Maxwell's Equations

Overview and Motivation: Maxwell's (M's) equations, along with the Lorentz force law constitute essentially all of classical electricity and magnetism (E and M). One phenomenon that arises from M's equations is electromagnetic radiation, that is, electromagnetic waves. Here we introduce M's equations, discuss how they should be viewed, and see how M's equations imply the wave equation. We will look at a plane-wave solution to M's equations. We will also see that the conservation of electric charge is a direct result of Maxwell's equations.

Key Mathematics: We will gain some practice with "del" (∇) used in calculating the divergence and the curl of the electric and magnetic vector fields.

I. Maxwell's Equations

The basic Maxwell's equations are typically written, in SI units[1], as

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0\,, \tag{2}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0\, \mathbf{j}(\mathbf{r},t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{4}$$

These are coupled first-order, linear, partial differential equations for the electric and magnetic vector fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, respectively. The two constants in the equations, ε_0 and μ_0, are the fundamental constants of E and M, respectively. You probably first encountered these two constants in your introductory physics class when you studied the electric force from a point charge and the magnetic force form a long, straight wire carrying a constant current. The other two quantities in these equations are the electric charge density $\rho(\mathbf{r},t)$ and electric-charge current density $\mathbf{j}(\mathbf{r},t)$.

[1] There have been no fewer than 5 systems of units traditionally used for E and M: electrostatic (esu), electromagnetic (emu), Gaussian (cgs), Heaviside-Lorentz, and Rationalized MKSA (now known as SI). Beware when reading the literature!

So what is the meaning of Eqs. (1) – (4)? The way these equations are written you should think of the quantity on the rhs as giving rise to the quantity on the lhs of each equation. So Eq. (1) tells us that electric charge density is the source of electric field. Similarly, Eq. (2) tells us that there is no such corresponding magnetic charge density. Equation (3) tells us that a time varying magnetic field can produce an electric field, and Eq. (4) says that both an electric-charge current density and a time varying electric field can produce a magnetic field. Maxwell was actually only responsible for the last term in Eq. (4), but the term is key to E and M because without it there would be no electromagnetic radiation.

Often one considers the charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$ to be given quantities. That is, one assumes that there is something external to the problem that controls $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ and so they are simply treated as given source terms for the equations. However, in some problems the dynamics of $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are determined by the fields themselves through the Lorentz force equation

$$\mathbf{F} = q\left[\mathbf{E}(\mathbf{r},t) + \left(\vec{v}\times\mathbf{B}(\mathbf{r},t)\right)\right], \tag{5}$$

which describes the force on a particle with charge q and velocity v. In such cases Eqs. (1) – (5) must be solved self consistently for both the time varying fields and charge and current distributions.

II. The Conservation of Electric Charge

One of the consequences of M's equations is the conservation of electric charge. If we start with Eq. (1) and takes its time derivative, we obtain

$$\nabla\cdot\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial\rho(\mathbf{r},t)}{\partial t}, \tag{6}$$

after switching the order of the divergence and time derivative on the lhs. Equation (4) can be rearranged as

$$\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} = \frac{1}{\mu_0\varepsilon_0}\nabla\times\mathbf{B}(\mathbf{r},t) - \frac{1}{\varepsilon_0}\mathbf{j}(\mathbf{r},t). \tag{7}$$

Using this equation to substitute for $\partial\mathbf{E}/\partial t$ in Eq. (6) then yields

$$\nabla\cdot\left[\frac{1}{\mu_0\varepsilon_0}\nabla\times\mathbf{B}(\mathbf{r},t) - \frac{1}{\varepsilon_0}\mathbf{j}(\mathbf{r},t)\right] = \frac{1}{\varepsilon_0}\frac{\partial\rho(\mathbf{r},t)}{\partial t}. \tag{8}$$

But this simplifies considerably because $\nabla \cdot (\nabla \times \mathbf{V}) = 0$ for any vector field $\mathbf{V}$. Thus we have

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0, \tag{9}$$

the continuity equation for the charge density and the charge current density. As we discussed in an earlier lecture, the continuity equation is the local form of the statement of the conservation of the particular quantity (in this case electric charge) that corresponds to the given density and current density.

III. Wave Equations for the Electric and Magnetic Fields

As we now demonstrate, M's equations imply waves equations for both the electric and magnetic fields. To keep things simple we consider M's equations in a charge-free, current-free region of space. Then Eq. (1) – (4) become

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0, \tag{10}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{11}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{12}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{13}$$

These are know as the homogeneous M's equations because either **E** or **B** appear linearly in every (nonzero) term in the equations. Let's derive the wave equation for the magnetic field. To do this we first take the curl of Eq. (13) to produce

$$\nabla \times [\nabla \times \mathbf{B}(\mathbf{r},t)] = \mu_0 \varepsilon_0 \nabla \times \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \tag{14}$$

Using the identity $\nabla \times (\nabla \times \mathbf{V}(\mathbf{r})) = \nabla(\nabla \cdot \mathbf{V}(\mathbf{r})) - \nabla^2 \mathbf{V}(\mathbf{r})$ on the lhs and switching the order of the curl and time derivative on the rhs then produces

$$\nabla[\nabla \cdot \mathbf{B}(\mathbf{r},t)] - \nabla^2 \mathbf{B}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial [\nabla \times \mathbf{E}(\mathbf{r},t)]}{\partial t}. \tag{15}$$

Now using Eq. (11) to substitute for $\nabla \cdot \mathbf{B}(\mathbf{r},t)$ on the lhs and Eq. (12) to substitute for $\nabla \times \mathbf{E}(\mathbf{r},t)$ on the rhs yields

$$\nabla^2 \mathbf{B}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}(\mathbf{r},t)}{\partial t^2}, \tag{16}$$

the wave equation for the magnetic field $\mathbf{B}(\mathbf{r},t)$, where the standard constant c^2 in the wave equation is equal to $1/\mu_0\varepsilon_0$. One can similarly derive the corresponding wave equation for the electric field

$$\nabla^2 \mathbf{E}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2}. \tag{17}$$

Notice that each of these wave equations is for a vector quantity, and so in essence each of these equations is really three wave equations, one for each component of the electric or magnetic field. Another thing to note is that while M's equations imply the wave equations for $\mathbf{E}$ and $\mathbf{B}$, the two fields are not independent. That is, all solutions to Eqs. (16) and (17) will not necessarily satisfy M's equations. Another way to think about this is that by taking a derivative (the curl) of Eq. (13) (to derive the $\mathbf{B}$-field wave equation) we lost some information originally contained in that equation.

IV. Plane Wave Solutions to M's Equations

We already know that a plane wave is one possible type of solution to the 3D wave equation. So let's assume that we have an electric field of the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi). \tag{18}$$

Remember, the wave vector $\mathbf{k}$ points in the direction of propagation and $k \equiv |\mathbf{k}| = 2\pi/\lambda$, where λ is the wavelength. Substituting Eq. (18) into Eq. (17) gives us, as usual, the dispersion relation, which in this case is $\omega = ck$ where $c = 1/\sqrt{\mu_0\varepsilon_0}$ is known as the speed of light. As far as the wave equation is concerned, the dispersion relation is the only constraint that needs to be satisfied in order for Eq. (18) to be a solution.

However, M's equations put a further constraint on the electric field. Let's substitute Eq. (18) into Eq. (10). Then we obtain

$$\nabla\cdot\left[\mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\right] = 0 \tag{19}$$

To see what this equation implies for our plane-wave solution let's rewrite it using Cartesian coordinates,

$$\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\cdot\left[\left(E_{0x}\hat{\mathbf{x}}+E_{0y}\hat{\mathbf{y}}+E_{0z}\hat{\mathbf{z}}\right)\cos\left(k_x x+k_y y+k_z z-\omega t+\phi\right)\right]=0\,. \tag{20}$$

Calculating the derivatives produces

$$\left(E_{0x}k_x+E_{0y}k_y+E_{0z}k_z\right)\sin\left(k_x x+k_y y+k_z z-\omega t+\phi\right)=0 \tag{21}$$

which can be written in coordinate-independent notation as

$$\left(\mathbf{E}_0\cdot\mathbf{k}\right)\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi\right)=0\,. \tag{22}$$

For this to be true for all values of $\mathbf{r}$ and t, this then implies

$$\mathbf{E}_0\cdot\mathbf{k}=0\,. \tag{23}$$

So what does this tell us? Because $\mathbf{k}$ points in the direction of the wave's propagation, this tells us that the electric field must be perpendicular (or transverse) to the direction of propagation. That is, there is no longitudinal component to the electric field for a plane-wave solution to M's equations.

So what about the magnetic field? Somewhere we should have learned that electromagnetic radiation consists of both propagating electric and magnetic fields. In order to see what else M's tell us let's assume that the magnetic field associated with the electric field given by Eq. (18) is of the form

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0\cos\left(\mathbf{k}'\cdot\mathbf{r}-\omega' t+\phi'\right). \tag{24}$$

Let's now see what M's equations tell us about $\mathbf{B}_0$, $\mathbf{k}'$, ω', and ϕ'. As with the electric field, we can use Eq. (11) to tell us that $\mathbf{B}_0$ is also perpendicular to the direction of propagation. We can learn more by substituting Eqs. (18) and (24) into Eq. (13). After a bit of algebra and differentiation we end up with the result

$$\left(\mathbf{B}_0\times\mathbf{k}'\right)\sin\left(\mathbf{k}'\cdot\mathbf{r}-\omega' t+\phi'\right)=\frac{\omega}{c^2}\mathbf{E}_0\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi\right), \tag{25}$$

where we have used $\mu_0\varepsilon_0=1/c^2$. For this to hold for all values of $\mathbf{r}$ and t we must have the following relationships: $\mathbf{k}'=\mathbf{k}$, $\omega'=\omega$, $\phi'=\phi$, and $\left(\mathbf{B}_0\times\mathbf{k}\right)=\left(\omega/c^2\right)\mathbf{E}_0$. The first three relationships tell us that the electric and magnetic fields have the same wavelength, frequency, and phase, and propagate in the same direction. The last relationship is a bit more interesting. The last relationship tells us that $\mathbf{E}_0$ is perpendicular to both $\mathbf{k}$ (which we knew already) and $\mathbf{B}_0$. Thus, because both $\mathbf{E}_0$ and $\mathbf{B}_0$ are perpendicular to $\mathbf{k}$ all three vector are perpendicular to each other. Furthermore, because $(\mathbf{B}_0 \times \mathbf{k}) \| \mathbf{E}_0$ we must also have $(\mathbf{E}_0 \times \mathbf{B}_0) \| \mathbf{k}$ and $(\mathbf{k} \times \mathbf{E}_0) \| \mathbf{B}_0$. And lastly, because $\mathbf{B}_0$ and $\mathbf{k}$ are perpendicular, the relationship $(\mathbf{B}_0 \times \mathbf{k}) = (\omega/c^2)\mathbf{E}_0$ tells us that $B_0 k = (\omega/c^2)E_0$ or, because $\omega = ck$, $B_0 = E_0/c$. Thus, the magnetic field can be expressed as

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}(\hat{\mathbf{k}} \times \mathbf{E}_0)\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi), \tag{26}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. Or we can simply write, for our plane wave solution to M's equations,

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}(\vec{r},t).$$

Exercises

***31.1**. As was done in the notes for the magnetic field, derive the wave equation for the electric field.

***31.2**. The product $\mu_0\varepsilon_0$ appears in the wave equations where $1/c^2$ traditionally appears. Look up μ_0 and ε_0 and calculate $c = 1/\sqrt{\mu_0\varepsilon_0}$. What do you get?

***31.3**. Derive the dispersion relation $\omega = ck$ by substituting Eq. (18) into Eq. (17).

****31.4**. **Conditions on $\vec{E}$ and $\vec{B}$ for a plane wave**
(**a**) Substitute Eqs. (18) and (24) into Eq. (13) and derive Eq. (25).
(**b**) Substitute Eqs. (18) and (24) instead into Eq. (12) and again derive Eq. (25). This shows that in this situation the information in Eq. (12) is redundant.

****31.5**. **A traveling-wave solution to Maxwell's equations**. Show that the traveling wave fields

$$\mathbf{E}(\mathbf{r},t) = E_0\hat{\mathbf{x}}\cos(kz - \omega t) \text{ and } \mathbf{B}(\mathbf{r},t) = \frac{E_0}{c}\hat{\mathbf{y}}\cos(kz - \omega t)$$

satisfy all four homogeneous Maxwell equations.