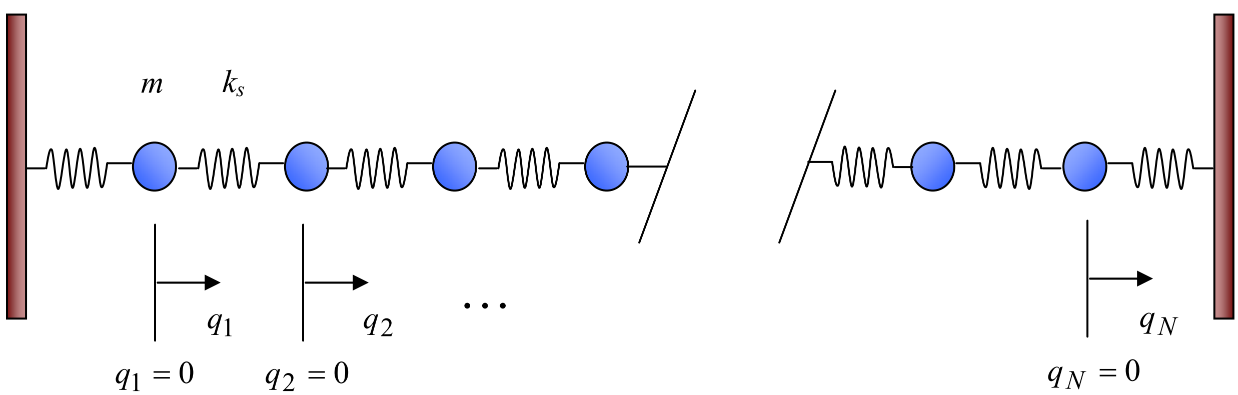
Linear Chain / Normal Modes

Overview and Motivation: We extend our discussion of coupled oscillators to a chain of N oscillators, where N is some arbitrary number. When N is large it will become clear that the normal modes for this system are essentially standing waves.

Key Mathematics: We gain some more experience with matrices and eigenvalue problems.

I. The Linear Chain of Coupled Oscillators

Because two oscillators are never enough, we now extend the system that we have discussed in the last two lectures to N coupled oscillators, as illustrated below. For this problem we assume that all objects have the same mass m and all springs have the same spring constant k_s.



Our first goal is to find the normal modes of this system. At the beginning we approach this problem in the same manner as for two coupled oscillators: we find the net force on each oscillator, find each equation of motion, and then assume a normal-mode type solution for the system. Let's consider some arbitrary object in this chain, say the j th object. The force on this object will depend upon the stretch of the two springs on either side of it. With a little thought, you should be able to write down the net force on this object as

$$F_j = -k_s(q_j - q_{j-1}) - k_s(q_j - q_{j+1}), \tag{1}$$

or, upon simplifying,

$$F_j = k_s(q_{j-1} - 2q_j + q_{j+1}). \tag{2}$$

You might worry that this equation is not valid for the first ($j=1$) and last ($j=N$) objects, but if we assume that the $j=0$ and $j=N+1$ objects (the walls) have infinite mass so that q_0 and q_{N+1} are identically zero, then Eq. (2) applies to all N objects. We shall refer to these two conditions, $q_0=0$ and $q_{N+1}=0$, as **boundary conditions** (bc's) on the chain of oscillators.

With the expression for the net force on each object we can write down the equation of motion (Newton's second law!) for each object as

$$\ddot{q}_j - \tilde{\omega}^2\left(q_{j-1} - 2q_j + q_{j+1}\right) = 0\,, \tag{3}$$

where $1 \le j \le N$ and, as before, $\tilde{\omega}^2 = k_s/m$. Notice that each equation of motion is coupled: the equation of motion for the jth object depends upon the displacement of both the ($j-1$) and ($j+1$) objects.

II. Normal Mode Solutions

We now look for normal-mode solutions (where all masses oscillate at the same frequency) by assuming that[1]

$$q_j = q_{0,j}e^{i\Omega t}\,. \tag{4}$$

If we substitute Eq. (4) into Eq. (3), after a bit of algebra the equations of motion become

$$\Omega^2 q_{0,j} + \tilde{\omega}^2\left(q_{0,j-1} - 2q_{0,j} + q_{0,j+1}\right) = 0 \tag{5}$$

Now, keep in mind that what we have here are N equations of motion, one for each value of j from 1 to N. As in the two-oscillator problem, the set of equations can be expressed in matrix notation

$$\begin{pmatrix} 2\tilde{\omega}^2 & -\tilde{\omega}^2 & 0 & 0 & \\ -\tilde{\omega}^2 & 2\tilde{\omega}^2 & -\tilde{\omega}^2 & 0 & \\ 0 & -\tilde{\omega}^2 & 2\tilde{\omega}^2 & -\tilde{\omega}^2 & \cdots \\ 0 & 0 & -\tilde{\omega}^2 & 2\tilde{\omega}^2 & \\ & & \vdots & & \end{pmatrix}\begin{pmatrix} q_{0,1} \\ q_{0,2} \\ q_{0,3} \\ q_{0,4} \\ \vdots \end{pmatrix} = \Omega^2\begin{pmatrix} q_{0,1} \\ q_{0,2} \\ q_{0,3} \\ q_{0,4} \\ \vdots \end{pmatrix} \tag{6}$$

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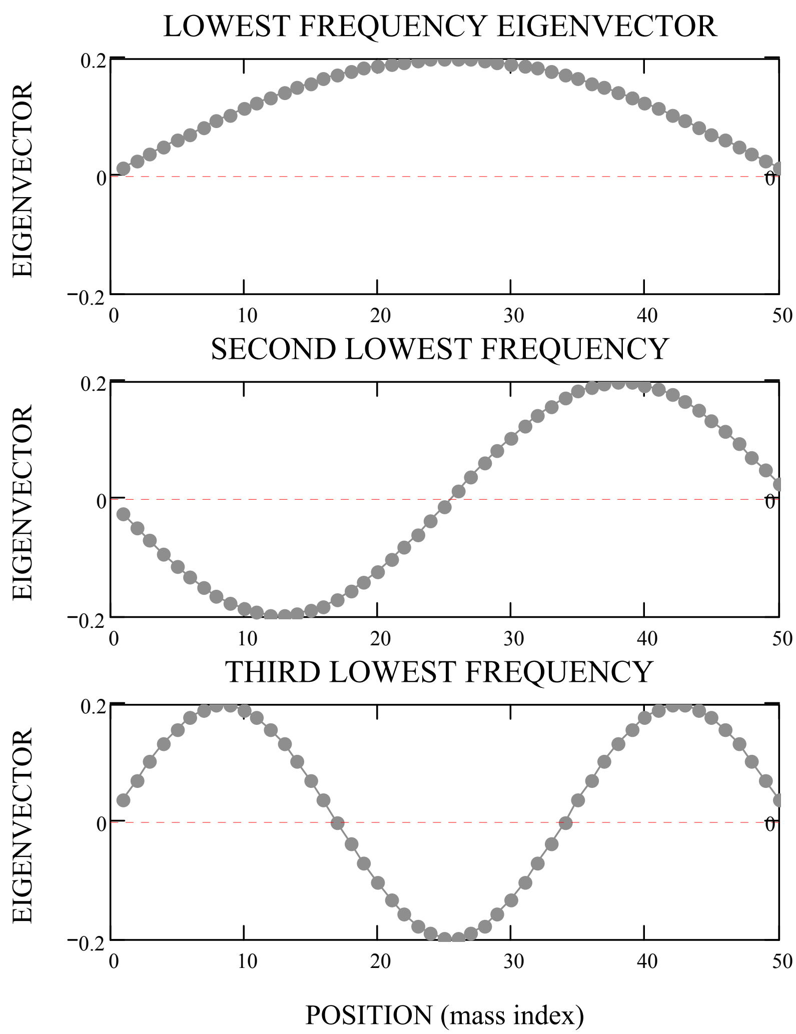
[1] We have slightly changed notation here. We now write the amplitudes $q_{0,j}$ with a comma between the zero and the mass index so that terms such as $q_{0,j+1}$ are unambiguous.

or

$$\begin{pmatrix} 2\tilde{\omega}^2-\Omega^2 & -\tilde{\omega}^2 & 0 & 0 & \\ -\tilde{\omega}^2 & 2\tilde{\omega}^2-\Omega^2 & -\tilde{\omega}^2 & 0 & \\ 0 & -\tilde{\omega}^2 & 2\tilde{\omega}^2-\Omega^2 & -\tilde{\omega}^2 & \cdots \\ 0 & 0 & -\tilde{\omega}^2 & 2\tilde{\omega}^2-\Omega^2 & \\ & & \vdots & & \end{pmatrix}\begin{pmatrix} q_{0,1} \\ q_{0,2} \\ q_{0,3} \\ q_{0,4} \\ \vdots \end{pmatrix} = 0 \qquad (7)$$

So, as before, finding the normal modes reduces to finding eigenvalues Ω^2 and eigenvectors $\begin{pmatrix} q_{0,1} \\ q_{0,2} \\ \vdots \end{pmatrix}$ [of the $N \times N$ matrix in Eq. (6)]. Recall that the eigenvalues are found by solving the (characteristic) equation that arises when we set the determinant of the $N \times N$ matrix in Eq. (7) to zero.



A. Eigenvectors

Now for N any larger than 3 solving the characteristic equation for the eigenvalues and eigenvectors by hand is not advisable. So let's turn to a computer mathematics program, such as Mathcad, and see what insight we can gain into this problem. Given the matrix in Eq. (6) (with a specific value for $\tilde{\omega}^2$), Mathcad can calculate the eigenvalues and eigenvectors of that matrix. In the graphs on the previous page we plot the eigenvectors corresponding to the three lowest eigenvalues for the $N = 50$ problem.

The key thing to notice is that these eigenvectors look like standing waves (on a string, for example). That is, as a function of position (i.e., mass index) j, the components $q_{0,j}$ of the eigenvector appear to be a sine function [which must equal zero at the ends of the chain ($j = 0$ and $j = N + 1$) because of the bc's].

This observation inspires the following ansatz for the eigenvectors

$$q_{0,j} = A\sin(\phi j), \tag{8}$$

where A is some arbitrary **amplitude** for this sine function (it could be complex because we are dealing with a complex form of the solutions), and ϕ is some real number that will be different for each normal mode.[2] Now Eq. (8) obviously satisfies the $q_0 = 0$ bc on the lhs of chain, but not necessarily the rhs bc $q_{N+1} = 0$. To satisfy this bc we must have

$$q_{0,N+1} = A\sin[\phi(N+1)] = 0, \tag{9}$$

which is true only for $\phi(N+1) = n\pi$, where n is an integer. That is, we must have

$$\phi_n = \frac{n\pi}{N+1} \tag{10}$$

where the integer n labels the (normal-mode) solution. Now because any integer n in Eq. (10) produces a value for ϕ that satisfies Eq. (9), it looks like ϕ_n can take on an infinite number of values; this seems to imply an infinite number of normal modes. Well, this can't be right because we know that there are only two normal modes for

[2] Recall, for a standing wave on a string the spatial part of the standing wave can be written as $\sin\left(\frac{2\pi}{\lambda}x\right)$, so the parameter ϕ is obviously related to the wavelength of the normal modes (in some manner – more detail on this later).

the two-oscillator problem. In fact, because the N oscillator problem involves an N dimensional eigenvalue problem, there are exactly N normal modes. The solution to this conundrum lies in the fact that the sine function [in Eq. (9)] is periodic. It can be shown that there are $N+1$ unique solutions, but because the $n=0$ solution is trivial, there are only N unique, nontrivial solutions. In fact, we can specify these N unique solutions by choosing n such that

$$1 \le n \le N . \tag{11}$$

Combining Eqs. (8), (10), and (11) we can write the N eigenvectors as

$$q_{0,j} = A\sin\left(\frac{n\pi}{N+1} j\right), \qquad 1 \le n \le N \tag{12}$$

B. Eigenvalues

So we have now specified the eigenvectors. What about the eigenvalues? We can obtain these by inserting Eq. (8) into Eq. (5), which produces

$$\Omega^2 \sin(\phi\, j) + \tilde{\omega}^2 \{\sin[\phi(j-1)] - 2\sin(\phi\, j) + \sin[\phi(j+1)]\} = 0 . \tag{13}$$

Now it looks like this equation depends upon j, but it does not. Using some trig identities it is not difficult to show that Eq. (13) simplifies and can be solved for Ω^2 as

$$\Omega^2 = 4\tilde{\omega}^2 \sin^2\left(\frac{\phi}{2}\right). \tag{14}$$

And remembering that ϕ only takes on the discrete values given by Eq. (10), we have the N eigenvalues

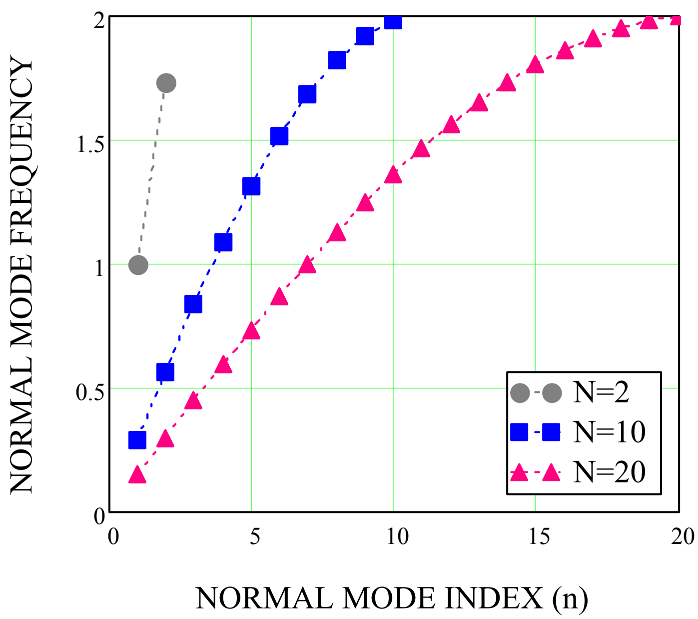
$$\Omega_n^2 = 4\tilde{\omega}^2 \sin^2\left(\frac{n\pi}{2(N+1)}\right), \tag{15}$$

where $n = 1,2,3,\cdots,N$. The normal-mode frequencies are thus given by

$$\Omega_{n\pm} = \pm 2\tilde{\omega} \sin\left(\frac{n\pi}{2(N+1)}\right). \tag{16}$$

It is interesting to plot the (positive) frequencies as a function of mode number n. Such a graph is shown below for several values of N. As with previous graphs we have set $m=1$ and $k_s = 1$.

As we will discuss in the next lecture, these graphs are essentially graphs of frequency vs inverse wavelength, and as such are known as **dispersion relations** or **dispersion curves**. As we will see as we work our way through the course, the dispersion relation is extremely useful in understanding the propagation of waves associated with that dispersion relation. Also, as we will discuss in the next lecture, the dispersion relation also contains information on the interactions between (the springs connecting)



the oscillating objects. Notice that the frequencies plotted for two oscillators ($N = 2$) equal our previous results $\Omega_1 = 1$ and $\Omega_2 = \sqrt{3}$ (for the special case of $m = 1$ and $k_s = 1$).

III. The Initial Value Problem

Lastly, we discuss how the initial value problem can be solved using the normal modes. Quite generally, using the above results and including both the Ω_{n+} and Ω_{n-} frequencies, we can write the nth normal-mode solution as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \\ q_N(t) \end{pmatrix}_n = \begin{pmatrix} \sin\left(\frac{n\pi}{N+1}1\right) \\ \sin\left(\frac{n\pi}{N+1}2\right) \\ \sin\left(\frac{n\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{n\pi}{N+1}N\right) \end{pmatrix} \left(A_n e^{i\Omega_n t} + B_n e^{-i\Omega_n t}\right) \tag{17}$$

where $\Omega_n = \Omega_{n+}$, and the constants A_n and B_n (which have replaced A above) are arbitrary complex numbers.[3] The general solution can thus be written as a linear combination of the normal modes as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \\ q_N(t) \end{pmatrix} = \sum_{n=1}^{N} \left[\begin{pmatrix} \sin\left(\frac{n\pi}{N+1}1\right) \\ \sin\left(\frac{n\pi}{N+1}2\right) \\ \sin\left(\frac{n\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{n\pi}{N+1}N\right) \end{pmatrix} \left(A_n e^{i\Omega_{n+}t} + B_n e^{-i\Omega_{n+}t}\right) \right]. \tag{18}$$

[Equation (18) is the extension of Eq. (3) of the Lecture (4) notes.] As before, the arbitrary amplitudes A_n and B_n depend upon the initial conditions of all the oscillating objects.

To see exactly how the A_n and B_n are determined, let's consider the $N = 3$ case. As in the two-oscillator case, let's make the normal modes explicitly real by setting $B_n = A_n^*$. For three oscillators Eq. (18) then becomes

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = \sum_{n=1}^{3} \begin{pmatrix} \sin\left(\frac{n\pi}{4}1\right) \\ \sin\left(\frac{n\pi}{4}2\right) \\ \sin\left(\frac{n\pi}{4}3\right) \end{pmatrix} \left(A_n e^{i\Omega_n t} + A_n^* e^{-i\Omega_n t}\right) \tag{19}$$

As in Lecture Notes 4 for the two-oscillator problem, we can rewrite $A_n e^{i\Omega_n t} + A_n^* e^{-i\Omega_n t}$ as $2[\mathrm{Re}(A_n)\cos(\Omega_n t) - \mathrm{Im}(A_n)\sin(\Omega_n t)]$ and apply the initial conditions, which gives us

$$\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{pmatrix} = \sum_{n=1}^{3} 2 \begin{pmatrix} \sin\left(\frac{n\pi}{4}1\right) \\ \sin\left(\frac{n\pi}{4}2\right) \\ \sin\left(\frac{n\pi}{4}3\right) \end{pmatrix} \mathrm{Re}(A_n), \tag{20}$$

and

$$\begin{pmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \\ \dot{q}_3(0) \end{pmatrix} = \sum_{n=1}^{3} -2\Omega_n \begin{pmatrix} \sin\left(\frac{n\pi}{4}1\right) \\ \sin\left(\frac{n\pi}{4}2\right) \\ \sin\left(\frac{n\pi}{4}3\right) \end{pmatrix} \mathrm{Im}(A_n). \tag{21}$$

[3] Notice that the column vector of the rhs of Eq. (17) is the n th eigenvector of the associated eigenvalue problem.

So we see that the real part of the amplitudes A_n depend upon the initial positions of the three objects, while the imaginary part of the amplitudes depend upon their initial velocities. So where do we go from here? You may remember that for the two-oscillator problem we applied the normal-mode transformation to the equivalent of Eqs. (20) and (21), which allowed us to find the amplitudes (see p. 5-6 of the Lecture 4 notes). There is an equivalent transformation here that will allow us to find the A_n's. To most easily see what it is, let's explicitly write out Eq. (20) as

$$\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{pmatrix} = 2\left[\begin{pmatrix} \sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{3\pi}{4}\right) \end{pmatrix}\mathrm{Re}(A_1) + \begin{pmatrix} \sin\left(\frac{\pi}{2}\right) \\ \sin(\pi) \\ \sin\left(\frac{3\pi}{2}\right) \end{pmatrix}\mathrm{Re}(A_2) + \begin{pmatrix} \sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{9\pi}{4}\right) \end{pmatrix}\mathrm{Re}(A_3)\right], \tag{22}$$

and evaluate the sine functions, which gives us

$$\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{pmatrix} = \left[\begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix}\mathrm{Re}(A_1) + \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}\mathrm{Re}(A_2) + \begin{pmatrix} \sqrt{2} \\ -2 \\ \sqrt{2} \end{pmatrix}\mathrm{Re}(A_3)\right]. \tag{23}$$

Now notice what now happens if we multiply this equation by the first eigenvector $\begin{pmatrix} \sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{3\pi}{4}\right) \end{pmatrix}$ when it is written as a row vector $\left(\sin\left(\frac{\pi}{4}\right) \quad \sin\left(\frac{\pi}{2}\right) \quad \sin\left(\frac{3\pi}{4}\right)\right) = \frac{1}{2}\left(\sqrt{2} \quad 2 \quad \sqrt{2}\right)$. We obtain

$$\frac{1}{2}\left(\sqrt{2}\, q_1(0) + 2q_2(0) + \sqrt{2}\, q_3(0)\right) = 4\,\mathrm{Re}(A_1). \tag{24}$$

Notice the very nice result that the terms containing the amplitudes A_2 and A_3 produce zero when multiplied by the first eigenvector (in row form). We can now solve for the real part of A_1 in terms of the initial positions as

$$\mathrm{Re}(A_1) = \frac{1}{8}\left(\sqrt{2}\, q_1(0) + 2q_2(0) + \sqrt{2}\, q_3(0)\right). \tag{25}$$

This equation is equivalent to the first row of Eq. (18) [or Eq. (20a)] in the Lecture 4 notes for the two-oscillator problem. To obtain $\mathrm{Re}(A_2)$ and $\mathrm{Re}(A_3)$ it should be obvious that one needs to multiply Eq. (23) by the respective (row) eigenvectors.

Further, to find the imaginary parts of the amplitudes, one similarly multiplies Eq. (21) by the row eigenvectors.

This "trick" of multiplying by the row eigenvector to obtain the corresponding amplitude is probably the most important part of this lecture. We will repeat it many times throughout the course: when we discuss Fourier series we will use this trick to find the Fourier coefficients of a function; when we talk about vector spaces this trick will be recognized as the "inner product"; and when we talk about Fourier transforms this trick will be known as "inversion". As we discuss these topics, you should keep in mind this little trick that allowed us to find the amplitudes A_n.

So we know how to find the A_n's, but what about the normal-mode transformation mentioned above? Well, it is lurking about here. If we now create an $N \times N$ matrix by stacking the row eigenvectors, then we indeed have that transformation. So for the three-oscillator problem, the transformation matrix would be

$$\frac{1}{2}\begin{pmatrix} \sqrt{2} & 2 & \sqrt{2} \\ 2 & 0 & -2 \\ \sqrt{2} & -2 & \sqrt{2} \end{pmatrix}. \tag{26}$$

If one multiples Eqs. (20) and (21) by Eq. (26) then one obtains two equations that are equivalent to Eqs. (18) and (19) of the Lecture 4 notes for the two-oscillator case. Also, if one multiplies the column vector $\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}$ by Eq. (26) then one obtains the normal-mode coordinates $\begin{pmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{pmatrix}$ for the three-oscillator case.

Exercises

*__5.1__ Using the appropriate trig formulae, obtain Eq. (14) for Ω^2 from Eq. (13).

***5.2 Only N unique eigenvalues and eigenvectors**
Eqs. (12) and (15) specify the N eigenvector and eigenvalues for the N oscillator problem. The condition $1 \le n \le N$ implies that values of n outside of this range simply give a solution that is the same as one of the solutions inside the range $1 \le n \le N$.

(**a**) Starting with Eq. (15) and using the angle-addition formula for the sine function, show, for example, that Ω_{N+2}^2 is the same as Ω_N^2. [Hint: write N as $(N+1)-1$ and $N+2$ as $(N+1)+1$.]
(**b**) Starting with Eq. (12) show also that the eigenvector for $n=N+2$ is the same as the eigenvector for $n=N$.

*__5.3__ Similar to the second figure in the notes, graph the three eigenvectors for three coupled oscillators.

*__5.4 Four coupled oscillators__
(a) What are the eigenvalues Ω_n^2 for four coupled oscillators?
(b) Similar to the second figure in the notes, find and then graph the four eigenvectors for four coupled oscillators.

*__5.5__ Similar to Eq. (25), find the imaginary part of A_2 for the $N=3$ system of coupled oscillators.

*__5.6__ Apply Eq. (26), the normal-mode transformation, to Eqs. (20) and (21) to obtain the two equations for three oscillators that are equivalent to Eqs. (18) and (19) of the Lecture 4 notes for two oscillators.

*__5.7__ Show that the square of Eq. (26), the normal-mode transformation, is proportional to the identity matrix. Thus find the inverse of the normal-mode transformation.

*__5.8__ For the three-oscillator problem find the normal-mode coordinates $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ in terms of the displacements $q_1(t)$, $q_2(t)$, and $q_3(t)$.