Long Wavelength Limit / Normal Modes

Overview and Motivation: Today we look at the long wavelength limit of the coupled-oscillator system. In this limit the equations of motion for the coupled oscillators can be transformed into the partial differential equation known as the wave equation, which has wide applicability beyond the coupled-oscillator system We also look at the normal modes and the dispersion relation for the coupled-oscillator system in this limit.

Key Mathematics: We again utilize the Taylor-series expansion.

I. Derivation of the Wave Equation

A. The Long Wavelength Limit (LWL)

So why look at the coupled-oscillator system at long wave lengths? Perhaps the main motivation comes from the fact that we are often interested in waves in systems where the wavelength is much longer than the distance between the coupled objects. For example, let's consider audible ($\nu =$ 20 to 20,000 Hz) sound waves in a solid. In a typical solid the speed of sound c is ~2000 m/s, so audible frequencies correspond to wavelengths ($\lambda = c/\nu$) between approximately 0.1 and 100 m. These wavelengths are obviously much greater that the typical interatomic spacing d of 2 × 10^{-10} m. As we will see, one benefit of the long wavelength limit is that we will no longer need to refer to the displacement of each interacting object: the index j will be "traded in" for the continuous position variable x, so that we will be considering displacements as a function of x and t.

Let's consider the equation of motion for the j th oscillator (which can be any oscillator in the N coupled-oscillator system),

$$\frac{d^2 q_j(t)}{dt^2} = \tilde{\omega}^2 \left[q_{j+1}(t) - 2q_j(t) + q_{j-1}(t) \right]. \tag{1}$$

Let 's go ahead and trade in the discrete object index j for the continuous position variable x via $x = jd$, where d is the equilibrium distance between objects in the chain. Then we can rewrite Eq (1) as

$$\frac{\partial^2 q(x,t)}{\partial t^2} = \tilde{\omega}^2 \left[q(x+d,t) - 2q(x,t) + q(x-d,t) \right]. \tag{2}$$

Notice that the time derivative is now a partial derivative because we are now thinking of the displacement q as a function of two continuous variables, x and t.

Now because d is a small parameter,[1] let's Taylor series expand the two functions $q(x+d,t)$ and $q(x-d,t)$ in a Taylor series in d and $-d$, respectively, about the point x [so in doing this expansion we are thinking about x as some fixed point along the chain and $q(x \pm d,t)$ as a function of $\pm d$]. The two Taylor series are

$$q(x+d,t)=q(x,t)+\frac{\partial q(x,t)}{\partial x}d+\frac{1}{2}\frac{\partial^2 q(x,t)}{\partial x^2}d^2+\ldots \tag{3a}$$

and

$$q(x-d,t)=q(x,t)+\frac{\partial q(x,t)}{\partial x}(-d)+\frac{1}{2}\frac{\partial^2 q(x,t)}{\partial x^2}(-d)^2+\ldots \tag{3b}$$

Keeping all terms up to order d^2 (we consider the validity of this approximation at the end of Sec. II) and substituting Eq. (3) into Eq. (2) gives us

$$\frac{\partial^2 q(x,t)}{\partial t^2}=\tilde{\omega}^2 d^2\frac{\partial q^2(x,t)}{\partial x^2} \tag{4}$$

This is the **wave equation**, which as you can see is a homogeneous, linear, second-order, **partial differential equation**. There is one more thing we need to do, however, in order to make Eq (4) more universally applicable. The term

$$\tilde{\omega}^2 d^2=\frac{k_s}{m}d^2 \tag{5}$$

on the rhs of Eq. (5) is a combination of the fundamental parameters k_s, m, and d of the coupled-oscillator problem. Let's define another constant $c^2=(k_s/m)d^2$ so that we can more generically write the wave equation as

$$\frac{\partial^2 q(x,t)}{\partial t^2}=c^2\frac{\partial q^2(x,t)}{\partial x^2}. \tag{6}$$

[1] Technically, the small parameter in the long wave length limit is the ratio d/λ. Generally, a parameter can only be large or small if it is unitless. Otherwise, whether it is small or large (or somewhere in between) will depend upon the system of units being used.

B. General Applicability of the Wave Equation

The is the standard form of the wave equation that we will use in this class. As we shall see in further study of the wave equation, c is the propagation speed of waves described by Eq. (6).

Now if Eq. (6) were only useful for studying the long wavelength motion of the coupled-oscillator system, it really wouldn't be that interesting. Fortunately, it is applicable in a wide variety of situations, including sound waves in fluids and solids, transverse waves on a string, and electromagnetic waves in vacuum (or other nondispersive media). In each situation the constant c^2 can be related to the underlying physics. For example, for transverse waves on a string, $c^2 = \tau/\mu$, where τ is the tension in the string and μ is the mass per unit length of the string.

II. LWL of Coupled Oscillator Solutions

A. Normal Modes

Let's now look at the coupled-oscillator normal modes in the LWL. As we discussed in Lecture 6, we can write the normal-mode solutions (indexed by n) as a function of x and t as

$$q_{(n)}(x,t) = \sin(k_n x)\left(A_n e^{i\omega(k_n)t} + B_n e^{-i\omega(k_n)t}\right) \tag{7}$$

where the wave vector is given by

$$k_n = \frac{n\pi}{N+1}\frac{1}{d}, \tag{8}$$

and the dispersion relation is

$$\omega(k_n) = 2\tilde{\omega}\sin\left(\frac{d}{2}k_n\right). \tag{9}$$

As written, Eqs. (8) and (9) are expressed in terms of the fundamental (or microscopic) parameters d, $\tilde{\omega}$, and N of the coupled-oscillator problem. Let's see if we can re-express them in terms of c and other more generic (or macroscopic) parameters. Well, the first thing to notice is that the length L of the system (a more generic, macroscopic parameter) can be written in terms of the fundamental parameters N and d as $L = (N+1)d$. The wave vector can thus be simply expressed as

$$k_n = \frac{n\pi}{L}. \tag{10}$$

Well, what about Eq. (9), the dispersion relation? It is not yet clear what to do with $\tilde{\omega}$ in order to obtain a more generic description. This is where the long wavelength limit comes into play again. Recall that the wave vector is related to the wavelength via

$$k_n = \frac{2\pi}{\lambda_n}. \tag{11}$$

This allows us to write Eq. (9) as

$$\omega(\lambda_n) = 2\tilde{\omega}\sin\left(\pi\frac{d}{\lambda_n}\right). \tag{12}$$

Now, the long wavelength limit is exactly the limit $d/\lambda << 1$. Thus, in this limit we can replace the sine function by its (very small) argument, so that Eq. (12) can be expressed as

$$\omega(\lambda_n) = \tilde{\omega}\, d\frac{2\pi}{\lambda_n}. \tag{13}$$

And now, using $\tilde{\omega} d = c$ and Eq (11), we have the long-wavelength-limit dispersion relation

$$\omega(k_n) = ck_n. \tag{14}$$

Or, in a more general form that is applicable to any harmonic wave described by the wave equation (not just the normal modes for the coupled-oscillator system where $k = k_n$ is discrete),

$$\omega(k) = ck. \tag{15}$$

So we see that, because c is constant, the dispersion relation for waves described by the wave equation is linear vs k. Looking back at the dispersion curves for the coupled-oscillator system that are plotted in the Lecture 6 notes, you should notice that for small k_n/k_N (which is equivalent to small d/λ_n) that the dispersion curves are indeed linear vs k_n. (The small wave vector limit is thus the same as the long wavelength limit.)

Putting all of this together, we can write the normal modes [Eq. (7)] for long wavelengths as

$$q_{(n)}(x,t) = \sin\left(\frac{n\pi}{L}x\right)\left(A_n e^{ic(n\pi/L)t} + B_n e^{-ic(n\pi/L)t}\right) \tag{16}$$

A remark about the normal modes should be made at this point. The function $\sin(n\pi x/L)$ in Eq. (16) came about because of the boundary conditions, which we can now express as $q(0,t)=0$ and $q(L,t)=0$ [You can check that these boundary conditions are indeed satisfied by $\sin(n\pi x/L)$]. For other boundary conditions we would generally get some linear combination of $\sin(k_n x)$ and $\cos(k_n x)$, and perhaps also different allowed wave vectors k_n (or equivalently, allowed wavelengths λ_n).[2]

B. Neglect of Higher Order Terms in Equation of Motion in the LWL

Now that we have an expression for the normal modes in the LWL, we can check to see that it is OK to neglect the terms in Eq. (3) that are of order higher than d^2. The first thing to notice is that all the odd terms in Eqs. (3a) and (3b) cancel each other when inserted into Eq. (2). Thus we need only consider the even terms. The second thing to notice is that for even m

$$\frac{\partial^m q_{(n)}(x,t)}{\partial x^m} = (-1)^{m/2}\left(\frac{2\pi}{\lambda_n}\right)^m q_{(n)}(x,t). \tag{17}$$

{This is obtainable using Eq. (16) in Eq. (3) [in Eq. (2)]}. Therefore, for example, the ratio of the 4th order term to the 2nd order term, neglecting numerical factors, is given by d^2/λ^2, which is much smaller than 1. Thus, we are justified in neglecting this term (and other higher order terms, which have an even smaller ratio) in Eq. (3), the Taylor series expansion of the equations of motion.

III. The Continuum Limit vs the Long Wavelength Limit

Recall that for N oscillators that there are N normal modes. Well, we have been clever and gotten rid of N in all of our expressions. So what do we do? It depends upon the situation that is being described by the wave equation. As a long wavelength limit of a truly discrete system, one must simply make sure that the waves being described have wavelengths that are much longer that the appropriate interparticle spacing. However, there are times when the wave equation is used, such as in electromagnetism, where there are no underlying oscillators and thus no underlying spacing d to be compared with the wavelength. In those cases, where the wave equation is believed to describe all waves, the long wavelength limit is replaced by a more mathematically technical limit, known as the continuum limit. In this limit one

[2] Think about the normal modes for sound waves in a pipe that is open on both ends vs a pipe with one end open and one end closed. This is explored in Exercise 7.4.

actually takes the limit $d \to 0$. The result is that the terms in Eq. (3) of order greater than d^2 can be neglected exactly. Furthermore, in the limit $d \to 0$ (for fixed L), we see that because $L = (N+1)d$, this limit is equivalent to $N \to \infty$. That is, in the continuum limit the number of normal modes becomes infinite. From here on out, when working with normal modes of the wave equation we will assume that there are, indeed, an infinite number of normal modes. (More details on the continuum limit can be found in Dr. Torre's text, *FWP*.)

Exercises

****7.1 The wave equation for NN and NNN coupled oscillators**

(a) Analogous to Eq, (2), write down the equation of motion for the coupled-oscillator system that has both NN and NNN springs (see the Lecture 6 notes).

(b) Similar to what was done here for the NN system, take the long wavelength limit of this equation of motion and derive the wave equation.

(c) For this system, what is the constant c^2 in terms of the fundamental parameters $\tilde{\omega}$, $\tilde{\omega}'$, and d?

***7.2 The long wavelength limit**

(**a**) Derive Eq. (17), which follows from Eq. (16), the expression for the normal mode solutions $q_{(n)}(x,t)$.

(**b**) Given your result in (**a**), Show that the ratio of the d^4 to d^2 terms in the Taylor series expansion of $[q(x+d,t) - 2q(x,t) + q(x-d,t)]$, which appears on the rhs of the equation of motion as expressed in Eq. (2), is equal to $-\frac{1}{12}(2\pi)^2 (d/\lambda_n)^2$.

(**c**) Thus argue that, compared to the d^2 term, the d^4 term can be neglected in the equation of motion in the LWL.

***7.3** Referring to the Lecture 6 notes, show for large N that the ratio k_n/k_N is equal to $2d/\lambda_n$, thus proving that the long-wavelength and small-wave-vector limits are equivalent.

****7.4 Normal modes for an open-closed pipe**

Consider the following general form for a 1D standing wave,

$$q_S(x,t) = [a\sin(kx) + b\cos(kx)]\sin(\omega t + \phi).$$

(**a**) Show that this is a solution to the wave equation only if ω and k are related by the dispersion relation $\omega = \pm ck$.

(**b**) Let's assume that the normal modes for sound waves in a pipe are of the form $q_S(x,t)$ given above. For a pipe that is open on one end (at $x = 0$) and closed on the other (at $x = L$) the appropriate boundary conditions are $\frac{\partial q(0,t)}{\partial x} = 0$ and $q(L,t) = 0$. Starting with the above general form of the standing wave solutions $q_S(x,t)$, use the $x = 0$ boundary condition to show that the normal modes of this system have the more specific form

$$q(x,t) = b\cos(kx)\sin(ckt + \phi).$$

(**c**) Further, using the $x = L$ boundary condition, show that the wave vector k can only take on the discrete values $k_n = n\pi/(2L)$ (where $n = 1, 3, 5, \ldots$). [Notice that these wave vectors are not the same as the normal-mode wave vectors for the coupled oscillator problem, given by Eq. (10)]. Thus show that the normal modes of the open-closed pipe can be written as

$$q(x,t) = b\cos\left(\frac{n\pi}{2L}x\right)\sin\left(\frac{n\pi c}{2L}t + \phi\right)$$

(**d**) Thus show that the wavelengths of the normal modes of the open-closed pipe are $\lambda_n = 4L/n$.