Traveling Waves, Standing Waves, and the Dispersion Relation

Overview and Motivation: We review the relationship between traveling and standing waves. We then discuss a general relationship that is important in all of wave physics – the relationship between oscillation frequency and wave vector – which is known as the dispersion relation.

Key Mathematics: We get some more practice with trig identities and eigenvalue problems.

I. Traveling and Standing Waves

A. Basic Definitions

The simplest definition of a 1D **traveling wave** is a function of the form

$$q_1(x,t) = g(x-ct) \tag{1a}$$

or

$$q_2(x,t) = f(x+ct), \tag{1b}$$

where c is some positive constant.[1] The constant c is the **speed of propagation** of the wave. The wave in Eq. (1a) propagates in the positive x direction, while the wave in Eq. (1b) propagates in the negative x direction. Now the functions g and f in Eq. (1) can essentially be any (well behaved) function, but often we are interested in **harmonic waves**. In this case the functions g and f in Eq. (1) take on the form

$$g(x-ct) = A\sin\left[\frac{2\pi}{\lambda}(x-ct)+\phi\right] \tag{2a}$$

and

$$f(x+ct) = B\sin\left[\frac{2\pi}{\lambda}(x+ct)+\psi\right], \tag{2b}$$

where A and B are the amplitudes, ϕ and ψ are the **phases**, and λ is the **wavelength** of the wave. Now the x- and t-dependent parts of the sine-function arguments are often written as $k\,x \pm \omega t$, and so we can identify the **wave vector** k as

[1] As we shall see, the functions in Eq. (1) are the general solutions to the wave equation, which we will study in short order. However, we shall also see, when we study the Schrödinger equation, that not all waves have these functional forms.

$$k = \frac{2\pi}{\lambda} \tag{3}$$

and the **angular frequency** ω as[2]

$$\omega = \frac{2\pi c}{\lambda}. \tag{4}$$

You should recall from freshman physics that the speed c, frequency $\nu = \omega/(2\pi)$ and wavelength λ are related by $\nu = c/\lambda$.

So what is a **standing wave**? Simply put, it is the superposition (i.e., sum) of two equal-amplitude, equal-wavelength (and thus equal-frequency) harmonic waves that are propagating in opposite directions. Using Eqs. (2), (3), and (4) (and for simplicity setting $\phi = \psi = 0$) we can write such a wave as

$$q_S(x,t) = A[\sin(kx - \omega t) + \sin(kx + \omega t)]. \tag{5}$$

With a bit of trigonometry, specifically the angle-addition formula for the sine function, Eq. (5) can be rewritten as

$$q_S(x,t) = 2A\sin(kx)\cos(\omega t) \tag{6}$$

So instead of being a function of $kx \pm \omega t$, a standing wave is a *product* of a function of x and a function of t. Equation (6) also show us how to identify the wave vector k and angular frequency ω in the case of an harmonic standing wave: whatever multiplies x is the wave vector and whatever multiplies t is the angular frequency.

In fact, for nonharmonic standing waves it is probably safe to define a standing wave as a wave where all parts of the system oscillate in phase, as is the case of the harmonic standing wave defined by Eq. (6).

B. Connection to the Coupled Oscillator Problem

Let's now go back to the coupled oscillator problem, and reconsider the nth normal-mode solutions to that problem, which we previously wrote as

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[2] Do not confuse this definition of ω (the angular frequency of the wave) with our earlier definition of $\tilde{\omega}$ (=$\sqrt{k/m}$) that arises when discussing single or coupled harmonic oscillators. It is easy to confuse the two definitions because for a single harmonic oscillator $\tilde{\omega}$ is also an angular frequency.

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \\ q_N(t) \end{pmatrix}_n = \begin{pmatrix} \sin\left(\frac{n\pi}{N+1}1\right) \\ \sin\left(\frac{n\pi}{N+1}2\right) \\ \sin\left(\frac{n\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{n\pi}{N+1}N\right) \end{pmatrix} \left(A_n e^{i\Omega_n t} + B_n e^{-i\Omega_n t}\right), \tag{7}$$

where $\Omega_n = 2\tilde{\omega}\sin[n\pi/2(N+1)]$. As we mentioned last time, these modes are essentially standing waves. Let's see that this is the case by writing Eq. (7) in the form of Eq. (6). After we do this, let's also identify the wave vector k and frequency ω for the normal modes. As written, Eq. (7) explicitly lists the motion of each individual oscillator. But the nth normal mode can also be written as a function of object index j and time t as

$$q_{(n)}(j,t) = \sin\left(\frac{n\pi}{N+1} j\right)\left(A_n e^{i\Omega_n t} + B_n e^{-i\Omega_n t}\right), \tag{8}$$

where j labels the oscillator. Although j is a discrete index, we have included it in the argument of the normal-mode function because it is the variable that labels position along the chain. Equation (8) is almost in the form of Eq. (6). In fact, if we take the specialized case of $A_n = B_n = A$, where A is real, then Eq. (8) can be written as

$$q_{(n)}(j,t) = 2A\sin\left(\frac{n\pi}{N+1} j\right)\cos(\Omega_n t). \tag{9}$$

This is very close, except in Eq. (6) the position variable is the standard distance variable x, while in Eq. (9) we are still using the object index j to denote position. However, if we define the equilibrium distance between nearest-neighbor objects as d, then we can connect x and j via $x = jd$, and so we can rewrite Eq. (9) as

$$q_{(n)}(x,t) = 2A\sin\left(\frac{n\pi}{N+1}\frac{x}{d}\right)\cos(\Omega_n t). \tag{10}$$

Now, remembering that whatever multiplies x is the wave vector k and that whatever multiplies t is the angular frequency ω, we have

$$k = \frac{n\pi}{N+1}\frac{1}{d} \tag{11}$$

and

$$\omega = \Omega_n = 2\tilde{\omega}\sin\left(\frac{n\pi}{2(N+1)}\right) \tag{12}$$

for the coupled-oscillator standing waves.

III. Dispersion Relations

A. Definition and Some Simple Examples

Simply stated, a **dispersion relation** is the function $\omega(k)$ for an harmonic wave. For the simplest of waves, where the speed of propagation c is a constant, we see from Eqs. (3) and (4) that their dispersion relation is simply

$$\omega(k) = ck\,. \tag{13}$$

That is, the frequency ω is a linear function of the wave vector k. We also see from Eq.(13) that the ratio ω/k is simply the propagation speed c. As we will discuss in more detail in a later lecture, the ratio ω/k is technically known as the **phase velocity**.[3]

Now you may be thinking, what is the big deal here? – Eq. (13) is so simple, what could be interesting about it? Now it is simple if the phase velocity c is independent of k. But this is usually not the case. Recall, for example, the propagation of light in a dielectric medium (such as glass) where the index of refraction $n = c_0/c$ (where c_0 is the speed of light in a vacuum) depends upon the wavelength (and thus the wave vector).[4] In this case Eq. (13) becomes

$$\omega(k) = \frac{c_0}{n(k)}k\,. \tag{14}$$

The dispersion relation now has the possibility of being quite interesting.

Another example of an interesting, nonlinear dispersion relation is found in modern physics. In your modern-physics class you (hopefully!) studied solutions to the Schrödinger equation. The wave (function) that describes a free particle (one with no

[3] OK, it should probably be called the phase speed, but it isn't. Sorry, even in physics all quantities are not logically named.

[4] The dependence of the phase velocity on wave vector leads to dispersion of light by a prism, for example. Thus the name "dispersion relation".

net force acting on it) propagating in the x direction with momentum p can be written as

$$\psi_p(x,t) = \psi_0 e^{i\left(\frac{p}{\hbar}x - \frac{p^2}{2m\hbar}t\right)}, \tag{15}$$

where m is the mass of the particle and $\hbar$ is Planck's constant. Comparing Eq. (15) with Eq. (5), for example, we identify the wave vector

$$k = \frac{p}{\hbar} \tag{17}$$

and the frequency

$$\omega = \frac{p^2}{2m\hbar}. \tag{18}$$

The dispersion relation is thus

$$\omega(k) = \frac{\hbar k^2}{2m}. \tag{19}$$

Notice that this dispersion relation is quadratic in the wave vector k. As we will study later, a nonlinear dispersion relation has profound consequences for the propagation of a localized wave (often called a pulse or wave packet) associated with that dispersion relation. First, for a nonlinear dispersion relation the propagation speed of the pulse will not be equal to the phase velocity. Second, a nonlinear dispersion relation typically leads to the spreading of the pulse with time. (This spreading is also known as dispersion!)

B. Connection to the Coupled Oscillator Problem

So what is the dispersion relation for our coupled oscillator system? By combining Eqs. (11) and (12) we see that we can write

$$\omega(k) = 2\tilde{\omega}\sin\left(\frac{d}{2}k\right) \tag{20}$$

This is another example of a nonlinear dispersion relation.

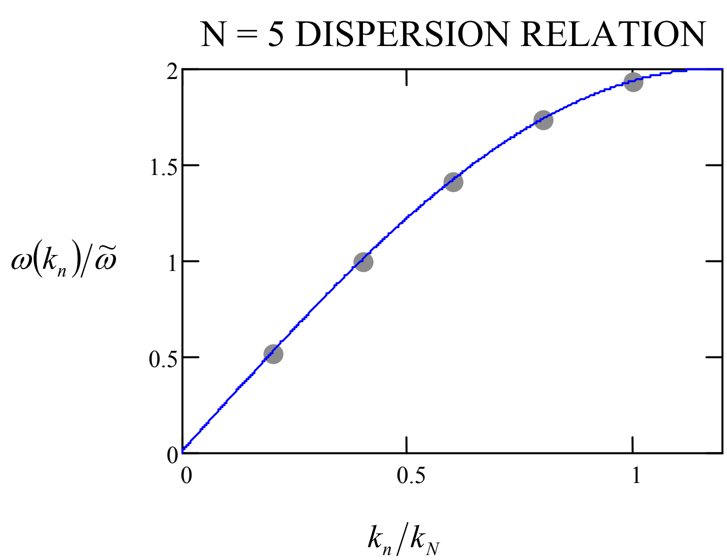
Let's make some graphs of the dispersion relation for the coupled-oscillator system. Now Eq. (20) is correct, but not particularly useful for doing this. That is because for the coupled oscillator problem k can only have the discrete values

$$k_n = \frac{n\pi}{N+1}\frac{1}{d}, \tag{21}$$

where $n = 1, 2, 3, \ldots, N$, are allowed [see Eq. (11)].[5] So we must be a bit clever here. Let's rewrite Eq. (12) as

$$\omega(k_n) = \tilde{\omega}\sin\left(\frac{\pi}{2}\frac{N}{N+1}\frac{n}{N}\right) = 2\tilde{\omega}\sin\left(\frac{\pi}{2}\frac{N}{N+1}\frac{k_n}{k_N}\right). \tag{22}$$

(The rhs follows because $k_n \propto n$. Thus, $n/N = k_n/k_N$.) This makes Eq. (12) look like the dispersion relation that we want, but in constructing a graph, we can simply plot ω vs n/N (keeping in mind that n/N is the same as k_n/k_N.) The following graph plots $\omega(k_n)/\tilde{\omega}$ vs k_n/k_N for $N = 5$.[6]



For $N = 5$, k_n is obviously a discrete variable (as the graph shows), but there are times when it is useful (and appropriate) to think of k_n as a continuous variable (even if it

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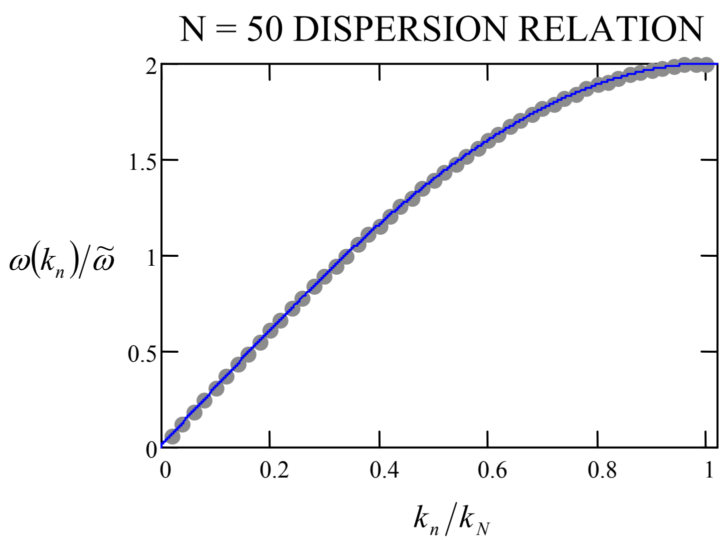
[5] Of course you already knew this because of your familiarity with standing waves on a string (although in that case $N = \infty$ -- more on this later!)

[6] When making a graph, it is often useful to use normalized, unitless quantities for the axes. This makes the graph more widely applicable.

isn't). To see when this is the case, let's consider Eq. (22) for larger N, as illustrated in the next figure, where the dispersion relation is plotted for $N = 50$. The key observation here is that that the spacing between adjacent values of k_n/k_N becomes smaller as N becomes larger. In fact, it is not hard to show that the relative spacing between allowed values of k is given by

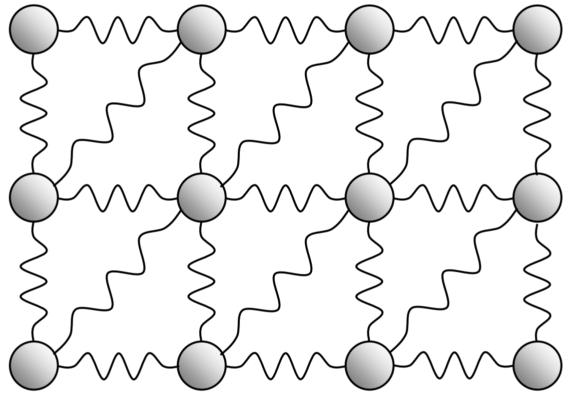


$$\frac{\Delta k}{k_N} = \frac{1}{N}. \tag{20}$$

For very large N, say $N = 10^{23}$ (as one might be interested if one were modeling atoms in a solid as coupled oscillators), the spacing $\Delta k/k_N$ is indeed truly negligible, and one is justified in thinking of k as continuous.

III. Interparticle Interactions and Dispersion Relations

Now you may think that our model of coupled oscillators is nothing more than an exercise in classical mechanics. This model, however, contains the essence of vibrational dynamics in solid-state materials. How can this be? Surely the interactions between atoms in a solid are much more complicated than the quadratic potentials of a bunch of springs. Yes, that is true. However, let's think back to the first lecture where we discussed the Taylor-series expansion of an arbitrary potential energy function near a minimum. We found out that if the object does not stray too far from equilibrium, then the potential is effectively quadratic – that is, the object is pulled back towards equilibrium as if it were attached to a spring. Well, the same thing is true for atoms in a solid. At most temperatures they never stray too far from their

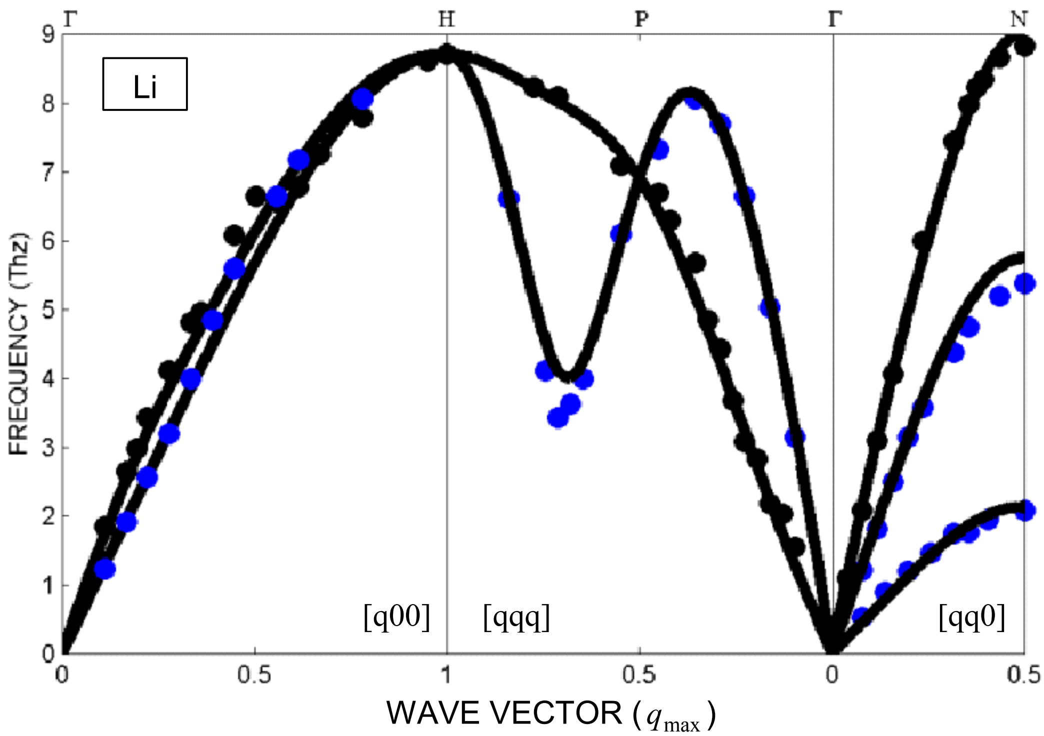


equilibrium positions. What this means is that a model where the atoms are hooked to other atoms with springs is, indeed, a pretty good model. Thus, when thinking of vibrations (i.e., oscillations) of atoms (really, the nuclei because the nucleus contain nearly all the mass of an atom) we can think of the nuclei as if they are attached to other nuclei with springs, as the picture above suggests.

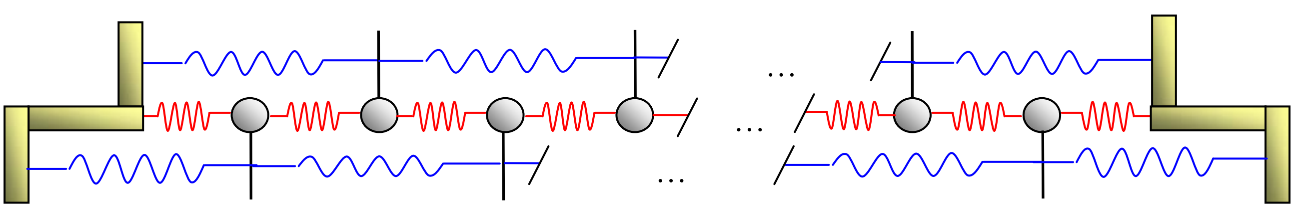
Now because the nuclei are essentially connected together with springs, there is a set of normal modes for the system. Further, as with our case of couple oscillators, there is a dispersion relation associated with the normal modes of vibration. In fact, for every wave vector (which is indeed a vector $\mathbf{k}$ because of the 3 dimensional nature of the solid) there are three normal modes (because each nucleus can vibrate in three dimensional space).

With regards to solid-state physics, the most important thing about dispersion relations is that they can be measured (and thus compared with theory). The figure on the next page shows both experimental (the discrete points) and calculated (the continuous lines) dispersion curves for Li.

For the wave-vector directions of propagation shown in the graph, there are two normal modes with transverse polarization and one with longitudinal polarization for each value of $\mathbf{k}$, although along the (100) and (111) directions the two transverse modes are degenerate (have the same frequency). Notice that along the (100) and (110) directions that the dispersion curves are similar to the dispersion curve for our simple 1D coupled oscillator system. So what is the point? Well, the measured curves provide insight into the microscopic interactions between the atoms: in order to theoretically calculate the dispersion curves one must know which atoms are coupled with springs and what the spring constants are for the different springs.

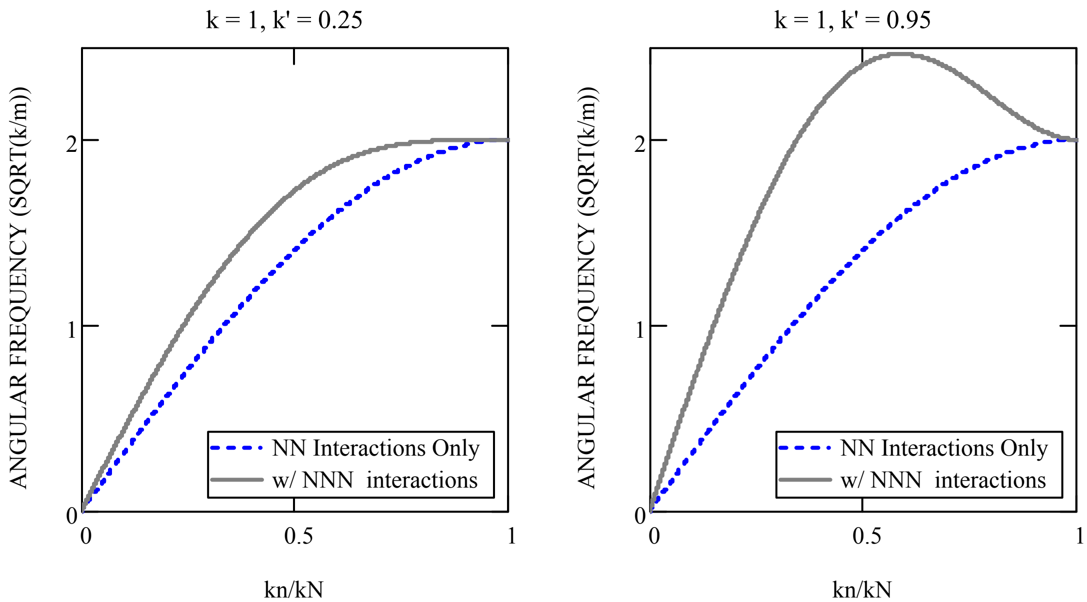


To get a better sense of how the interactions between atoms affect the dispersion curves, let's go back to our simple 1D system and modify it by adding some more springs to see how the dispersion curve is affected. In particular, let's add some next-nearest-neighbor (NNN) springs in addition to the nearest-neighbor (NN) springs that we already have. This modification is illustrated in the following picture.



Of course, so that we have something to play with, we let the NN and NNN springs have different spring constants: we continue to let the NN springs have spring constant k_s while the NNN springs have spring constant k_s'. (Of course, this is what we would expect: why should NN's and NNN's have the same interactions?)

The following picture shows the dispersion curves that result when these NNN springs are added. (These curves are for large N and so are plotted as continuous functions.) Notice that the dispersion curves are quite sensitive to the value of k_s', the spring constant for the NNN interaction. Also notice that the dispersion curve in the graph on the right is similar to the dispersion curve for the longitudinal mode in Li along the (111) direction.



Exercises

***6.1** Using the appropriate trig identities, derive Eq. (6) from Eq. (5).

***6.2** Under the condition $A_n = B_n = A$, show that Eq. (9) is equivalent to Eq. (8).

***6.3** Show that the superposition (sum) of the two standing waves $A\sin(kx)\sin(\omega t)$ and $A\cos(kx)\cos(\omega t)$ is a traveling wave. What is the direction of propagation of this traveling wave?

****6.4 Linear chain with NN and NNN interactions**. Here you will find the normal-mode frequencies for a linear chain of coupled oscillators with both NN springs (k) and NNN springs (k').
(**a**) Write down the equation of motion for the j th oscillator for this system. Consulting the picture on p. 9 may be helpful.
(**b**) Analogous to the steps in Sec. II of the Lecture 5 notes, find the normal-mode frequencies Ω_n and show that they can be written as

$$\frac{\Omega_{n\pm}}{\tilde{\omega}} = \pm 2\left[\sin^2\left(\frac{\phi_n}{2}\right) + \left(\frac{\tilde{\omega}'}{\tilde{\omega}}\right)^2 \sin^2(\phi_n)\right]^{1/2}$$

where $\tilde{\omega} = \sqrt{k_s/m}$, $\tilde{\omega}' = \sqrt{k_s'/m}$, and $\phi_n = n\pi/(N+1)$.
(**c**) For this problem plot (using computer software) the dispersion curve $\omega(k_n)/\tilde{\omega}$ vs k_n/k_N for $m=1$, $k_s=1$, and $k_s' =$ 0.2, 0.5, and 0.8. You may assume that there are many oscillators so that your dispersion curves are effectively continuous.

***6.5** Show that the spacing between allowed wave vectors for the coupled oscillator problem is given by Eq. (20).