Normal Mode Coordinates / Initial Value Problem

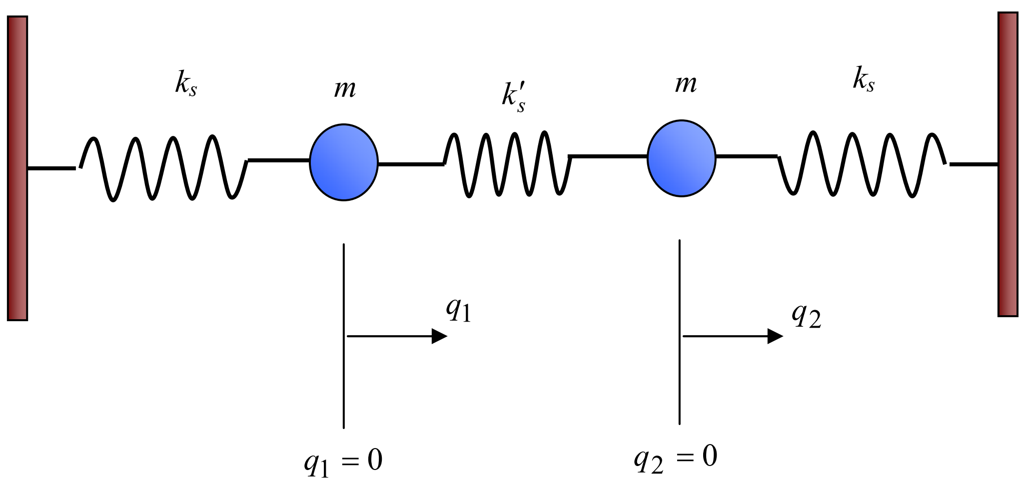
Overview and Motivation: We continue to look at the coupled-oscillator problem. We extend our analysis of this problem by introducing functions known as normal-mode coordinates. These coordinates make the coupled-oscillator problem simple because they transform the coupled equations of motion into two uncoupled equations of motion. Using the normal modes, we then solve the general initial-value problem for this system.

Key Mathematics: We gain experience with linear transformations and initial value problems.

I. Normal Mode Solutions

A. Summary from Last Lecture

The problem that we studied last time is shown in the following diagram. There are two objects, each with mass m and three springs. The springs on the ends have spring constant k_s and the one in the center has spring constant k_s'.



Last time we found the two normal-mode solutions, which can be written as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(A_1 e^{i\Omega_1 t} + B_1 e^{-i\Omega_1 t}\right) \tag{1}$$

and

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(A_2 e^{i\Omega_2 t} + B_2 e^{-i\Omega_2 t} \right), \tag{2}$$

where $\Omega_1 = \tilde{\omega}$ and $\Omega_2 = \sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}$ are the normal-mode frequencies of oscillation. As we will show below, any solution to this problem can be written as a linear combination of these two normal modes. Thus, we can write the most general solution to this problem as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(A_1 e^{i\Omega_1 t} + B_1 e^{-i\Omega_1 t} \right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(A_2 e^{i\Omega_2 t} + B_2 e^{-i\Omega_2 t} \right) \tag{3}$$

B. Normal Mode Coordinates

Let's now consider the following **linear transformation**[1] of the displacements $q_1(t)$ and $q_2(t)$,

$$\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}. \tag{4}$$

Calculating the rhs of Eq. (4) produces[2]

$$\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_1(t) + q_2(t) \\ q_1(t) - q_2(t) \end{pmatrix}. \tag{5}$$

Written more pedantically, we have

$$Q_1(t) = \frac{q_1(t) + q_2(t)}{2} \tag{6a}$$

and

$$Q_2(t) = \frac{q_1(t) - q_2(t)}{2}. \tag{6b}$$

For reasons that will soon become apparent, the functions $Q_1(t)$ and $Q_2(t)$ are known

[1] Any linear transformation of an N-vector can be represented as lhs multiplication of that vector by an $N \times N$ matrix.

[2] Note that the normal-mode coordinate Q_2 as defined here is the negative of Q_2 as defined in Dr. Torre's text *FWP*. We define it here with this change so as to be more consistent with the later treatment of N coupled oscillators. To be honest, it also makes some of the equations look prettier!

as **normal-mode coordinates**. (Why aren't they called normal-mode functions? Don't ask me!)

Let's now apply the linear transformation in Eq. (4) to the rhs of Eq. (3) and see what it tells us. Applying the transformation and equating the result to the lhs of Eq. (4) yields

$$\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(A_1 e^{i\Omega_1 t} + B_1 e^{-i\Omega_1 t}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(A_2 e^{i\Omega_2 t} + B_2 e^{-i\Omega_2 t}\right). \tag{7}$$

This equation may look complicated, but, in fact, it is very simple. It says that $Q_1(t)$ harmonically oscillates at the first normal-mode frequency Ω_1 and that $Q_2(t)$ harmonically oscillates at the second normal-mode frequency Ω_2. Pretty cool! In fact, if Eq. (3) is the general solution to this problem (more on this below), Eq. (7) say that no matter what the motion, the sum $q_1(t)+q_2(t)$ always oscillates at Ω_1, and the difference $q_1(t)-q_2(t)$ always oscillates at Ω_2.

C. Equations of Motion

Let's now go back to the coupled equations of motion,

$$\ddot{q}_1 + \tilde{\omega}^2 q_1 + \tilde{\omega}'^2(q_1 - q_2) = 0 \tag{8a}$$

and

$$\ddot{q}_2 + \tilde{\omega}^2 q_2 + \tilde{\omega}'^2(q_2 - q_1) = 0, \tag{8b}$$

and see what happens if we write them in terms of the normal-mode coordinates $Q_1(t)$ and $Q_2(t)$. To do that we need the **inverse** of the transformation in Eq. (4). Using

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9}$$

(make sure that you understand this!) we apply this inverse transformation to Eq. (4), which gives us (after switching the rhs and lhs)

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} Q_1(t) + Q_2(t) \\ Q_1(t) - Q_2(t) \end{pmatrix}. \tag{10}$$

That is, $q_1(t) = Q_1(t) + Q_2(t)$ and $q_2(t) = Q_1(t) - Q_2(t)$. Substituting these results into Eq. (8) produces

$$\ddot{Q}_1 + \ddot{Q}_2 + \tilde{\omega}^2(Q_1 + Q_2) + 2\tilde{\omega}'^2 Q_2 = 0 \tag{11a}$$

and

$$\ddot{Q}_1 - \ddot{Q}_2 + \tilde{\omega}^2(Q_1 - Q_2) - 2\tilde{\omega}'^2 Q_2 = 0\,. \tag{11b}$$

Looks pretty ugly, eh? Well , it is about to get much simpler. If we take the sum and difference of Eqs. (11a) and (11b) we get the following two equations,

$$\ddot{Q}_1 + \tilde{\omega}^2 Q_1 = 0 \tag{12a}$$

and

$$\ddot{Q}_2 + \left(\tilde{\omega}^2 + 2\tilde{\omega}'^2\right) Q_2 = 0\,. \tag{12b}$$

First, notice that these two equations are uncoupled: the equation of motion for Q_1 doesn't depend upon Q_2 and vice-versa. Furthermore, you should now able to recognize each of these equations as the equation of motion for a single harmonic oscillator! Thus, Eq. (12a) tells us that Q_1 harmonically oscillates at $\tilde{\omega}$ $(=\Omega_1)$, and Eq. (12b) tells us that Q_2 harmonically oscillates at $\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}$ $(=\Omega_2)$. Of course, this is exactly what was expressed earlier by Eq. (7).

We can also infer something very important from this transformation. Because the normal coordinates are governed by Eq. (12), which is simply an harmonic oscillator equation for each coordinate, we know that the general solution for $Q_1(t)$ and $Q_2(t)$ is given by Eq. (7). Thus, the general solution for $q_1(t)$ and $q_2(t)$ is given by the inverse transformation of Eq. (7), which is simply Eq. (3). This proves that the general solution for $q_1(t)$ and $q_2(t)$ is, indeed, a linear combination of the normal mode coordinates $Q_1(t)$ and $Q_2(t)$. This is a general result that we will use throughout the course.

II. Initial Value Problem

Let's now solve the initial-value problem for the coupled-oscillator system. That is, we want to write Eq. (3), the general solution to the coupled oscillator problem, in terms of the initial conditions $q_1(0)$, $\dot{q}_1(0)$, $q_2(0)$, and $\dot{q}_2(0)$ (which are all real quantities).

Now Eq. (3),

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(A_1 e^{i\Omega_1 t} + B_1 e^{-i\Omega_1 t}\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(A_2 e^{i\Omega_2 t} + B_2 e^{-i\Omega_2 t}\right), \tag{3}$$

uses the complex form of the harmonic oscillator solution. Looking back on p. 8 of the Lecture 2 notes, we see that we have three choices about how to deal with making the IVP solution real. Let's use the first approach and make Eq. (3) manifestly real by setting $B_1 = A_1^*$ and $B_2 = A_2^*$. Then we have

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(A_1 e^{i\Omega_1 t} + A_1^* e^{-i\Omega_1 t}\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(A_2 e^{i\Omega_2 t} + A_2^* e^{-i\Omega_2 t}\right). \tag{13}$$

This expression is real because each term in parenthesis is the sum of a complex number and its complex conjugate. To rewrite Eq. (13) in a form that is explicitly real we use the relationship

$$Ae^{ix} + A^* e^{-ix} = 2\,\mathrm{Re}\left(Ae^{ix}\right) = 2\left[\mathrm{Re}(A)\cos(x) - \mathrm{Im}(A)\sin(x)\right], \tag{14}$$

so that Eq. (13) becomes

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left[\mathrm{Re}(A_1)\cos(\Omega_1 t) - \mathrm{Im}(A_1)\sin(\Omega_1 t)\right] + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\left[\mathrm{Re}(A_2)\cos(\Omega_2 t) - \mathrm{Im}(A_2)\sin(\Omega_2 t)\right]. \tag{15}$$

We now apply the initial conditions to Eq. (15), which gives us

$$\begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathrm{Re}(A_1) + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathrm{Re}(A_2), \tag{16}$$

and

$$\begin{pmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{pmatrix} = -2\Omega_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathrm{Im}(A_1) - 2\Omega_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathrm{Im}(A_2). \tag{17}$$

The easiest way to solve for the four unknowns [$\mathrm{Re}(A)$, $\mathrm{Im}(A)$, $\mathrm{Re}(C)$, and $\mathrm{Im}(C)$] is to apply the normal-mode transformation $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ to each side of these two equations, which produces

$$\frac{1}{2}\begin{pmatrix} q_1(0)+q_2(0) \\ q_1(0)-q_2(0) \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}(A_1) + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{Re}(A_2), \tag{18}$$

and

$$\frac{1}{2}\begin{pmatrix} \dot{q}_1(0)+\dot{q}_2(0) \\ \dot{q}_1(0)-\dot{q}_2(0) \end{pmatrix} = -2\Omega_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Im}(A_1) - 2\Omega_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{Im}(A_2). \tag{19}$$

From these last two equations we immediately see that

$$\mathrm{Re}(A_1) = \frac{q_1(0)+q_2(0)}{4}, \tag{20a}$$

$$\mathrm{Im}(A_1) = -\frac{\dot{q}_1(0)+\dot{q}_2(0)}{4\Omega_1}, \tag{20b}$$

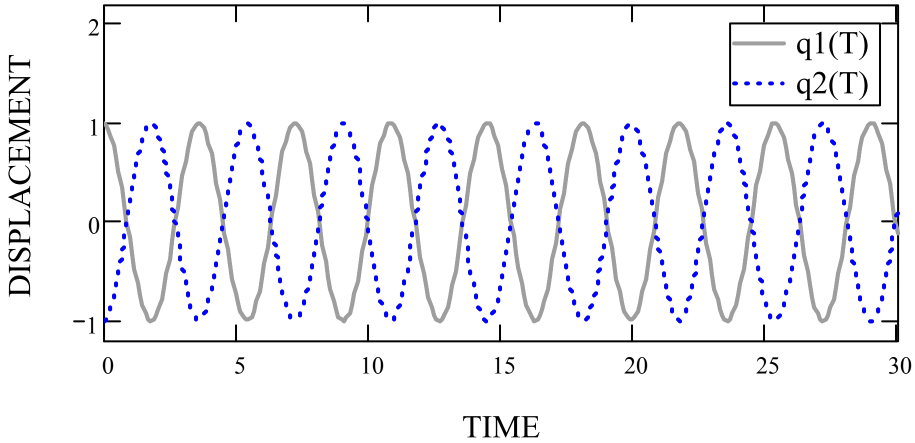
$$\mathrm{Re}(A_2) = \frac{q_1(0)-q_2(0)}{4}, \text{ and} \tag{20c}$$

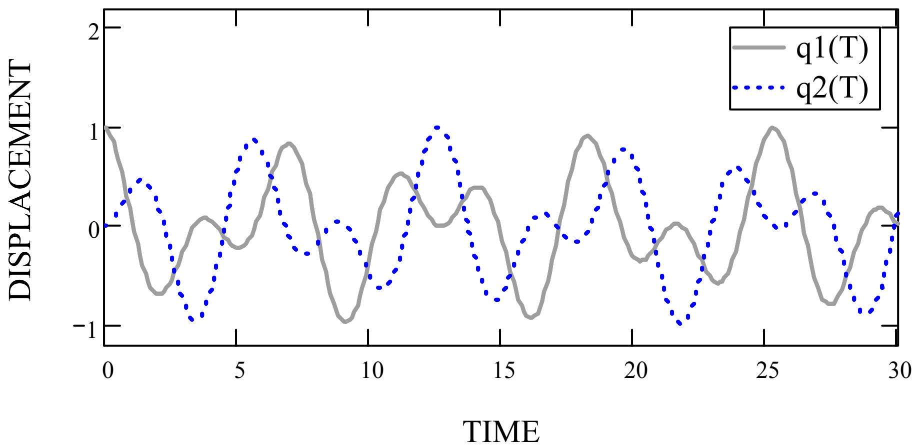
$$\mathrm{Im}(A_2) = \frac{\dot{q}_2(0)-\dot{q}_1(0)}{4\Omega_2}. \tag{20d}$$

If we now substitute Eq. (20) into Eq. (15) we finally obtain the solution to the IVP,

$$\begin{aligned}\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}&\left\{[q_1(0)+q_2(0)]\cos(\Omega_1 t) + \left[\frac{\dot{q}_1(0)+\dot{q}_2(0)}{\Omega_1}\right]\sin(\Omega_1 t)\right\} \\ +\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}&\left\{[q_1(0)-q_2(0)]\cos(\Omega_2 t) + \left[\frac{\dot{q}_1(0)-\dot{q}_2(0)}{\Omega_2}\right]\sin(\Omega_2 t)\right\}\end{aligned}. \tag{21}$$

The graphs on the following page illustrate the motion that results for two sets of initial conditions. For both graphs $m=1$, $k_s=1$, and $k_s'=1$ (so that $\Omega_1=1$ and $\Omega_2=\sqrt{3}$). In the top graph $q_1(0)=1$, $q_2(0)=-1$, $\dot{q}_1(0)=0$, and $\dot{q}_2(0)=0$. What special motion is this? In the second graph the initial conditions are $q_1(0)=1$, $q_2(0)=0$, $\dot{q}_1(0)=0$, and $\dot{q}_2(0)=0$. This motion is quite complicated. In fact, it is not even periodic: it never repeats, even though the normal-mode coordinates are simply harmonically oscillating at their respective frequencies. The nonrepetitive nature of the motion occurs because (in this example) the ratio is of the two normal-mode frequencies is not a rational number.





Exercises

***4.1** Apply the inverse transformation $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to Eq. (7) to recover Eq. (3).

***4.2** Assuming $q_1(0)=0$ and $q_2(0)=0$, find the general condition on the initial velocities $\dot{q}_1(0)$ and $\dot{q}_2(0)$ that results in only the first normal mode being excited.

***4.3** Assuming $q_1(0)=0$ and $q_2(0)=0$, find the general condition on the initial velocities $\dot{q}_1(0)$ and $\dot{q}_2(0)$ that results in only the second normal mode being excited.

****4.4** The general solution to the two-coupled oscillator problem can alternatively be expressed in terms of real quantities as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[A\cos(\Omega_1 t) + B\sin(\Omega_1 t) \right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[C\cos(\Omega_2 t) + D\sin(\Omega_2 t) \right]$$

Starting with this form of the general solution find the (real) parameters A, B, C, and D in terms of the initial conditions $q_1(0)$, $\dot{q}_1(0)$, $q_2(0)$, and $\dot{q}_2(0)$. Check to see that your solution agrees with Eq. (21).

*__4.5__ Use Euler's relation to derive Eq. (14).