General Solution with Boundary Conditions

Overview and Motivation: Last time we wrote down the general form of the solution to the 1D wave equation. We then solved the initial-value problem for an infinitely long system. Today we use the same form of the solution and solve the initial-value problem for a finite system with boundary conditions.

Key Mathematics: We again use chain rule for taking derivatives and utilize the Gaussian function.

I. Review of the Initial Value Problem (for an Infinite System)

Last time we considered waves on a one-dimensional system of infinite extent. We wrote down the solution to the wave equation

$$\frac{\partial^2 q(x,t)}{\partial t^2} = c^2 \frac{\partial^2 q(x,t)}{\partial x^2} \tag{1}$$

as

$$q(x,t) = f(x+ct) + g(x-ct), \tag{2}$$

where f and g are any well-behaved functions. In terms of the initial conditions $q(x,0) = a(x)$ and $\partial q(x,0)/\partial t = b(x)$ the functions f and g can be written as

$$f(x) = \frac{1}{2}\left[a(x) + \frac{1}{c}\int_{x_0}^{x} b(x')dx'\right] \tag{3a}$$

and

$$g(x) = \frac{1}{2}\left[a(x) - \frac{1}{c}\int_{x_0}^{x} b(x')dx'\right] \tag{3b}$$

which results in the solution to the initial-value problem

$$q(x,t) = \frac{1}{2}\left[a(x+ct) + a(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} b(x')dx'\right]. \tag{4}$$

We then looked at two examples where the initial conditions were related to the Gaussian function.

Today we are going to again consider the initial-value problem, but this time on a system of finite extent, where we will impose some boundary conditions. As we shall see, the bc's impose constraints on the functions $f(x+ct)$ and $g(x-ct)$, with one result being that the motion of the system is periodic.

II. Imposing the Boundary Conditions

Let's assume that the extent of the physical system is from $x=0$ to $x=L$. We consider bc's equivalent to those for the coupled oscillator system

$$q(0,t)=0 \tag{5a}$$

and

$$q(L,t)=0\,. \tag{5b}$$

These bc's are also appropriate for transverse waves on a string where the end supports are fixed or for sound waves that travel along the axis of a pipe that is closed at both ends. For a pipe these bc's are often referred to as closed-closed bc's.

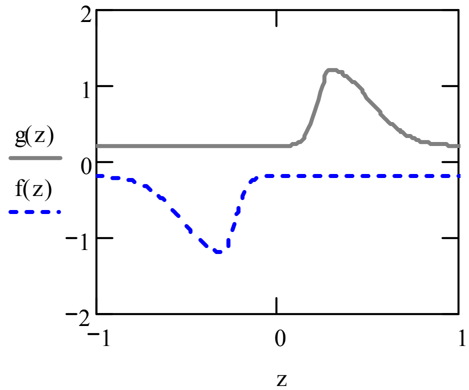
Applying the first bc $q(0,t)=0$ to the form of $q(x,t)$ expressed in Eq. (2) gives us

$$f(ct)=-g(-ct). \tag{6}$$

So what does this equation tell us? Well, because this bc applies for all times t, Eq. (6) is valid for any value of ct, and so we can introduce another variable $z=ct$ and re-express Eq. (6) as

$$f(z)=-g(-z), \tag{7}$$

which must hold for all z. So irrespective of anything else (like the initial conditions), we see that the functions f and g are intimately related. The following picture illustrates the relationship expressed by Eq. (7). The solid curve is (some arbitrary) $g(x)$. The dashed curve is $f(z)$ corresponding to $g(z)$ consistent with Eq. (7).



Let's now consider the second bc $q(L,t)=0$. When applied to Eq. (2) this bc gives

$$f(L+ct)=-g(L-ct). \tag{8}$$

Now, again, because this equation is valid for any value of t, it is valid for any value of ct, so this time let's let $L+ct=z$. Then Eq. (8) can be re-expressed as

$$f(z)=-g(2L-z). \tag{9}$$

And using Eq. (7) to replace $f(z)$ in Eq. (9) gives us

$$g(-z)=g(2L-z), \tag{10}$$

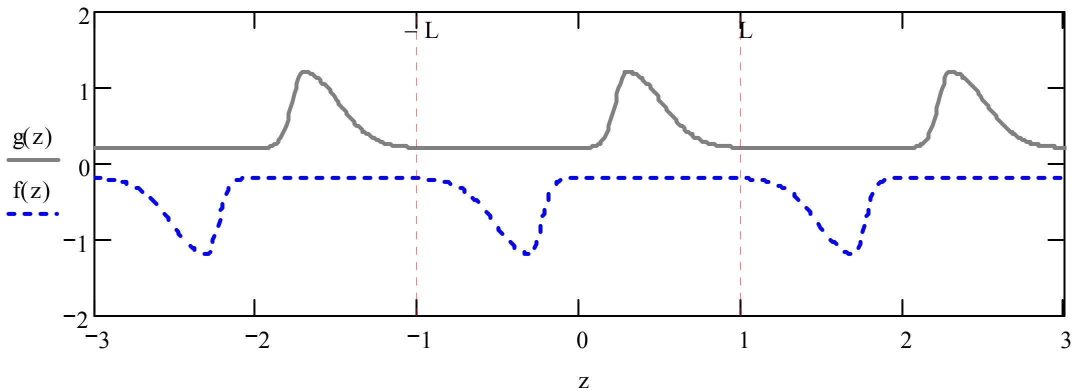
which is valid for all values of z, so let's replace z by $-z$, which results in

$$g(z+2L)=g(z). \tag{11}$$

Now this is very interesting. It says that $g(z)$ is periodic with period $2L$. Of course, because $f(z)=-g(-z)$, $f(z)$ is also periodic with period $2L$. Thus we also have

$$f(z+2L)=f(z). \tag{12}$$

Summarizing, the two bc's $q(0,t)=0$ and $q(L,t)=0$ have imposed the constraints given by Eqs. (7), (11), and (12) on the functions f and g. So our previous illustration of f and g must be modified, as shown in the following picture (where we have set $L=1$).



OK, so something like the following may be bothering you: if f and g are periodic with period $2L$ then, for $t = 0$, for example, $f(x+ct) = f(x)$ and $g(x+ct) = g(x)$ are defined outside the physical boundaries of the system, which lies between 0 and L. That is indeed true, but so what? There is no problem in defining f and g as functions of infinite extent; in fact they must be defined over an infinite domain because $f(x+ct)$ and $g(x+ct)$ must be defined for all times t. We just need to remember that they only describe the physical system via $q(x,t) = f(x+ct) + g(x-ct)$ for x between 0 and L.

III. An Initial Value Problem

Let's now look at an initial-value problem with the boundary conditions discussed above. As we did in the last lecture, let's see what happens with an initial Gaussian displacement and no initial velocity, which we write as

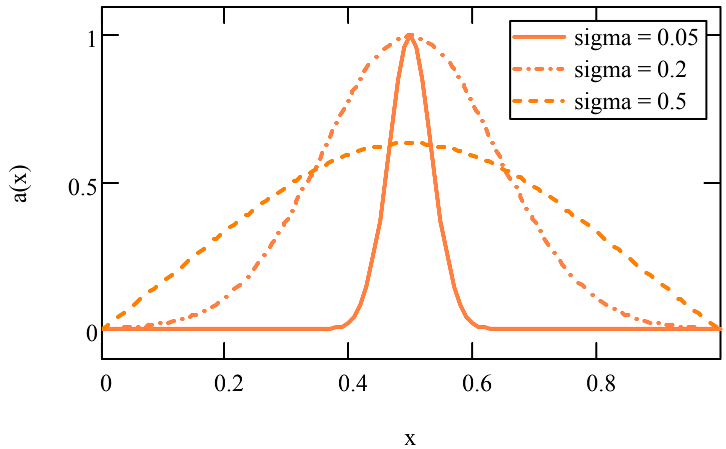
$$a(x) = A\left\{\exp\left[-\left(\frac{x-(L/2)}{\sigma}\right)^2\right] - \exp\left[-\left(\frac{(L/2)}{\sigma}\right)^2\right]\right\} \tag{13a}$$

and

$$b(x) = 0. \tag{13b}$$

If you compare Eq. (13a) to the similar initial condition that we discussed in Lecture 9, you will notice that it is slightly more complicated. First, the Gaussian function is centered at $x = L/2$ rather than $x = 0$. Second, we have subtracted off a constant from the Gaussian: this insures that the two bc's are satisfied by the initial condition. This particular initial condition is illustrated in the next picture for three values of σ,

$\sigma = 0.05$, $\sigma = 0.2$, and $\sigma = 0.5$. ($A = 1$ and $L = 1$ for both initial conditions). We must also keep in mind that outside the interval $0 < x < 1$, $a(x)$ and $b(x)$ are not defined.



Let's see what this initial condition tells us about the functions f and g. Referring to Eq. (3) we see that

$$f(x) = g(x) = \frac{1}{2}a(x), \tag{14}$$

but because $a(x)$ is only defined on the interval $0 \le x \le L$, this equation is only valid in that domain. We must use Eqs. (7), (11), and (12) to define $f(x)$ and $g(x)$ outside this domain. Using Eq. (7) we can define both functions for $-L \le x < 0$. Eq. (7) and Eq. (14) together imply

$$f(-x) = -f(x), \tag{15a}$$

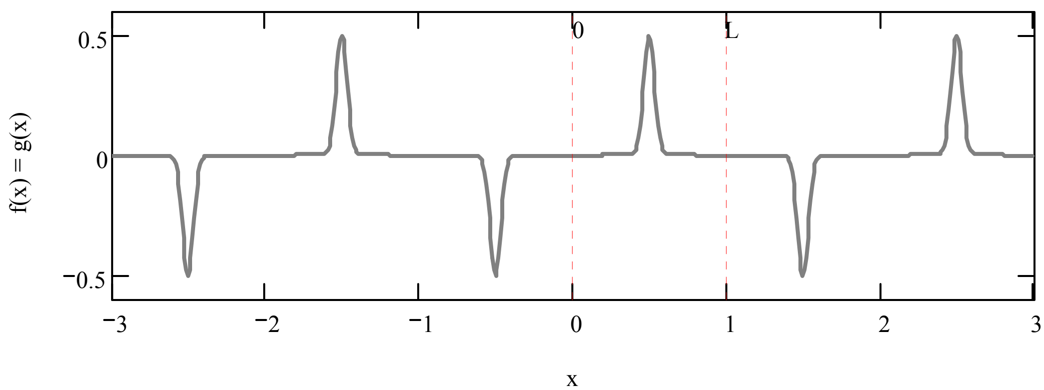
and

$$g(-x) = -g(x). \tag{15b}$$

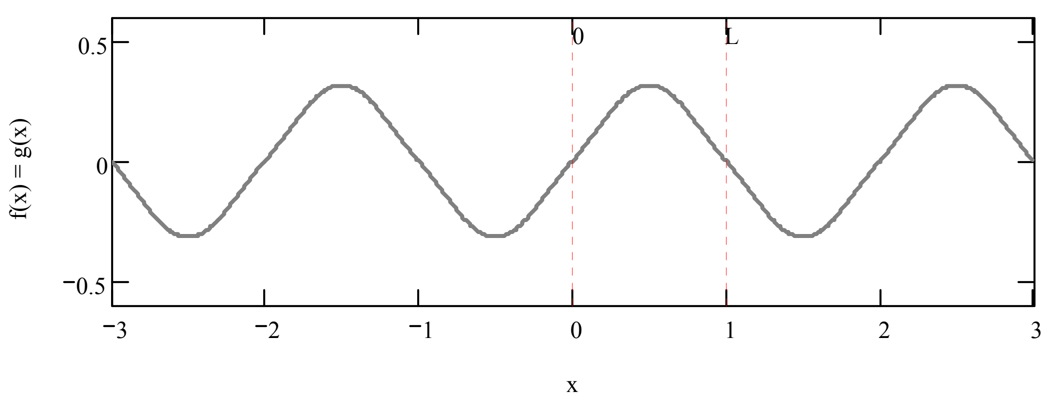
That is, $f(x)$ and $g(x)$ are both odd about $x = 0$. We now know what $f(x)$ and $g(x)$ are for $-L \le x < L$. We can use Eqs. (11) and (12), which tell us that both functions are periodic with period $2L$, to define $f(x)$ and $g(x)$ outside this interval. Putting all of this together, we can formally write the functions $f(x)$ and $g(x)$ as

$$f(x) = g(x) = \frac{1}{2}\sum_{m=-\infty}^{\infty}\left[a(x - 2mL) - a(-x + 2mL)\right], \tag{16}$$

but we must remember that $a(x)$ is only defined on the interval $0 < x < L$.
For the initial condition described by Eq. (13) with $a_0 = 0.05$, the following figure plots $f(x) = g(x)$.



So now that we know $f(x)$ and $g(x)$ for all values of their arguments, we have the solution to the initial value problem via Eq. (2). Instead of a picture to illustrate the time-dependent motion, go check out Video 1 for Lecture 9 on the class web site.[1] Notice that $f(x)$ and $g(x)$ are indeed constructed so that the bc's are satisfied. Also notice that the effect of the bc's is to make the Gaussian pulses flip over when they reflect from the boundaries. Further, notice that the motion is indeed periodic in time. Can you figure out what the period is?



[1] In the video the functions f and g have been displaced vertically for clarity.

Let's also look at the case of the initial condition with $\sigma = 0.5$, illustrated on p. 5? The picture at the bottom of the preceding page plots $f(x) = g(x)$ for this case. These functions look similar to harmonic functions, but they are not – they too are described by Eq. (13). As you can see in Video 2 for Lecture 9, the resulting motion is similar to an harmonic standing wave.

Summarizing, we have seen that in a finite system with boundary condition, the solution to the wave equation can again be written in the form of Eq. (2), the sum of waves traveling at speed c and propagating in the $-x$ and $+x$ directions. The boundary conditions, however, put constraints on the traveling-wave functions f and g. These constraints, in turn, make the motion of the system periodic.

Exercises

***9.1** Show that Eq. (13a) satisfies the bc's [Eq. (5a) and (5b)] for the problem discussed in the notes.

***9.2** For the problem discussed in the notes (waves on a string located between $x = 0$ and $x = L$) find the temporal period of the motion in terms of the parameters c and L.

****9.3** Because $q(x,t) = Ae^{-(x-ct)^2/\sigma^2}$ is a function of $x - ct$, it is a solution to the wave equation (on an infinite domain).
(**a**) What are the initial conditions [$a(x)$ and $b(x)$] that give rise to this form of $q(x,t)$?
(**b**) If $f(x)$ is constant, then Eq. (2) shows that solution is only a function of $x - ct$. For the condition that $f(x)$ is constant find $b(x)$ in terms of $a(x)$. [Hint: consider Eq. (3a).]
(**c**) Show that the initial conditions you found in part (**a**) satisfy the relationship that you found in part (**b**).