplitude vector orthogonally (*good exercise*):

$$\mathbf{k} \cdot \mathbf{E}_0 = 0. \quad (18.7)$$

Thus the direction of the electric field in the electromagnetic plane wave is always perpendicular to the direction of propagation of the wave.

The remaining Maxwell equations still need to be imposed, and they involve the magnetic field. We can build a divergence-free ($\nabla \cdot \mathbf{B} = 0$) magnetic field plane wave just as we did for \mathbf{E} , but we must keep in mind the other 2 Maxwell equations

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 0 \quad (18.8)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (18.9)$$

Let us first look at (18.8). From (18.5) the time derivative of \mathbf{E} yields a sine function, and this is to be equated to a curl of \mathbf{B} , which involves first spatial derivatives. This suggests we should also try a cosine for the magnetic field:

$$\mathbf{B} = \mathbf{B}_0 \cos(\mathbf{k}' \cdot \mathbf{r} - \omega' t + \psi). \quad (18.10)$$

To satisfy the wave equation we need the dispersion relation (*exercise*)

$$\omega' = k' c. \quad (18.11)$$

Because we want $\nabla \cdot \mathbf{B} = 0$, we have to choose the wave vector and amplitude such that

$$\mathbf{k}' \cdot \mathbf{B}_0 = 0. \quad (18.12)$$

It is easy to check that (*exercise*)

$$\begin{aligned} \nabla \times \mathbf{B} &= -\mathbf{B}_0 \times \nabla [\cos(\mathbf{k}' \cdot \mathbf{r} - \omega' t + \psi)] \\ &= (\mathbf{B}_0 \times \mathbf{k}') \sin(\mathbf{k}' \cdot \mathbf{r} - \omega' t + \psi). \end{aligned} \quad (18.13)$$

Next, we compute

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{\omega}{c} \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi). \quad (18.14)$$

Thus (18.8) reduces to

$$\frac{\omega}{c} \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) = \mathbf{B}_0 \times \mathbf{k}' \sin(\mathbf{k}' \cdot \mathbf{r} - \omega' t + \psi).$$

Since this must hold at each location and time, it follows (*exercise*) that

$$\phi = \psi, \quad (18.15)$$

$$\mathbf{k}' = \mathbf{k}, \quad (18.16)$$

and

$$\frac{\omega}{c} \mathbf{E}_0 = -\mathbf{k} \times \mathbf{B}_0. \quad (18.17)$$

Let \mathbf{n} be the unit vector in the direction of wave propagation:

$$\mathbf{n} = \frac{\mathbf{k}}{k}, \quad (18.18)$$

then we can write (*exercise*)

$$\mathbf{E}_0 = -\mathbf{n} \times \mathbf{B}_0. \quad (18.19)$$

Thus we see that the electric and magnetic fields must have the same wavelengths ($\lambda = 2\pi/k$) and propagation directions $\frac{\mathbf{k}}{k}$. Furthermore, from (18.19) and the dispersion relation (18.6) it follows that:

- (1) The magnitudes of the electric and magnetic fields are equal (*exercise*),

$$E_0 = B_0. \quad (18.20)$$

- (2) \mathbf{E} and \mathbf{B} are orthogonal (perpendicular), $\mathbf{E} \cdot \mathbf{B} = 0$ (*exercise*).

- (3) Each field is orthogonal to the propagation direction, $\mathbf{E} \cdot \mathbf{n} = 0 = \mathbf{B} \cdot \mathbf{n}$ (*exercise*).

Note that when \mathbf{E}_0 and \mathbf{B}_0 satisfy (18.19), they automatically satisfy (18.7) and (18.12).

Recall that, thus far, we have only dealt with (18.8). As an exercise you can show that the other evolution equation (18.9) gives identical information except we find that

$$\mathbf{B}_0 = \mathbf{n} \times \mathbf{E}_0. \quad (18.21)$$

However, this is equivalent to our earlier relation (18.19). To see this we first note that, given that \mathbf{E}_0 and \mathbf{n} are orthogonal (as follows from (18.7)), the relation (18.21) above says that: (1) (18.20) holds and (2) \mathbf{B}_0 is perpendicular to both \mathbf{E}_0 and \mathbf{n} and has a direction given by the right-hand rule. The relation (18.19) also implies (18.20) and that the three vectors (\mathbf{E}_0 , \mathbf{B}_0 , \mathbf{n}) are mutually orthogonal; the only thing to check is that the right hand rule gives the same relation among the three vectors—which you can check. It is also possible to massage (18.21) directly into (18.19) using some vector identities; this will be explored in the homework.

At this point we have taken care of all the Maxwell equations. Let us gather all our results together. We can build a plane wave solution to Maxwell equations by the following procedure:

- (i) Pick a wave propagation direction \mathbf{n} and a wavelength λ ; then define

$$\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{n}. \quad (18.22)$$

- (ii) Pick an electric field amplitude \mathbf{E}_0 , which is any (constant) vector (field) such that $\mathbf{E}_0 \cdot \mathbf{n} = 0$.

- (iii) Set $\omega = kc$ and define

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi), \quad (18.23)$$

$$\mathbf{B}(\mathbf{r}, t) = (\mathbf{n} \times \mathbf{E}_0) \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi). \quad (18.24)$$

Equations (18.23) and (18.24) give the electric and magnetic fields arising in a plane electromagnetic wave. This wave is *linearly polarized* (see §20). It can be shown that the general solution to the source-free Maxwell equations can be expressed as a superposition of these plane wave solutions over all possible values of the amplitude, phase, propagation direction, and wave-number.

19. Electromagnetic Energy.

In a previous physics course you should have encountered the interesting notion that the electromagnetic field carries energy and momentum. If you have ever been sunburned, you have experimental confirmation of this fact! We are now in a position to explore this idea quantitatively. In physics, the notions of energy and momentum are of interest mainly because they are conserved quantities. We can uncover the energy and momentum quantities associated with the electromagnetic field by searching for conservation laws. As before, such conservation laws will appear embodied in a continuity equation. Thus we begin by investigating a continuity equation for energy and momentum. As was mentioned earlier, there are systematic methods to search for continuity equations associated with a system of differential equations such as the Maxwell equations, but we will not get into that. Instead, we simply write down the continuity equation associated to energy-momentum conservation.

We will restrict our attention to the source-free case ($\rho = 0$, $\mathbf{j} = 0$). With sources, there can be an exchange of energy-momentum of the electromagnetic field with that of the sources and the story is a little longer than justified for this course*. Since we have used the symbols ρ and \mathbf{j} to denote the electric charge density and electric current density, to avoid confusion we use the symbols U and \mathbf{S} to denote the energy density and energy current density (also known as the “Poynting vector”) for the electromagnetic field. These are defined by (still in Gaussian units)

$$U = \frac{1}{2}(E^2 + B^2), \quad (19.1)$$

$$\mathbf{S} = c\mathbf{E} \times \mathbf{B}. \quad (19.2)$$

The claim is that (U, \mathbf{S}) satisfy

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = 0 \quad (19.3)$$

when \mathbf{E} and \mathbf{B} satisfy the (source-free) Maxwell equations.

* As you might expect on physical grounds, in order to get a conservation law in this case, one must keep track of the energy and momentum of the sources.

To verify this, we need to use some vector identities. If \mathbf{A} and \mathbf{B} are any two vector fields that depend upon a variable t , we have that (*exercise*)

$$\begin{aligned}\frac{\partial}{\partial t}(A^2) &= \frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{A}) \\ &= 2\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t},\end{aligned}\tag{19.4}$$

and (a longer *exercise* – you needn't do this one unless you are ambitious)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}.\tag{19.5}$$

Using these vector identities and the Maxwell equations, we have (*exercise*)

$$\begin{aligned}\frac{\partial U}{\partial t} &= \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= c\mathbf{E} \cdot \nabla \times \mathbf{B} - c\mathbf{B} \cdot \nabla \times \mathbf{E},\end{aligned}\tag{19.6}$$

and

$$\nabla \cdot \mathbf{S} = c(\mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}).\tag{19.7}$$

from which the result (19.3) follows immediately. Note we only needed the evolution equations to obtain (19.3).

We interpret $U(\mathbf{r}, t)$ as the *electromagnetic energy density at the point \mathbf{r} and time t* and the Poynting vector* \mathbf{S} as the energy current density. Define the total electromagnetic energy \mathcal{E} in the volume V to be the integral of the energy density U :

$$\mathcal{E} := \int_V U dV = \int_V \frac{1}{2} (E^2 + B^2) dV.\tag{19.8}$$

According to the continuity equation (19.3), the time rate of change of \mathcal{E} is controlled by the flux of the Poynting vector \mathbf{S} through the boundary A of V :

$$\frac{d\mathcal{E}}{dt} = - \int_A \mathbf{S} \cdot \mathbf{n} dA.\tag{19.9}$$

It is worth computing the energy density and Poynting vector for the plane wave constructed above in (18.23) and (18.24). You will find (*exercise*)

$$U = (E_0)^2 \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi),\tag{19.10}$$

$$\mathbf{S} = c(E_0)^2 \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) \mathbf{n}.\tag{19.11}$$

Note that the Poynting vector lies along the direction of propagation \mathbf{n} of the plane wave. Thus the flow of energy is along \mathbf{n} . Of course, if you integrate U over a volume you will

* You should keep in mind that the Poynting vector is really a vector *field*.

find that the total energy so-constructed is changing according to the net flux of \mathbf{S} through the volume. This you will experiment with in a homework problem.

If the volume V in (19.8) is taken to be all space, we can view the boundary A as a sphere of infinite radius. If we consider an *isolated system*, so that the electric and magnetic fields vanish sufficiently rapidly at large distances (*i.e.*, “at infinity”), then the flux of the Poynting vector will vanish as the radius of A is taken to infinity. Thus the total electromagnetic energy of an isolated (and source-free) electromagnetic field is conserved.

20. Polarization.

Our final topic in this brief study of electromagnetic waves concerns the phenomenon of *polarization*, which occurs thanks to the vector nature of the waves. More precisely, the polarization of an electromagnetic plane wave concerns the direction of the electric (and magnetic) vector fields. Let us first give a rough, qualitative motivation for the phenomenon. An electromagnetic plane wave is a traveling sinusoidal disturbance in the electric and magnetic fields. Let us focus on the behavior of the electric field since we can always reconstruct the behavior of the magnetic field from the electric field. Because the electric force on a charged particle is along the direction of the electric field, the response of charges to electromagnetic waves is sensitive to the direction of the electric field in a plane wave. Such effects are what we refer to when we discuss polarization phenomena involving light.

Based upon our elementary discussion above, it may appear to you that plane electromagnetic waves will always have a *linear polarization*, that is, a constant electric (and hence magnetic) field direction. However, consider superimposing two plane waves with the same propagation direction and wavelength but with different phases and directions for the electric and magnetic fields. Thanks to the linear-homogeneous nature of the source-free Maxwell equations, we know that this superposition will also be a solution of those equations. And, as we shall see, this superposition will be another plane wave of the type we have studied. Even though the direction of the electric field in each of the constituent waves is constant, the superposition of the two can have a time varying electric (and magnetic) field direction because the two constituent electric fields need not be in phase with each other. The net effect is a time varying electric (and magnetic) field direction and the resulting phenomena of circular and elliptical polarization. We now want to see how to describe this mathematically.

Let us choose (arbitrarily) our z -axis along the direction of propagation of the wave. Let us construct the wave by superimposing a 2 plane waves: $(\mathbf{E}_1, \mathbf{B}_1)$, with \mathbf{E}_1 directed along the x -axis, and $(\mathbf{E}_2, \mathbf{B}_2)$, with \mathbf{E}_2 directed along the y -axis. Further, let us work with the (by now, familiar) representation of the waves as complex-valued exponentials.

In what follows, keep in mind that we should take the real part of the electric field at the end of the day. This keeps the trigonometry from getting in our way; in particular, the phase information is contained in the complex amplitudes. We write therefore*

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = (E_1 \mathbf{i} + E_2 \mathbf{j}) e^{i(kz - \omega t)}. \quad (20.1)$$

Here E_1 and E_2 are just two complex numbers, that is, for real numbers R_1, R_2 , α , and β we have

$$E_1 = R_1 e^{i\alpha} \quad (20.2)$$

$$E_2 = R_2 e^{i\beta}. \quad (20.3)$$

We see that all we have given is the complex representation of the superposition of two real waves (*exercise*):

$$\mathbf{E}_1 = R_1 \cos(kz - \omega t + \alpha) \mathbf{i} \quad (20.4)$$

$$\mathbf{E}_2 = R_2 \cos(kz - \omega t + \beta) \mathbf{j}. \quad (20.5)$$

Different polarizations occur for different choices of the phase difference $\alpha - \beta$ and the amplitudes R_1 and R_2 .

Note that even though we began by assuming the two electric fields were orthogonal, even if they weren't orthogonal we would have ended up with a similar result upon superposition. Specifically, in (20.1) E_1 would have been the x component of the superposition and E_2 would have been the y component of the superposition.

Let us now consider the behavior of the electric field for various choices of the relative phases and amplitudes.

20.1 Linear Polarization

The linear polarization case, which we studied in previous sections, occurs when $\beta = \alpha \pm n\pi$, where $n = 0, 1, 2, 3, \dots$. We place no restrictions upon R_1 and R_2 . In this case you can check (*exercise*) that the complex form of the total electric field is

$$\mathbf{E} = (R_1 \mathbf{i} \pm R_2 \mathbf{j}) e^{i(kz - \omega t + \alpha)}. \quad (20.6)$$

Taking the real part gives (*exercise*)

$$\mathbf{E} = (R_1 \mathbf{i} \pm R_2 \mathbf{j}) \cos(kz - \omega t + \alpha). \quad (20.7)$$

This is a wave in which the magnitude of the electric field oscillates in time and space, but with its direction held fixed.

* Don't confuse the unit vector \mathbf{i} with the complex unit i !

20.2 Circular Polarization

Here we set $R_1 = R_2 \equiv R$ and $\beta = \alpha \pm \frac{\pi}{2}$. We find for the complex field (*exercise*)

$$\mathbf{E} = R(\mathbf{i} \pm \mathbf{j})e^{i(kz - \omega t + \alpha)}. \quad (20.8)$$

Take the real part to find (*exercise*)

$$\mathbf{E} = \mathbf{i}R \cos(kz - \omega t + \alpha) \mp \mathbf{j}R \sin(kz - \omega t + \alpha). \quad (20.9)$$

Let us consider the direction of \mathbf{E} as measured at a fixed point (x_0, y_0, z_0) in space. Physically, we move to this point and hold out our electric field probe. As time passes, what do we find? Well, of course we find that \mathbf{E} always lies in the x - y plane, as you can see from (20.9). More interestingly, we find that the magnitude of \mathbf{E} is constant while its direction moves in a circle with constant angular velocity ω . It is easy to verify that the magnitude of \mathbf{E} is constant in time:

$$E^2 = R^2(\cos(kz - \omega t + \alpha))^2 + R^2(\sin(kz - \omega t + \alpha))^2 = R^2. \quad (20.10)$$

To see that the electric field direction moves with uniform circular motion at angular velocity ω , simply recall that such motion can always be mathematically characterized in the general form

$$u(t) = r \cos(\omega t + \xi) \quad (20.11)$$

$$v(t) = r \sin(\omega t + \xi), \quad (20.12)$$

where (u, v) are Cartesian coordinates in a two dimensional space (*exercise*). At a fixed location, the x and y components \mathbf{E} , given above in (20.9), are precisely of this form.

20.3 Elliptical Polarization

Finally, we consider the most general case in which $R_1 \neq R_2$ and $\alpha \neq \beta$. From our previous special case, you can see that it might be profitable to make the definition

$$\gamma = \beta - \frac{\pi}{2}, \quad (20.13)$$

in which case you can show as a nice *exercise* that the real electric field is given by:

$$\mathbf{E} = \mathbf{i}R_1 \cos(kz - \omega t + \alpha) + \mathbf{j}R_2 \sin(kz - \omega t + \gamma). \quad (20.14)$$

From this result it follows that the electric field direction—as measured at a fixed z —traces out an *ellipse*. There are a number of ways to see this. First of all, for fixed z , the x and y components of \mathbf{E} are mathematically identical to a pair of general solutions

to a one-dimensional harmonic oscillator (*exercise*). Put differently, the motion of the x - y components of \mathbf{E} is mathematically identical to the x and y motions of a two-dimensional (isotropic) harmonic oscillator. It is a familiar result from classical mechanics that the superposition of these x - y motions is an ellipse in the x - y plane. In case this result is unfamiliar to you, let us recall the definition of an ellipse. One definition is the locus of points in the x - y plane such that, for some constants a and b

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (20.15)$$

where a and b are some constants. You will be asked to show in the Problems that such a relationship is satisfied by the x and y components of \mathbf{E} .

PROBLEM SET 9

Problem 9.1

Compute the divergence and curl of the following vector fields:

- (a) $\mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}$, $r = \sqrt{x^2 + y^2 + z^2} > 0$ (Coulomb electric field)
- (b) $\mathbf{B}(\mathbf{r}) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$, $x^2 + y^2 > 0$, (magnetic field outside a long wire)
- (c) $\mathbf{A}(\mathbf{r}) = -y\mathbf{i} + x\mathbf{j}$, (vector potential for a uniform magnetic field).

Problem 9.2

A common but erroneous claim, which stems from the notation $\nabla \times \mathbf{V}$ for the curl of a vector field \mathbf{V} , is that the curl of a vector field \mathbf{V} is everywhere orthogonal to \mathbf{V} . Disprove this claim by exhibiting a counter-example.

Problem 9.3

Show that

- (a) $\nabla \times (\nabla f) = 0$, where f is a function;
- (b) $\nabla \cdot (\nabla \times \mathbf{V}) = 0$, where \mathbf{V} is a vector field;
- (c) $\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$.

Hints: Work in Cartesian coordinates. To prove equality between vectors (in (a) and (c)) it is enough to show that the identities hold for one component, *e.g.*, the x component.

Problem 9.4

We derived the continuity equation for electric charge from the inhomogeneous Maxwell equations. Show that an analogous computation with the homogeneous Maxwell equations yields no new equations.

Problem 9.5

We used half the Maxwell equations to derive a wave equation for the magnetic field \mathbf{B} . Using the other half of the equations, perform an analogous computation to derive a wave equation for \mathbf{E} .

Problem 9.6

Suppose that

$$A \sin(\mathbf{k} \cdot \mathbf{r} + \alpha t) = B \sin(\mathbf{k}' \cdot \mathbf{r} + \beta t)$$

for all \mathbf{r} and t . Show that $A = B$, $\alpha = \beta$, and $\mathbf{k} = \mathbf{k}'$.

Problem 9.7

For an electromagnetic plane wave show, from (18.9), that

$$\mathbf{B}_0 = \mathbf{n} \times E_0.$$

Problem 9.8

Verify (19.4) – (19.7)

Problem 9.9

Using the Maxwell equations (17.2)–(17.5), and charge continuity equation, (17.8), show that, when the Maxwell equations are satisfied,

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{E} - 4\pi\rho) = 0,$$

and

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0.$$

This implies that if the constraint equations are satisfied at one time, then they are satisfied for all time by virtue of the evolution equations.

Problem 9.10

Show that the electric field in (20.14) traces out an ellipse.

Problem 9.11

Show that $\mathbf{V} = x\mathbf{i} - z\mathbf{k}$ has vanishing divergence. Find a vector whose curl is \mathbf{V} . Show that $\mathbf{W} = xy^2 \cos z\mathbf{i} + x^2y \cos z\mathbf{j} - \frac{1}{2}x^2y^2 \sin z\mathbf{k}$ has vanishing curl. Find a function whose gradient is \mathbf{W} .

Problem 9.12

Let S be a closed surface and let C be a closed curve. Prove that

(a) $\oint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = 0,$

(b) $\oint_C \nabla f \cdot d\mathbf{l} = 0.$

Problem 9.13

Let S be a closed surface enclosing a charge Q . Let A be a surface through which a current I passes; A has the (closed) boundary curve L . Using the divergence theorem and Stokes' theorem, derive the integral form of Maxwell's equations:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi Q, \quad (\text{Gauss}),$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0,$$

$$\oint_L \mathbf{B} \cdot d\mathbf{l} - \frac{1}{c} \frac{\partial}{\partial t} \int_A \mathbf{E} \cdot d\mathbf{A} = \frac{4\pi}{c} I, \quad (\text{Ampere} - \text{Maxwell}),$$

$$\oint_L \mathbf{E} \cdot d\mathbf{l} + \frac{1}{c} \frac{\partial}{\partial t} \int_A \mathbf{B} \cdot d\mathbf{A} = 0, \quad (\text{Faraday}).$$

Problem 9.14

Let C be a circle of unit radius in the x - y plane, enclosing the unit disk D , and let $\mathbf{V} = y\mathbf{i}$. Compute (i) the line integral of \mathbf{V} around C and (ii) the flux of $\nabla \times \mathbf{V}$ through D .

Hint: You should find that

$$\oint_C \mathbf{V} \cdot d\mathbf{l} = \int_D (\nabla \times \mathbf{V}) \cdot d\mathbf{S}.$$

21. Non-linear Wave Equations and Solitons.

In 1834 the Scottish engineer John Scott Russell observed at the Union Canal at Hermiston a well-localized* and unusually stable disturbance in the water that propagated for miles virtually unchanged. The disturbance was stimulated by the sudden stopping of a boat on the canal. He called it a “wave of translation”; we call it a *solitary wave*. As it happens, a number of relatively complicated – indeed, non-linear – wave equations can exhibit such a phenomenon. Moreover, these solitary wave disturbances will often be stable in the sense that if two or more solitary waves collide after the collision they will separate and take their original shape. Solitary waves which have this stability property are called *solitons*. The terminology stems from a combination of the word solitary and the suffix “on” which is used to signify a particle (think of the proton, electron, neutron, *etc.*). We shall discuss a little later the sense in which a soliton is like a particle. Solitary waves and solitons have become very important in a variety of physical setting, for example: hydrodynamics, non-linear optics, plasmas, meteorology, and elementary particle physics, to name a few. Our goal in this chapter is to give a very brief — and relatively superficial — introduction to solitonic solutions of non-linear wave equations.

First, however, let us note that the relatively simple wave equation provides an illustration of some of the phenomena we want to explore. We have already seen that the solutions to the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} \right) q(x, t) = 0 \quad (21.1)$$

take the general form

$$q(x, t) = f(x + vt) + g(x - vt), \quad (21.2)$$

where the functions f and g are determined by initial conditions. Let us suppose that we choose our initial conditions so that the solution has $f = 0$, so that the wave is simply the displacement profile $q = g(x - vt)$, that is, a traveling disturbance with the shape dictated by the curve $y = g(x)$ translating rigidly to the “right” (toward positive x) at speed v . Let us also suppose that g is a function that is localized in some region, so that it has a finite width. We have a “pulse”, which travels to the right, unchanged in shape. Thus the pulse is a solitary wave. To visualize this, imagine that you and your friend are holding a rope taut between you and you shake the end of the rope one time. The result is a pulse which travels toward your friend (with a speed depending upon the density and tension of the rope). This “pulse” has its shape described by the function g . Now suppose we also allow f to be a localized non-trivial function, you then get a pulse traveling to the left superposed with the pulse traveling to the right. Suppose the two pulses are initially separated, the one described by f sitting off at large, positive x , and the one described

* About 10 meters long and half a meter in height.

by g sitting off at large negative x , say. The pulses will approach each other at a relative speed of $2v$ and at some point they will overlap, giving a wave profile which is, evidently, the algebraic sum of the individual pulses. Eventually, the pulses will become separated with the pulse described by g moving off toward large positive values of x and the pulse described by f moving to large negative values of x . The pulses “collide”, but after the collision they retain their shape – their “identity”, if you will. To visualize this, return to the rope experiment. You shake the rope once, and your friend also shakes the rope once. Each of you produce a pulse which travels toward the other person, overlap for a time, then separate again unscathed. Try it! (You may need a long rope to see this work.) This is a simple example of solitonic behavior. In this example the solitonic behavior results from the linearity of the wave equation (superposition!) and its dispersion relation.

Our model of coupled oscillators that led to the wave equation is about as simple as it can be. As such, it does not take into account many details of the behavior of the medium that the waves propagate in. More realistic or alternate wave equations do not necessarily exhibit this solitary wave behavior because they lack the superposition property and or they lack the requisite dispersion relation. For example, let us consider the Schrödinger equation for a free, non-relativistic, quantum mechanical particle with mass m moving in one dimension:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}. \quad (21.3)$$

This equation, being linear, respects the superposition property, but you will recall it does not have the simple dispersion relation possessed by the wave equation. As we saw, the general solution of (21.3) is a superposition of traveling waves

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(k) e^{i(kx - \omega(k)t)} dk, \quad (21.4)$$

where $C(k)$ is determined by a choice of initial wave function, $\psi(x, 0)$, and

$$\omega(k) = \frac{\hbar k^2}{2m}. \quad (21.5)$$

The traveling waves appearing in the superposition have different speeds. To see this, just note that the wave with wavenumber k has a speed given by

$$v(k) = \frac{\omega(k)}{k} = \frac{\hbar k}{2m}. \quad (21.6)$$

The consequence of this dispersion relation is that, at times $t \neq 0$ a well-localized initial wave function does not retain its shape. Indeed, the wave pulse will spread as time passes because its different frequency (or wavelength) components do not travel at the same speed. (Physically, this is a manifestation of the uncertainty principle: localizing the particle at $t = 0$ within the support of the initial pulse leads to an uncertainty in momentum which,

at later times, reduces the localization of the particle.) One says that the Schrödinger waves exhibit *dispersion* because the wave profiles “disperse” as time runs. So we see that solitary wave behavior is not a universal feature of wave phenomena.

Let us look at another linear wave equation that exhibits dispersion. The following equation is known as the *Klein-Gordon equation* (in one spatial dimension):

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) q(x, t) + m^2 c^2 q(x, t) = 0. \quad (21.7)$$

This equation can be used to describe a relativistic quantum particle with rest mass determined by m (moving in one dimension). In this context, c is the speed of light – it is *not* the speed of the waves. You can think of it as a relativistic generalization of the Schrödinger equation.

Just like the Schrödinger equation, the general solution of Klein-Gordon equation can be constructed by superimposing sinusoidal wave solutions over amplitudes, phases and wavelengths. To see that dispersion arises, we simply compute the dispersion relation that arises for a sinusoidal wave. Consider a (complex) solution of the form

$$q(x, t) = A e^{i(kx - \omega t)}. \quad (21.8)$$

It is not hard to see that this wave solves the Klein-Gordon equation if and only if

$$\omega^2 = c^2 k^2 + m^2 c^4 \iff \omega = \pm c \sqrt{k^2 + m^2 c^2}. \quad (21.9)$$

As you can see, the wave speed ω/k again depends upon k , leading to dispersion.

Normally, solitons are discussed in the context of *non-linear wave equations*. A detailed study of such equations is beyond the scope of this text. Still, let us explore one such equation now and get a glimpse of how solitons can arise. A relatively simple non-linear equation is given by a modification of the Klein-Gordon equation (in one spatial dimension) for a scalar field $\phi(x, t)$:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{m^3}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{m} \phi\right) = 0. \quad (21.10)$$

For simplicity in what follows we have chosen units in which $c = 1$. The new parameter λ is going to be a “self-coupling constant”. The equation (21.10) is sometimes called the “sine-Gordon equation”, showing that even mathematicians have a sense of humor. We will use this terminology, also. The sine-Gordon equation is a wave equation that includes a “self-interaction” thanks to the sine term. Because of this term the equation is not linear, as one says, it is *non-linear*. Consequently, the superposition property (wherein one can linearly combine two solutions to make a third solution) is *not* present. To see the relationship between the sine-Gordon equation and the Klein-Gordon equation, suppose

that we restrict attention to solutions which always have a bounded, small magnitude for ϕ . In this case, we can make the approximation:

$$\sin\left(\frac{\sqrt{\lambda}}{m}\phi\right) \approx \frac{\sqrt{\lambda}}{m} \left(\phi - \frac{1}{6} \frac{\lambda}{m^2} \phi^3 + \cdots \right). \quad (21.11)$$

Using this Taylor expansion in the sine-Gordon equation, you will see that the first term in the expansion gives the Klein-Gordon equation (with $c = 1$) while the next (and higher) terms provide non-linearities. Physically, these describe a “self-interaction” of ϕ . The strength of the self-interaction is defined by λ . Indeed, if you consider the limit as $\lambda \rightarrow 0$ in (21.10) you will recover the Klein-Gordon equation (*exercise*).

There is an extensive body of literature that analyzes the sine-Gordon equation and methods for its solution. Here we merely point out that the sine-Gordon equation admits the solution

$$\phi(x, t) = 4 \frac{m}{\sqrt{\lambda}} \arctan \left(e^{m(x-x_0)} \right), \quad (21.12)$$

where x_0 is an arbitrary constant. This is the *static soliton solution*. To verify this you should first note that this putative solution does not depend upon time (it is a *static* solution), so the time derivatives of ϕ in (21.12) vanish and we only need to compare the x derivatives to the sine term. The key thing to check is that

$$\frac{d^2}{dy^2} (4 \arctan(e^y)) = \sin(4 \arctan(e^y)). \quad (21.13)$$

To check this, you will need the math facts:

$$\frac{d}{dy} \arctan(y) = \frac{1}{1+y^2}, \quad (21.14)$$

$$\sin(4 \arctan(y)) = -4 \frac{y(y^2 - 1)}{(1 + y^2)^2}. \quad (21.15)$$

(These calculations are a good place to try your skills with some algebraic computing software!) With these results in hand, it is a simple matter to see that (21.12) does solve the sine-Gordon equation (*exercise*).

The solution we have exhibited to the sine-Gordon equation is not, at first glance, a solitary wave such as we discussed for the wave equation. To see this, just plot the graph of soliton solution (see Problems) to see that the wave profile is rather spread out. However, it has the property that its *energy density* is localized about $x = x_0$. The energy density of ϕ is defined as follows:

$$\mathcal{E} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{m^4}{\lambda} (1 - \cos(\frac{\sqrt{\lambda}}{m}\phi)). \quad (21.16)$$

This definition is used because it leads to the continuity equation

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j}{\partial x} = 0 \quad (21.17)$$

when ϕ satisfies the sine-Gordon equation. Here we define the energy current density just as we did for the wave equation:

$$j = \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x}. \quad (21.18)$$

(You will be asked to verify this continuity equation in the Problems.) For simplicity, set $x_0 = 0$ and compute:

$$\mathcal{E}_{x_0=0} = 16 \frac{m^4}{\lambda} \frac{e^{2mx}}{(1 + e^{2mx})^2}. \quad (21.19)$$

This function is peaked about $x = 0$ and decays rapidly as $|x|$ grows. Thus the static soliton solution can be viewed as defining a “lump” of energy at $x = x_0$. It is this feature of the energy density that justifies the description of (21.12) as a “solitary wave”.

It is not too hard to see that the sine-Gordon equation actually allows for a (time-dependent) solution in which the soliton we have exhibited moves at any constant speed V . To see this, we employ an elegant trick, which is based upon the observation that if $\phi(x)$ is a solution to the equation then so is

$$\Phi(x, t) = \phi\left(\frac{x \pm Vt}{\sqrt{1 - V^2}}\right). \quad (21.20)$$

To check this, we just need the chain rule. We have

$$\frac{\partial \Phi(x, t)}{\partial x} = \frac{1}{\sqrt{1 - V^2}} \phi'\left(\frac{x \pm Vt}{\sqrt{1 - V^2}}\right) \quad (21.21)$$

and so

$$\frac{\partial^2 \Phi(x, t)}{\partial x^2} = \frac{1}{1 - V^2} \phi''\left(\frac{x \pm Vt}{\sqrt{1 - V^2}}\right). \quad (21.22)$$

We also have

$$\frac{\partial \Phi(x, t)}{\partial t} = \pm \frac{V}{\sqrt{1 - V^2}} \phi'\left(\frac{x \pm Vt}{\sqrt{1 - V^2}}\right), \quad (21.23)$$

and

$$\frac{\partial^2 \Phi(x, t)}{\partial t^2} = \frac{V^2}{1 - V^2} \phi''\left(\frac{x \pm Vt}{\sqrt{1 - V^2}}\right), \quad (21.24)$$

Here we using the notation

$$\phi'(z) = \left(\frac{d\phi(y)}{dy} \right)_{y=z}, \quad \phi''(z) = \left(\frac{d^2\phi(y)}{dy^2} \right)_{y=z}. \quad (21.25)$$

So the formulas with ϕ' , *etc.* are the derivatives of the function ϕ evaluated at the point $\frac{x \pm Vt}{\sqrt{1-V^2}}$. Now, using these formulas, plug the result into the sine-Gordon equation to see that Φ satisfies this equation because, as we showed earlier,

$$\phi'' = \frac{m^3}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{m}\phi\right). \quad (21.26)$$

The interpretation of this result is that one has solutions (21.20) of the sine-Gordon equation which are “lumps” of energy, propagating without change in shape at any constant speed V . One can view these solutions – these solitary waves – as a continuum model of a free particle.

The change of variables

$$x \rightarrow \frac{x \pm Vt}{\sqrt{1-V^2}}, \quad (21.27)$$

when complemented with

$$t \rightarrow \frac{t \pm Vx}{\sqrt{1-V^2}}, \quad (21.28)$$

is an example of a *Lorentz transformation*. It defines the relation between time and space as determined in two inertial reference frames that are moving at constant relative velocity $\pm V$, according to Einstein’s special theory of relativity. Using a computation similar to that performed above, it can be shown that *any* solution of the sine-Gordon equation, $\phi(x, t)$ is transformed to another solution $\Phi(x, t)$ of the sine-Gordon equation by the Lorentz transformation:

$$\Phi(x, t) = \phi\left(\frac{x \pm Vt}{\sqrt{1-V^2}}, \frac{t \pm Vx}{\sqrt{1-V^2}}\right). \quad (21.29)$$

This state of affairs is characterized by the statement that *Lorentz transformations are symmetries of the sine-Gordon equation*. You can think of two solutions related by (21.29) as a single solution being viewed in two different reference frames moving at constant relative velocity.

So far we have only shown how to get solitary wave solutions to the sine-Gordon equation, and this is all we shall do here. However, it can be shown that there exist solutions to the sine-Gordon equation that have the following properties, which we also saw when we looked at the wave equation and which justify calling these solutions “solitons”. At very early times (mathematically: $t \rightarrow -\infty$) the solution takes the form of two solitary waves of the type just described, very far apart and approaching each other at relative speed $2V$. This (more complicated) solution is usually called a “two soliton solution”. For the sake of our further discussion, let us call the solitary wave moving toward the left as “soliton 1” and the one moving to the right as “soliton 2”. As time runs the solution has a relatively complicated wave profile as the two solitary waves overlap and “interact”. At late times (mathematically: $t \rightarrow \infty$), the solution again takes the form of two solitons, with

soliton 1 now moving off toward $x = -\infty$ and soliton 2 moving off toward $x = +\infty$. Thus this solution can be viewed as a continuum model of two particles which approach each other, interact, then continue on their way unscathed. Moreover, multi-soliton solutions of this type also exist. It is this stability of the solutions as they propagate and interact with each other which is the defining feature of the soliton solutions.

PROBLEM SET 10

Problem 10.1

Show that if ϕ is any solution of the sine-Gordon equation then $\tilde{\phi} = -\phi$ and $\tilde{\phi} = \phi + 2n\pi$, where n is an integer, are also solutions to the sine-Gordon equation. (These are *discrete symmetries* of the sine-Gordon equation.)

Problem 10.2

Verify the continuity equation (21.17) for the sine-Gordon equation.

Problem 10.3

Plot the soliton solution (21.12) and its energy density as a function of x , verifying that the soliton is a localized “lump” of energy. How do these graphs change as you vary λ and m ? (Here is a good place to use a computer.)

Problem 10.4

Verify the identities (21.14) and (21.15).

Problem 10.5

The results of this problem will be used in the next problem. Define

$$u = \frac{m}{2}(t - x), \quad v = \frac{m}{2}(t + x), \quad \tilde{\phi}(u, v) = \frac{\sqrt{\lambda}}{m}\phi(t(u, v), x(u, v)).$$

Show that the sine-Gordon equation can be written as

$$\frac{\partial^2 \tilde{\phi}}{\partial u \partial v} + \sin(\tilde{\phi}) = 0. \tag{21.30}$$

Problem 10.6

Using the notation of the previous problem, show that the quantities

$$\rho = -\frac{1}{2} \left(\frac{\partial^2 \tilde{\phi}}{\partial u^2} \right)^2 + \frac{1}{8} \left(\frac{\partial \tilde{\phi}}{\partial u} \right)^4,$$

and

$$j = \frac{\partial^2 \tilde{\phi}}{\partial u^2} \frac{\partial^2 \tilde{\phi}}{\partial u \partial v} \sin(\tilde{\phi}) + \frac{\partial^2 \tilde{u}}{\partial u^2} \sin(\tilde{\phi}) - \frac{1}{2} \left(\frac{\partial \tilde{\phi}}{\partial u} \right)^2 \cos(\tilde{\phi}).$$

satisfy a continuity equation

$$\frac{\partial \rho}{\partial v} + \frac{\partial j}{\partial u} = 0,$$

when $\tilde{\phi}$ satisfies (21.30). The resulting conservation law is but one of an infinite hierarchy of conserved quantities that are intimately related to the stability of the soliton.

Appendix A. Taylor's Theorem and Taylor Series

Taylor's theorem and Taylor's series constitute one of the more important tools used by mathematicians, physicists and engineers. They provides a means of approximating a function in terms of polynomials. To begin, we present *Taylor's theorem*, which is an identity satisfied by any function $f(x)$ that has continuous derivatives of, say, order $(n+1)$ on some interval $a \leq x \leq b$. Taylor's theorem asserts that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_{n+1}, \quad (\text{A.1})$$

where R_{n+1} – the *remainder* – can be expressed as

$$R_{n+1} = \frac{1}{(n+1)!}(x-a)^{n+1}f^{(n+1)}(\xi), \quad (\text{A.2})$$

for some ξ with $a \leq \xi \leq b$. Here we are using the notation

$$f^{(k)}(c) = \left. \frac{d^k f}{dx^k} \right|_{x=c}. \quad (\text{A.3})$$

The number ξ is not arbitrary; it is determined (though not uniquely) via the mean value theorem of calculus. For our purposes we just need to know that it lies between a and b . The equation (A.1) is an identity; it involves no approximations.

The idea is that for many functions the value of n can be chosen so that the remainder is sufficiently small compared to the polynomial terms that we can omit it. In this case we get Taylor's approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots + \frac{1}{n!}f^{(n)}(a)(x-a)^n. \quad (\text{A.4})$$

Typically, the approximation is reasonable provided x is close enough to a and none of the derivatives of f get too large in the region of interest. As you can see, if $(x-a)$ is small, *i.e.*, $x-a \ll 1$, successive powers of $(x-a)$ become smaller and smaller so that one need only keep a few terms in the polynomial expansion to get a good approximation.

If you can prove that

$$\lim_{n \rightarrow \infty} R_n = 0, \quad (\text{A.5})$$

then it makes sense to consider expressing $f(x)$ as a *power series*:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(x-a)^n, \quad (\text{A.6})$$

which is known in this context as the *Taylor series* for f . (Note that here we use the definitions $0! = 1$ and $f^{(0)}(x) = f(x)$.) Normally the Taylor series of a function will

converge in some neighborhood of $x = a$ and diverge outside of this neighborhood.* In any case, for a sufficiently “well-behaved” function, one can get a good approximation to it using (A.4) by keeping n relatively small. How small n needs to be depends, in large part, on how big $(x - a)$ is. Often times one can get away with just choosing $n = 1$ or perhaps $n = 2$ for x sufficiently close to a .

As a simple example, consider the sine function $f(x) = \sin(x)$. Let us approximate the sine function in the vicinity of $x = 0$, so that we are taking $a = 0$ in the above formulas. The *zeroth-order* approximation amounts to using $n = 0$ in (A.4). We get

$$\sin(x) \approx \sin(0) = 0. \quad (\text{A.7})$$

This is obviously not a terribly good approximation. But you can check (using your calculator *in radian mode*) that if x is nearly zero, so is $\sin(x)$. A better approximation, the *first-order* approximation, arises when $n = 1$ in (A.4). We get (exercise)

$$\sin(x) \approx \sin(0) + \cos(0)x = x. \quad (\text{A.8})$$

Again, you can check this approximation on your calculator. If x is kept sufficiently small (in radians), this approximation does a pretty good job. As x gets larger the approximation gets less accurate. For example, at $x = 0.1$ the error in the approximation is about 0.2%. At $x = 0.75$, the error is about 10%. The *second-order* approximation is identical to the first-order approximation, as you can check explicitly (exercise). The third-order approximation (exercise),

$$\sin(x) \approx x - \frac{1}{6}x^3 \quad (\text{A.9})$$

is considerably better than the first-order approximation. It gives good results out to, say, $x = 1.7$, where the error is about 11%. Incidentally, the remainder term for the sine function satisfies (A.5) (exercise), and we can represent the analytic function $\sin(x)$ by its (everywhere convergent) Taylor series.

Appendix B. Vector Spaces

Throughout this text we have noted that various objects of interest form a *vector space*. Here we outline the basic structure of a vector space. You may find it useful to refer to this Appendix when you encounter this concept in the text.

B.1. What are vector spaces?

In physics and engineering one of the first mathematical concepts that gets introduced is the concept of a vector. Usually, a vector is defined as a quantity that has a direction and

* Functions that can be represented by a convergent Taylor series are called *analytic*.

a magnitude, such as a position vector, velocity vector, acceleration vector, *etc.* However, the notion of a vector has a considerably wider realm of applicability than these examples might suggest. The set of all real numbers forms a vector space, as does the set of all complex numbers. The set of functions on a set (*e.g.*, functions of one variable, $f(x)$) form a vector space. Solutions of linear homogeneous equations form a vector space. There are many more examples, some of these are highlighted in the text. We begin by giving the abstract rules for forming a space of vectors, also known as a vector space.

A *vector space* V is a set equipped with an operation of “addition” and an additive identity. The elements of the set are called vectors, which we shall denote as \vec{u} , \vec{v} , \vec{w} , *etc.* For now, you can think of them as position vectors in order to keep yourself sane. *Addition*, is an operation in which two vectors, say \vec{u} and \vec{v} , can be combined to make another vector, say, \vec{w} . We denote this operation by the symbol “+”:

$$\vec{u} + \vec{v} = \vec{w}. \quad (\text{B.1})$$

Do not be fooled by this simple notation. The “addition” of vectors may be quite a different operation than ordinary arithmetic addition. For example, if we view position vectors in the x - y plane as “arrows” drawn from the origin, the addition of vectors is defined by the parallelogram rule. Clearly this rule is quite different than ordinary “addition”. In general, any operation can be used to define addition if it has the commutative and associative properties:

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}, \quad (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}), \quad (\text{B.2})$$

The requirement of an additive identity means that there exist an element of V , called the zero vector and denoted by $\vec{0}$, such that for any element $\vec{v} \in V$,

$$\vec{v} + \vec{0} = \vec{v}. \quad (\text{B.3})$$

As an exercise you can check that the set of position vectors relative to the origin in the x - y plane forms a vector space with (i) the vectors being viewed as arrows with the parallelogram rule for addition, and (ii) the position of the origin being the zero vector.

In applications to physics and engineering one is normally interested in a vector space with just a little more structure than what we defined above. This type of vector space has an additional operation, called *scalar multiplication*, which is defined using either real or complex numbers, called *scalars*. Scalars will be denoted by a , b , c , *etc.* . When scalar multiplication is defined using real (complex) numbers for scalars, the resulting gadget is called a real (complex) vector space.* Scalar multiplication is an operation in which a scalar a and vector \vec{v} are combined to make a new vector, denoted by $a\vec{v}$. Returning to our example of position vectors in the plane, the scalar multiplication operation is defined by

* One often gets lazy and calls a real/complex vector space just a “vector space”.

saying that the vector $a\vec{v}$ has the same direction as \vec{v} , *provided* $a \geq 0$, but the length of \vec{v} is scaled by the amount a . So, if $a = 2$ the vector is doubled in length, and so forth. If the scalar is negative, then the vector is reversed in direction, and its length is scaled by $|a|$. In general, any rule for scalar multiplication is allowed provided it satisfies the properties:

$$(a + b)\vec{v} = a\vec{v} + b\vec{v}, \quad a(b\vec{v}) = (ab)\vec{v}, \quad a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}, \quad 1\vec{v} = \vec{v}, \quad 0\vec{v} = \vec{0}. \quad (\text{B.4})$$

Again, you can check that the scalar multiplication we use for position vectors satisfies all these properties.

As an exercise, prove that the vector $-\vec{v}$, defined by

$$-\vec{v} = (-1)\vec{v} \quad (\text{B.5})$$

is an *additive inverse* of \vec{v} , that is,

$$\vec{v} + (-\vec{v}) = \vec{0}. \quad (\text{B.6})$$

We often use the notation

$$\vec{v} + (-\vec{w}) \equiv \vec{v} - \vec{w}, \quad (\text{B.7})$$

so that

$$\vec{w} - \vec{w} = \vec{0}. \quad (\text{B.8})$$

One of the most important features of a (real or complex) vector space is the existence of a *basis*. To define it, we first introduce the notion of linear independence. A subset of vectors $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k)$ is *linearly independent* if no non-trivial *linear combination* of them vanishes, *i.e.*, a relation

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \dots + a_k\vec{e}_k = \vec{0} \quad (\text{B.9})$$

exists between the elements of the set only if $a_1 = a_2 = \dots = a_k = 0$. If such a relation (B.9) exists, the subset is called *linearly dependent*. For example, if \vec{v} and \vec{w} are position vectors, then they are linearly dependent if they have parallel or anti-parallel directions, *i.e.*, they are colinear. If they are not colinear, then they are linearly independent (exercise). Note that in a linearly dependent subset of vectors it will be possible to express some of the vectors as *linear combinations* of the others. In general, there will be a unique maximal size for sets of linearly independent vectors. If all sets with more than n vectors are linearly dependent, then we say that the vector space is n -dimensional, or has n dimensions. In this case, any set of n linearly independent vectors, say $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$, is

said to form a *basis*.^{*} The utility of a basis is that every element of V can be uniquely expressed as a linear combination of the basis vectors:

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 + \dots + v^n \vec{e}_n. \quad (\text{B.10})$$

The scalars v^i , $i = 1, 2, \dots, n$ are called the *components* of \vec{v} in the basis \vec{e}_i . (Note that in expressions like v^1, v^2, v^3, \dots the superscripts are simply numerical labels — not exponents!) Thus a vector can be characterized by its components in a basis. As a nice exercise, you can check that, in a given basis, the components of the sum of two vectors \vec{v} and \vec{w} are the ordinary sums of the components of \vec{v} and \vec{w} :

$$(\vec{v} + \vec{w})^i = v^i + w^i. \quad (\text{B.11})$$

Likewise, you can check that the components of the scalar multiple $a\vec{v}$ are obtained by ordinary multiplication of each component of \vec{v} by the scalar a :

$$(a\vec{v})^i = av^i. \quad (\text{B.12})$$

Let us take a deep, relaxing breath and return to our running example, position vectors in the plane. As you know, in the x - y plane we can introduce a basis consisting of a (unit) vector \vec{e}_1 along the x direction and a (unit) vector \vec{e}_2 along the y direction. Every position vector can then be expressed as

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2, \quad (\text{B.13})$$

where v^1 is the “ x -component” of \vec{v} (sometimes denoted by v^x) and v^2 is the “ y -component” of \vec{v} (sometimes denoted by v^y). Evidently, the set of position vectors in the plane is a 2-dimensional, real vector space.

B.2. Scalar Products

Often times we augment the properties of a vector space with an extra bit of structure called a *scalar product* (also known as an “inner product”). The scalar product is a way of making a scalar from a pair of vectors. We shall denote this scalar by (\vec{v}, \vec{w}) . The scalar product generalizes the familiar notion of a “dot product” of position vectors, velocity vectors, *etc.* Any rule for forming a scalar from a pair of vectors will be allowed as a scalar product provided it satisfies[†]

$$(\vec{v}, \vec{w}) = (\vec{w}, \vec{v}), \quad (a\vec{v} + b\vec{w}, \vec{u}) = a(\vec{v}, \vec{u}) + b(\vec{w}, \vec{u}), \quad (\vec{v}, \vec{v}) \geq 0, \quad (\text{B.14})$$

^{*} It can be shown that a vector space of dimension n admits infinitely many sets of basis vectors, but each basis will always consist of precisely n (linearly independent) vectors. Speaking a little loosely, if $n = \infty$ we say that the vector space is *infinite dimensional*.

[†] Here, for simplicity, we restrict ourselves to a real vector space, where the scalars are always real numbers.

and $(\vec{v}, \vec{v}) = 0$ if and only if $\vec{v} = \vec{0}$. As a good exercise you can check that the dot product of position vectors,

$$(\vec{v}, \vec{w}) \equiv \vec{v} \cdot \vec{w}, \quad (\text{B.15})$$

which you certainly should have some experience with by now, provides an example of a scalar product.

Borrowing terminology from, say, position vectors in the plane, a pair of vectors \vec{v} and \vec{w} are called *orthogonal* if

$$(\vec{v}, \vec{w}) = 0. \quad (\text{B.16})$$

If two vectors are orthogonal, then they are linearly independent. The converse is not true, however (exercise). Likewise we define the *length* or *norm* of a vector, $\|\vec{v}\|$ by

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}. \quad (\text{B.17})$$

We say that a vector is *normalized* if it has unit length. Any vector can be normalized by scalar multiplication (exercise). A basis \vec{e}_i , $i = 1, 2, \dots, n$ is called *orthonormal* if it has the following scalar products among its elements:

$$(\vec{e}_i, \vec{e}_j) = \delta_{ij}, \quad i, j = 1, 2, \dots, n. \quad (\text{B.18})$$

Here we have used the very convenient symbol δ_{ij} , known as the *Kronecker delta*, by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \quad (\text{B.19})$$

If \vec{e}_i , $i = 1, 2, \dots, n$ is an orthonormal basis, you can verify as an exercise that the components v^i of a vector \vec{v} ,

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 + \dots + v^n \vec{e}_n, \quad (\text{B.20})$$

can be computed by

$$v^i = (\vec{e}_i, \vec{v}). \quad (\text{B.21})$$

B.3. Linear Operators

An extremely useful notion that is employed in the context of vector spaces is that of a *linear operator* (sometimes just called an “operator”). Given a (real or complex) vector space V , a linear operator denoted, say, by A , is

- (i) an *operator*: it is a rule which assigns to every vector $\vec{v} \in V$ another (possibly the same) vector, denoted $A\vec{v}$.
- (ii) *linear*: for any two vectors \vec{v} and \vec{w} and any two scalars a and b ,

$$A(a\vec{v} + b\vec{w}) = aA\vec{v} + bA\vec{w}. \quad (\text{B.22})$$

This latter requirement can be viewed as insisting that the operator is compatible with the structure of the set V as a vector space.

We encounter several examples of linear operators in the text. All the differential operators appearing in the linear differential equations are linear operators. The matrices featuring in the coupled oscillator discussion define linear operators. As a simple example of a linear operator, consider the real vector space of position vectors in the x - y plane. We can define a linear operator by the rule that takes any vector and rotates it 10° clockwise about the origin. To see that this rule is linear is a very nice exercise — try it! As another exercise: Which vector(s) are left unchanged by this rule?

You may have encountered linear operators only in the context of matrices, matrix multiplication, *etc.* While the idea of a linear operator is somewhat more general than that of a matrix, for many applications one need only use matrices. This is because every linear operator on a finite dimensional vector space can be viewed as a matrix acting upon column vectors. The matrix and the column vectors are defined relative to a choice of basis.

B.4. Eigenvalue problems

Given a linear operator, A , one is often interested in its *eigenvalues* and *eigenvectors*. The eigenvalue is a scalar and the eigenvector is a vector; if we denote them by λ and \vec{e}_λ , respectively, they are solutions to

$$A\vec{e}_\lambda = \lambda\vec{e}_\lambda. \quad (\text{B.23})$$

It is a fundamental result of linear algebra (for finite dimensional vector spaces) that the eigenvectors and eigenvalues (over the complex numbers) completely characterize the linear operator A . Indeed, the word “eigen” in German means “inherent” or “characteristic”.

Assuming A is a linear operator on a finite dimensional vector space, one solves the *eigenvalue problem* (B.23) as follows. Let the vector space have dimension n . In a given basis, the operator A is represented by a square $n \times n$ matrix, which we shall also denote by A , while the eigenvector e_λ will be represented by a column vector with n rows. It is a fundamental theorem of linear algebra that (B.23) has a solution if and only if there exists a solution λ to

$$\det(A - \lambda I) = 0, \quad (\text{B.24})$$

where I is the identity matrix. This is the *characteristic equation* defined by A ; it says that λ is the root of an n^{th} -order polynomial. If we are working with complex numbers, then (B.24) always has a solution. But, as is often the case, if we are working with real numbers there may be no solution to (B.24), and hence there will be no eigenvalues/eigenvectors. Given a solution λ to (B.24), one can substitute it into (B.23) and solve for the eigenvector(s) \vec{e}_λ corresponding to λ . Note that if \vec{e}_λ is a solution to (B.23) for some λ , then so

is any scalar multiple of \vec{e}_λ (*exercise*). This means that (given a scalar product) one can always normalize the eigenvectors to have unit length.

Still assuming the real vector space of interest is finite-dimensional (so that, for example, our discussion need not apply to linear differential operators acting on vector spaces of functions) the solution to the eigenvalue problem for *symmetric* operators has very special properties. Recall that a symmetric operator has a matrix which satisfies $A^T = A$, where the superscript “T” means “transpose” (interchange rows and columns). For example, a 2×2 matrix M is symmetric if and only if it takes the form

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Symmetric operators always have real eigenvalues, *i.e.*, (B.24) always has solutions even when we are restricting to real numbers only. More significantly, the eigenvectors always form a basis. Moreover, eigenvectors belonging to two distinct eigenvalues are always orthogonal. With a little work, it can be shown that the eigenvectors of a symmetric operator can be chosen to form an orthonormal basis.