1D Wave Equation – General Solution / Gaussian Function

Overview and Motivation: Last time we derived the partial differential equation known as the (one dimensional) wave equation. Today we look at the general solution to that equation. As a specific example of a localized function that can be useful when studying waves, we introduce the Gaussian function.

Key Mathematics: We reacquaint ourselves with the chain rule (for taking derivatives) and look at the Gaussian function and the integral of the Gaussian function, which is known as the error function.

I. Solutions to the Wave Equation

A. General Form of the Solution

Last time we derived the wave equation

$$\frac{\partial^2 q(x,t)}{\partial t^2} = c^2 \frac{\partial^2 q(x,t)}{\partial x^2} \tag{1}$$

from the long wave length limit of the coupled oscillator problem. Recall that c^2 is a (constant) parameter that depends upon the underlying physics of whatever system is being described by the wave equation.

Now it may surprise you, but the solution to Eq. (1) can, quite generally, be written in very succinct form as

$$q(x,t) = f(x+ct) + g(x-ct), \tag{2}$$

where f and g are any "well-behaved" functions. We won't worry about the details of what well-behaved means, but certainly we certainly want their second derivatives to exist. As we previously discussed, $f(x+ct)$ travels in the $-x$ direction at the speed c and $g(x-ct)$ travels in the $+x$ direction at the same speed. To see that Eq. (2) is a solution to Eq. (1) let's calculate the second x and t derivatives of $f(x+ct)$ and $g(x-ct)$. To do this we need the chain rule, which can be written for the case at hand as

$$\frac{\partial h[k(x,t)]}{\partial x} = \frac{dh}{dk}\frac{\partial k}{\partial x} = h'\frac{\partial k}{\partial x}. \tag{3}$$

Applying this rule to $f(x+ct)$, for example, we have

$$\frac{\partial f(x+ct)}{\partial x} = f'(x+ct)\frac{\partial(x+ct)}{\partial x} = f'(x+ct), \tag{4}$$

where $f'(x+ct)$ is the derivative of $f(x+ct)$ with respect to its argument. Applying the chain rule again to calculate the second derivative of $f(x+ct)$ with respect to x give us

$$\frac{\partial^2 f(x+ct)}{\partial x^2} = \frac{\partial f'(x+ct)}{\partial x} = f''(x+ct)\frac{\partial(x+ct)}{\partial x} = f''(x+ct). \tag{5}$$

Similarly, we have for the t derivatives

$$\frac{\partial f(x+ct)}{\partial t} = f'(x+ct)\frac{\partial(x+ct)}{\partial t} = c\, f'(x+ct) \tag{6}$$

and

$$\frac{\partial^2 f(x+ct)}{\partial t^2} = c\frac{\partial f'(x+ct)}{\partial t} = cf''(x+ct)\frac{\partial(x+ct)}{\partial t} = c^2 f''(x+ct). \tag{7}$$

From Eqs. (5) and (7) we see that

$$\frac{\partial^2 f(x+ct)}{\partial t^2} = c^2\,\frac{\partial^2 f(x+ct)}{\partial x^2} \tag{8}$$

and so $f(x+ct)$ indeed solves the wave equation. Proof that $g(x-ct)$ also satisfies Eq. (1) follows from an essentially identical calculation.

B. More Specific Solutions

So the (totally unknown) functions $f(x+ct)$ and $g(x-ct)$ are solutions, but only in a very general sense. As we shall see, any solution has the form of Eq. (2), but how do we know what the solution will be in any given situation? Well, as with earlier problems that we have looked at in this class, the situation can be specified by **initial conditions** and the **boundary conditions**. Recall, for the coupled oscillator problem the initial conditions were specified by values for $q_j(0)$ and $\dot{q}_j(0)$. Now, however, the spatial variable is not the discrete index j but the continuous variable x. The corresponding initial conditions can thus be written as

$$q(x,0) = a(x) \tag{9a}$$

and

$$\frac{\partial q}{\partial t}(x,0) = b(x), \tag{9b}$$

where $a(x)$ and $b(x)$ are assumed to be known functions. We could have just stuck with $q(x,0)$ and $\frac{\partial q}{\partial t}(x,0)$, but in the long run we save a bit of notational cumbersomeness by using $a(x)$ and $b(x)$ instead. For the coupled-oscillator system the boundary conditions are $q_0(t) = 0$ and $q_{N+1}(t) = 0$. There will be similar instances for the wave equation where we will be interested in waves on a finite sized system. In such cases we will need to specify the condition on $q(x,t)$ at the system boundaries. Indeed, you have already seen an example of this in Exercise 7.4 from the last lecture notes.

II. Initial Value Problem (IVP) for an Infinite System

Here we write the most general solution to the wave equation, given the initial conditions $a(x)$ and $b(x)$. To keep things simple at this point we will assume that the ends of the (1D) system where these waves exist are at $-\infty$ and $+\infty$. In this way we do not have to deal with any boundary conditions. (We will deal with bc's in the next lecture!) From Eq. (2)

$$q(x,t) = f(x+ct) + g(x-ct), \tag{2}$$

we have for $t = 0$

$$a(x) = f(x) + g(x). \tag{10}$$

That is simple enough. What about the other initial condition. Well, taking the t derivative of Eq. (2) and setting it equal to $b(x)$ at $t = 0$ gives us

$$b(x) = c[f'(x) - g'(x)]. \tag{11}$$

Now Eq. (11) can be formally integrated, which gives us

$$\frac{1}{c}\int_{x_0}^{x} b(x')dx' = f(x) - g(x) \tag{12}$$

where x_0 can have any (constant) value. If we now take the sum and difference of Eqs. (10) and (12) we obtain

$$f(x)=\frac{1}{2}\left[a(x)+\frac{1}{c}\int_{x_0}^{x}b(x')dx'\right] \tag{13a}$$

and

$$g(x)=\frac{1}{2}\left[a(x)-\frac{1}{c}\int_{x_0}^{x}b(x')dx'\right] \tag{13b}$$

Using Eq. (13) in Eq. (2) finally gives us the general solution to the initial-value problem

$$q(x,t)=\frac{1}{2}\left[a(x+ct)+a(x-ct)+\frac{1}{c}\int_{x-ct}^{x+ct}b(x')dx'\right]. \tag{14}$$

Notice that the undetermined constant x_0 has disappeared. Eq. (14) is remarkably simple.

II. The Gaussian Function and Two Initial-Value-Problem Examples

A. Gaussian Function

A very useful function in physics is the **Gaussian**, which is defined as

$$G_\sigma(x)=e^{-x^2/\sigma^2}. \tag{15}$$

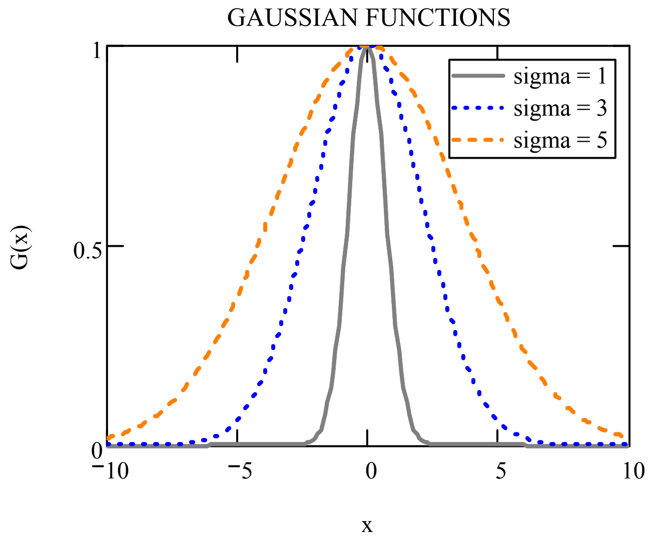
As shown in the picture on the top of the following page, the Gaussian is peaked at $x=0$ and has a width that is proportional to the parameter σ. In fact, the **full width at half maximum** (FWHM), which is the width of the peak at half its maximum height, is equal to $2\sqrt{\ln(2)}\sigma \approx 1.665\sigma$.

B. IVP Solution with Gaussian Initial Position

Let's see what the solution $q(x,t)$ looks like with the initial conditions $a(x)=AG_\sigma(x)$, (A is just some arbitrary amplitude) and $b(x)=0$. (Physically, how would you describe this set of initial conditions?) Using Eq. (14) we rather trivially obtain

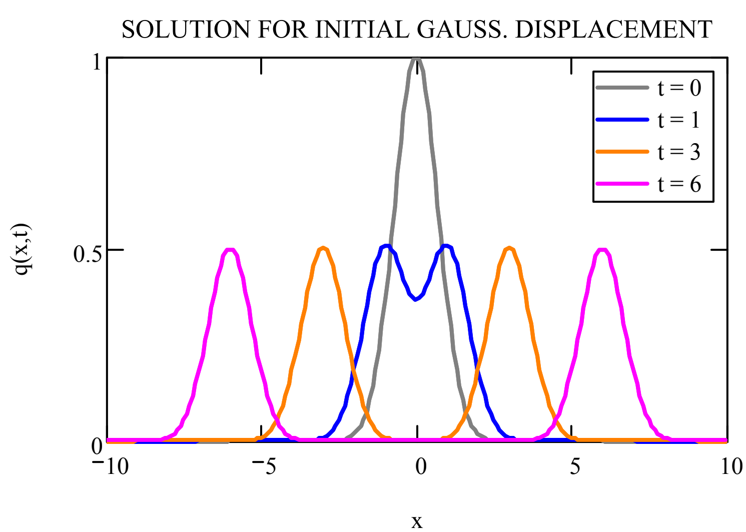
$$q(x,t) = \frac{A}{2}\left[G_\sigma(x+ct) + G_\sigma(x-ct)\right], \tag{16}$$

or more explicitly,



$$q(x,t) = \frac{A}{2}\left[e^{-(x+ct/\sigma)^2} + e^{-(x-ct/\sigma)^2}\right], \tag{17}$$

So the solution consists of two Gaussian functions, one moving in the $-x$ direction and one in the $+x$ direction, both at the speed c. The amplitude of each function is ½ the amplitude of the initial Gaussian displacement. The following picture illustrates this solution as a function of x for several times t. For simplicity we have set $A = 1$, $\sigma = 1$, and $c = 1$.



B. IVP Solution with Gaussian Initial Velocity / Error Function

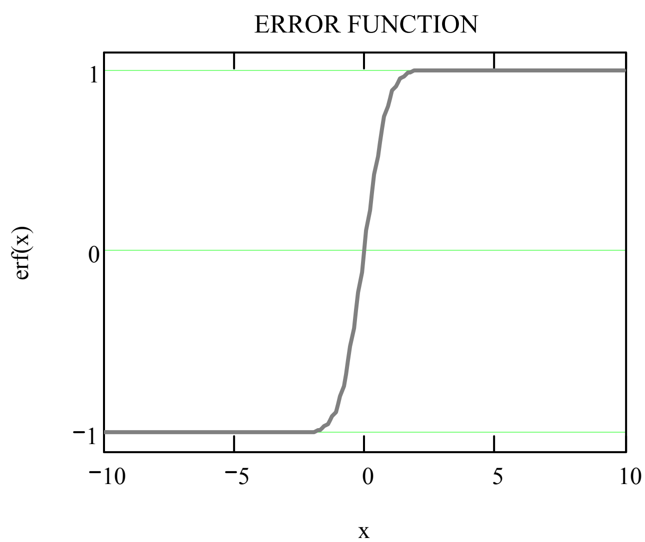
Let's look at another example using the Gaussian function. This time let's have the initial displacement of the system be zero so that $a(x)=0$, but let's have a Gaussian initial-velocity function so that $b(x)=B\,G_\sigma(x)$, where B is some arbitrary velocity amplitude. In this case we get from Eq. (14)

$$q(x,t)=\frac{B}{2c}\int_{x-ct}^{x+ct} e^{-(x'/\sigma)^2}\,dx' . \tag{18}$$

So what is the integral of a Gaussian function? Well, it is known as the error function. Specifically, the **error function** is defined as

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x'^2}dx' . \tag{19}$$

The following figure shows a plot of $erf(x)$ vs x. As the graph indicates, $erf(x)$ is defined such that $erf(x\to-\infty)=-1$, $erf(x\to\infty)=1$, and $erf(0)=0$.



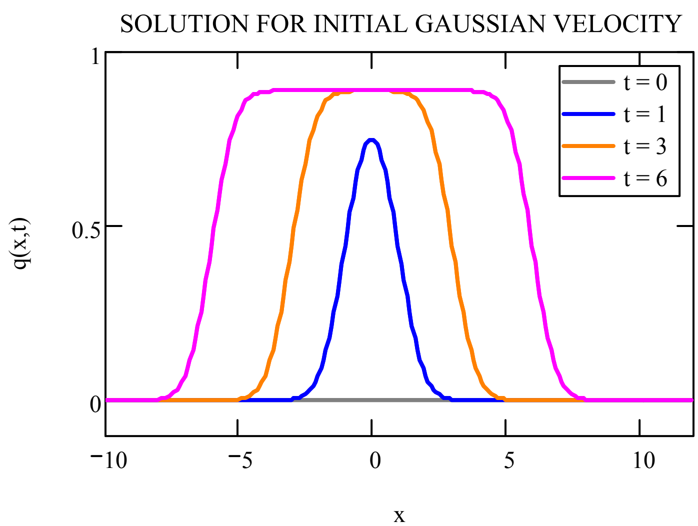
All this is fine and well, but what do we do about the σ in Eq. (18), which does not appear in Eq. (19)? We must do a little math (!) and change variables in Eq. (18). Let's define a new integration variable $y=x'/\sigma$, $dy=dx'/\sigma$. Then Eq. (18) becomes

$$q(x,t)=\frac{B\sigma}{2c}\int_{(x-ct)/\sigma}^{(x+ct)/\sigma} e^{-y^2}\,dy . \tag{20}$$

We can now use Eq. (19), the definition of the error function, to write

$$q(x,t) = \frac{\sqrt{\pi}B\sigma}{4c}\{erf[(x+ct)/\sigma] - erf[(x-ct)/\sigma]\}. \tag{21}$$

Notice that again we have the sum of two functions, each traveling in opposite directions at the speed c. The following picture plots the solution vs x for several values of t (with B, σ, and c all set to 1).



Exercises

***8.1 The chain rule.** Let $h(x,y) = x^2 + xy + y^2$.

(**a**) Directly calculate $\partial h/\partial x$ and $\partial h/\partial y$.

(**b**) Now define two new independent variables $u = (x+y)/2$ and $u = (x-y)/2$

(**c**) Rewrite $h(x,y)$ in terms of u and v. That is, find $h(u,v)$.[1]

(**d**) Now starting with $h(u,v)$ and thinking of it as $h(u(x,t),v(x,t))$, calculate $\partial h/\partial x$ using the chain rule. That is, calculate this derivative using $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}$. Does this agree with your answer in (**a**)?

(**e**) Follow a procedure analogous to that in (**d**) to calculate $\partial h/\partial y$. Again, does this agree with your answer in (**a**)?

[1] Technically speaking, we should give the function $h(u,v)$ another name [$\bar{h}(u,v)$, say], but being physicists, we are rather lazy and typically still call the new function h.

*$\mathbf{8.2}$ Verify for $t=0$ that Eq. (14) and its time derivative reduce to the initial conditions $q(x,0)=a(x)$ and $\frac{\partial q}{\partial t}(x,0)=b(x)$, respectively.

*$\mathbf{8.3}$ If the initial displacement is zero, then Eq. (14), the solution to the initial value problem, can be written as $q(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} b(x')dx'$. By calculating second derivatives in x and t, directly demonstrate that this function solves Eq. (1), the wave equation.

$\mathbf{8.4}$ **The error function.
(**a**) Using an appropriate change of integration variable in the equation
$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x'^2}dx'$, show that $erf(x/\sigma)=\frac{2}{\sqrt{\pi}\sigma}\int_0^x e^{-(x'/\sigma)^2}dx'$.
(**b**) Using a computer mathematic package, plot $erf(x/\sigma)$ over an appropriate range of x for σ = 1, 3, and 5.

$\mathbf{8.5}$ **The initial value problem.
Consider Eq. (14), the general solution to the initial value problem.
(**a**) Explain why Eq. (14) in not a function of the variable x'. (This is a basic feature of the definite integral. Consult a calculus book if necessary.)
(**b**) What are c, $a(x)$, and $b(x)$ in Eq. (14)?
(**c**) Consider the specific case where $a(x)=0$ and $b(x)=2cx/(1+x^4)$. Using a computer mathematics package and letting $c=1$, plot $b(x)$ over an appropriate range of x.
(**d**) Using the initial conditions given in (**c**), solve Eq. (14). (Do not set c to zero!) The integral can be done either with a change of variable, a computer mathematics package, or can be looked up in a table of integrals (such as found in the *CRC Handbook of Chemistry and Physics*).
(**e**) Show that your solution can be written in the form $q(x,t)=f(x+ct)+g(x-ct)$. Thus identify $f(x)$ and $g(x)$.
(**f**) Again, using a computer mathematics package and letting $c=1$, plot $q(x,t)$ as a function of x for t = 0, 10, 20, and 30. Be careful to let your graph include all interesting parts of the solution!