

Energy Density and the Poynting Vector

Overview and Motivation: We saw in the last lecture that electromagnetic waves are one consequence of Maxwell's (M's) equations. With electromagnetic waves, as with other waves, there is an associated energy density and energy flux. Here we introduce these electromagnetic quantities and discuss the conservation of energy in the electromagnetic fields. Further, we see how the expressions for the energy density and energy flux can be put into a form that is similar to expressions for the same quantities for waves on a string.

Key Mathematics: We will gain some more practice with the "del" operator ∇ . We will also discuss what is meant by a time-averaged quantity.

I. Energy Density and Energy-current Density in EM Waves

Recall from the last lecture the basic Maxwell's equations,

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0}, \quad (1)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad (2)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \quad (3)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{j}(\mathbf{r}, t) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t}. \quad (4)$$

As we discussed last time, for $\rho(\mathbf{r}, t) = 0$ and $\mathbf{j}(\mathbf{r}, t) = 0$, M's equations imply the wave equation for both $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. We know that waves transport energy. So how is the energy in an electromagnetic wave expressed? Well, you should have learned in your introductory physics course that the energy density contained in the electric field is given by¹

$$u_{el}(\mathbf{r}, t) = \frac{\epsilon_0}{2} \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\epsilon_0}{2} [\mathbf{E}(\mathbf{r}, t)]^2. \quad (5)$$

Typically this energy density is introduced in a discussion of the energy required to charge up a capacitor (which produces an electric field between the plates). Similarly, the energy density contained in the magnetic field is given by

¹ In keeping with standard EM notation, we use u for the energy density and \mathbf{S} for the energy flux.

$$u_{mag}(\mathbf{r}, t) = \frac{1}{2\mu_0} \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t) = \frac{1}{2\mu_0} [\mathbf{B}(\mathbf{r}, t)]^2. \quad (6)$$

Typically this relationship is introduced in a discussion of the energy required to establish a current in a toroid (which produces a magnetic field inside the toroid). Notice again that the two fundamental constants of E and M, ϵ_0 and μ_0 , appear in Eq. (5) and Eq. (6), respectively. Thus the total energy $u(\mathbf{r}, t)$ contained in a region of space with both electric and magnetic fields is

$$u(\mathbf{r}, t) = \frac{1}{2} \left\{ \epsilon_0 [\mathbf{E}(\mathbf{r}, t)]^2 + \frac{1}{\mu_0} [\mathbf{B}(\mathbf{r}, t)]^2 \right\}. \quad (7)$$

Because $c^2 = 1/(\mu_0 \epsilon_0)$, this can also be written as

$$u(\mathbf{r}, t) = \frac{1}{2\mu_0} \left\{ \left[\frac{\mathbf{E}(\mathbf{r}, t)}{c} \right]^2 + [\mathbf{B}(\mathbf{r}, t)]^2 \right\}. \quad (8)$$

Recall, for a traveling EM wave in vacuum the electric and magnetic field amplitudes are related by $B = E/c$. Equation (8) thus shows that equal amounts of energy are contained in the electric and magnetic fields in such a wave.

What about the energy current density (also known as the energy flux)? Well, another basic fact about electromagnetic radiation (that you may or may not have learned in your introductory physics course) is that the energy flux in a particular region of space is equal to

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \quad (9)$$

As we learned in the last lecture, the direction of propagation of an electromagnetic plane wave is in the direction of $\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)$. As expected, Eq. (9) indicates that the energy flux points in this same direction. In E and M the energy flux is known as the **Poynting vector** (convenient because it *points* in the direction of the energy flow).

II. Continuity Equation for u and \vec{S}

If u and \vec{S} are indeed the energy and energy-current densities, respectively, then we expect that they should be related by the continuity equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{S}(\mathbf{r}, t) = 0. \quad (10)$$

Let's see if M's equations indeed imply Eq. (10). We first start with Eq. (8) and calculate its time derivative, which gives us

$$\frac{\partial u}{\partial t} = \frac{1}{\mu_0} \left(\frac{1}{c^2} \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right). \quad (11)$$

In deriving Eq. (11) we must remember, for example, that \mathbf{E}^2 is really shorthand for $\mathbf{E} \cdot \mathbf{E}$ [see Eq. (5)]. Starting with Eq. (9) we can also calculate $\nabla \cdot \mathbf{S}$, which gives us, after using the vector identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$,

$$\nabla \cdot \mathbf{S} = \frac{1}{\mu_0} [\mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})]. \quad (12)$$

We can now use M's Eqs. (3) and (4) to replace the curls in Eq. (12), which produces, after a bit of manipulation and the use of $\mu_0 \epsilon_0 = 1/c^2$,

$$\nabla \cdot \mathbf{S} = -\frac{1}{\mu_0} \left(\frac{1}{c^2} \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) - \mathbf{E} \cdot \mathbf{j}. \quad (13)$$

Comparing Eqs. (11) and (13) we see that

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \quad (14)$$

So what happened? Why didn't M's equations give us Eq. (10)? Well, there is a very good reason. The energy density u and energy-current density \mathbf{S} are densities associated with the fields only. But energy can also exist in the (kinetic) energy of the charge density. The term $\mathbf{E} \cdot \mathbf{j}$ is known as **Joule heating**; it expresses the rate of energy transfer to the charge carriers from the fields. This is the (spatially) local version of an equation with which you are already familiar, $P = VI$. Notice that this term only contains the electric field because the magnetic field can do no work on the charges. The term appears with a negative sign in Eq. (14) because an increase in energy of the charge carriers contributes to a decrease in energy in the fields.

Obviously, if the homogeneous M's equations apply [$\rho(\mathbf{r},t)=0$ and $\mathbf{j}(\mathbf{r},t)=0$], then Eq. (10), the standard continuity equation is indeed valid.

III. The densities u and \mathbf{S} for an EM Plane Wave

In the last lecture we looked at the plane-wave solution

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) \quad (15)$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c} (\hat{\mathbf{k}} \times \mathbf{E}_0) \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) \quad (16)$$

to the homogeneous Maxwell's equations. Let's calculate u and $\bar{\mathbf{S}}$ for these fields. Substituting Eqs. (15) and (16) into Eqs. (8) and (9) produces

$$u(\mathbf{r},t) = \frac{1}{c} \sqrt{\frac{\epsilon_0}{\mu_0}} [E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)]^2, \quad (17)$$

and

$$\mathbf{S}(\mathbf{r},t) = \sqrt{\frac{\epsilon_0}{\mu_0}} [E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)]^2 \hat{\mathbf{k}}, \quad (18)$$

respectively. Comparing Eqs. (17) and (18) we see that

$$\mathbf{S}(\mathbf{r},t) = cu(\mathbf{r},t)\hat{\mathbf{k}}. \quad (19)$$

This agrees with the general expectation for a traveling wave that the energy current flux $\mathbf{j}_\epsilon(\mathbf{r},t)$ is related to its associated energy density $\rho_\epsilon(\mathbf{r},t)$ via $\mathbf{j}_\epsilon(\mathbf{r},t) = \rho_\epsilon(\mathbf{r},t)\mathbf{v}$, where \mathbf{v} is the velocity of $\rho_\epsilon(\mathbf{r},t)$.

The Poynting vector expressed in Eq. (18) is a space and time dependent quantity. Often, however, often we are more interested in the time-averaged value of this quantity. In general, the time-averaged value of a periodic function with period T is given by

$$\langle A(t) \rangle_t = \frac{1}{T} \int_0^T A(t) dt. \quad (20)$$

With this definition the time-averaged value of \mathbf{S} is

$$\langle \mathbf{S}(t) \rangle_t = E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\mathbf{k}} \left\{ \frac{\omega}{2\pi} \int_0^{\omega/2\pi} [\cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)]^2 dt \right\} \quad (21)$$

Because the average value of any harmonic function squared is simply 1/2, we have

$$\langle \mathbf{S}(t) \rangle_t = \frac{E_0^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\mathbf{k}}. \quad (22)$$

On last remark about $\langle \mathbf{S}(\mathbf{r}, t) \rangle_t$. In the optics world $\langle \mathbf{S}(\mathbf{r}, t) \rangle_t$ is known as the **intensity** associated with the electromagnetic wave. Its dot product with a normal vector to some surface gives the average power per unit area incident on that surface.

IV. An Analogy Between Mechanical and EM Waves

We previously studied the energy contained in mechanical waves. In particular, we looked at transverse waves on a string, which have an energy density and energy-current density that were essentially expressed as

$$\rho_\epsilon(x, t) = \frac{\tau}{2} \left\{ \left[\frac{1}{c} \frac{\partial q(x, t)}{\partial t} \right]^2 + \left[\frac{\partial q(x, t)}{\partial x} \right]^2 \right\}, \quad (23)$$

$$j_\epsilon(x, t) = -\tau \left[\frac{\partial q(x, t)}{\partial t} \right] \left[\frac{\partial q(x, t)}{\partial x} \right]. \quad (24)$$

As they stand, these equations do not look particularly like Eqs. (8) and (9) for the corresponding electromagnetic quantities.

The mechanical-waves expressions are written in terms of derivatives of the displacement while the electromagnetic quantities are written in terms of the fields. However, in the theory of electricity and magnetism we can introduce a quantity known as the vector potential $\mathbf{A}(\mathbf{r}, t)$ that, in the absence of $\rho(\vec{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$, can be defined such that it is related to the electric and magnetic fields via

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad (25)$$

and

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \quad (26)$$

Substituting these expressions into Eqs. (8) and (9) then gives us two equations that now look quite similar to Eqs. (23) and (24),

$$u(\mathbf{r}, t) = \frac{1}{2\mu_0} \left\{ \left[\frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \right]^2 + [\nabla \times \mathbf{A}(\mathbf{r}, t)]^2 \right\}, \quad (27)$$

$$\mathbf{S}(\mathbf{r}, t) = -\frac{1}{\mu_0} \left(\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \times [\nabla \times \mathbf{A}(\mathbf{r}, t)] \right). \quad (28)$$

For a more exact analogy, let's go back to Eqs. (15) and (16), the plane-wave solution to M's equations. Let's simplify things by choosing the coordinate system so that \mathbf{k} points in the $+x$ direction, \mathbf{E}_0 points along the $+y$ direction, which leaves \mathbf{B}_0 to point along the $+z$ direction. The electric and magnetic fields for the plane wave can then be written as

$$\mathbf{E}(x, t) = E_0 \hat{\mathbf{y}} \cos(kx - \omega t + \phi), \quad (29)$$

$$\mathbf{B}(x, t) = \frac{E_0}{c} \hat{\mathbf{z}} \cos(kx - \omega t + \phi). \quad (30)$$

These two fields are consistent with the vector potential

$$\mathbf{A}(x, t) = \frac{E_0}{\omega} \hat{\mathbf{y}} \sin(kx - \omega t + \phi) = A_y(x, t) \hat{\mathbf{y}}. \quad (31)$$

With this vector potential the time derivative and curl of \mathbf{A} simplify to $\partial \mathbf{A} / \partial t = (\partial A_y / \partial t) \hat{\mathbf{y}}$ and $\nabla \times \mathbf{A} = (\partial A_y / \partial x) \hat{\mathbf{z}}$ so that

$$\left(\frac{\partial \mathbf{A}}{\partial t} \right)^2 = \left(\frac{\partial A_y}{\partial t} \right)^2, \quad (32)$$

$$(\nabla \times \mathbf{A})^2 = \left(\frac{\partial A_y}{\partial x} \right)^2, \quad (33)$$

and

$$\frac{\partial \mathbf{A}}{\partial t} \times (\nabla \times \mathbf{A}) = \frac{\partial A_y}{\partial t} \frac{\partial A_y}{\partial x} \hat{\mathbf{x}}. \quad (34)$$

With these last three expressions we can express u and \vec{S} for our plane wave solution [Eqs. (31) and (32)] as

$$u(x,t) = \frac{1}{2\mu_0} \left\{ \left[\frac{1}{c} \frac{\partial A_y}{\partial t} \right]^2 + \left[\frac{\partial A_y}{\partial x} \right]^2 \right\}, \quad (35)$$

$$\mathbf{S}(x,t) = -\frac{1}{\mu_0} \frac{\partial A_y}{\partial t} \frac{\partial A_y}{\partial x} \hat{\mathbf{x}}. \quad (36)$$

These expressions are now essentially identical to Eqs. (23) and (24), the analogous expressions for mechanical waves on a string if the following correspondences are made: $q \leftrightarrow A_y$ and $\tau \leftrightarrow 1/\mu_0$.

Exercises

***32.1** Show that Eqs. (29) and (30) follow from Eq. (31).

****32.2 A traveling-wave solution to Maxwell's equations.** Consider the electric field $\mathbf{E}(r,t) = E_0 \hat{\mathbf{x}} \cos(kz - kct)$

- What is the corresponding magnetic field?
- Calculate the energy density $u(z,t)$ associated with each of these fields.
- Calculate the Poynting vector $\mathbf{S}(z,t)$ associated with these fields.
- Show that $u(z,t)$ and $\mathbf{S}(z,t)$ satisfy the appropriate continuity equation.

***32.3** Show in the absence of charge and current densities that – in general – the vector potential $\mathbf{A}(\mathbf{r},t)$ satisfies the wave equation. In addition to equations in the notes, you will need to use the fact that the vector potential satisfies $\nabla \cdot \mathbf{A} = 0$.