Divergence and Curl

Overview and Motivation: In the upcoming two lectures we will be discussing Maxwell's equations. These equations involve both the divergence and curl of two vector fields, the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{B}(\mathbf{r},t)$. Here we discuss some details of the divergence and curl.

Key Mathematics: The aim here is to gain some insight into the physical meanings of the divergence and curl of a vector field. We will also state some useful identities concerning these two quantities.

I. Review of "del"

We have already discussed the differential object ∇ (sometimes called "del"). In Cartesian coordinates ∇ can be written as

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}. \tag{1}$$

We have seen that this object can operate on a scalar function (or scalar field) $f(\mathbf{r})$ to produce the **gradient** of $f(\mathbf{r})$, denoted $\nabla f(\mathbf{r})$. If f is expressed as a function of Cartesian coordinates, then the gradient can be written as

$$\nabla f(x,y,x) = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{2}$$

Recall that $\nabla f(\mathbf{r})$ points in the direction of the greatest change in $f(\mathbf{r})$ and is perpendicular to surfaces of constant $f(\mathbf{r})$. Recall also that $\nabla f(\mathbf{r})$ is a vector function (or vector field), which assigns a vector to each point $\mathbf{r}$ in real space.

We have also previously used ∇ on a vector function $\mathbf{V}(\mathbf{r})$ to calculate the **divergence** of $\mathbf{V}(\mathbf{r})$, denoted $\nabla\cdot\mathbf{V}(\mathbf{r})$. If $\mathbf{V}$ is expressed as a function of Cartesian coordinates, then the divergence can be written as

$$\nabla\cdot\mathbf{V}(x,y,z) = \frac{\partial V_x(x,y,z)}{\partial x} + \frac{\partial V_y(x,y,z)}{\partial y} + \frac{\partial V_z(x,y,z)}{\partial z} \tag{3}$$

Recall that $\nabla\cdot\mathbf{V}(\mathbf{r})$ is a scalar function.

A special case is the divergence of a vector field that is itself the gradient of a scalar function, $\nabla\cdot[\nabla f(\mathbf{r})]$. In this case the vector field is ∇f and so Eq. (3) becomes

$$\nabla \cdot [\nabla f(x,y,z)] = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{4}$$

which in coordinate-system-independent notation we also write as $\nabla^2 f$.

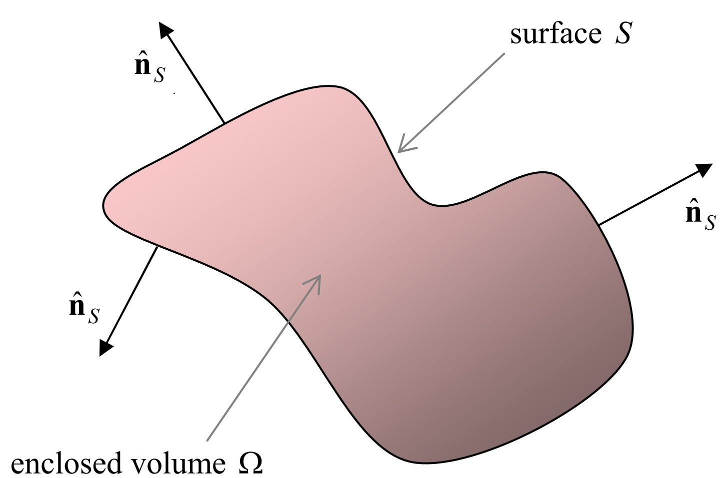
It may not be obvious, but one reason that the gradient and divergence are useful is that they are coordinate-system independent quantities. That is, as functions of position vector $\mathbf{r}$ they produce a quantity that exists independent of the coordinate system used to calculate them. However, this does *not* mean, for example, that the divergence of a vector field will be have the same functional form in each coordinate system. For example, consider the function that is equal to the distance from some reference point, the origin say. This function will have the form $\sqrt{x^2+y^2+z^2}$ when expressed in Cartesian coordinates, $\sqrt{\rho^2+z^2}$ when expressed in cylindrical coordinates, and simply r when expressed in spherical coordinates.

II. Interpretation of the Divergence.

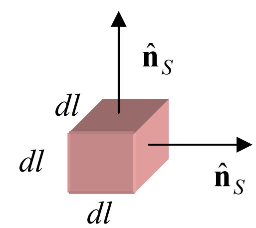
Although we previously defined the divergence of a vector function, we did not spend much (if any) time on its meaning. Let's see what we can say about $\nabla \cdot \mathbf{V}(\mathbf{r})$. To gain some insight into the divergence, let's consider the **divergence theorem**

$$\int_\Omega \nabla \cdot \mathbf{V}(\mathbf{r}) dv = \oint_S \mathbf{V}(\mathbf{r}_S) \cdot \hat{\mathbf{n}}_S(\mathbf{r}_S) dS, \tag{5}$$

which applies to a vector field $\mathbf{V}(\mathbf{r})$ defined both inside and on a closed surface S, as illustrated



In words, this equation says that the volume integral (over some volume Ω) of the divergence of a vector field $\mathbf{V}(\mathbf{r})$ equals the integral over the surface S enclosing that volume of the normal component of that same vector field $\mathbf{V}(\mathbf{r})$. Note that $\hat{\mathbf{n}}_S(\mathbf{r}_S)$ is a unit-vector field that is only defined on the surface S, and at every point on S it points *outward* and is normal to the surface. Now this theorem is true for any volume, but let's think about Eq. (5) when the volume is infinitesimally small. In fact, let's think about a cubic infinitesimal volume $dv = dl^3$, as shown.



If the volume is small enough (and the vector field is not too pathological) then we can consider $\nabla \cdot \mathbf{V}(\mathbf{r})$ to be approximately constant within the volume dl^3. On each face of the cube (with area dl^2) we can also consider $\mathbf{V}(\mathbf{r}_S)\cdot\hat{\mathbf{n}}_S = V_\perp(\mathbf{r}_S)$ to be constant. Then Eq. (5) can be rewritten approximately as

$$\nabla \cdot \mathbf{V}(\mathbf{r}) dl^3 = \sum_{\substack{6\ faces \\ (n)}} V_{\perp,n}(\mathbf{r}_S) dl^2 \tag{6}$$

or

$$\nabla \cdot \mathbf{V}(\mathbf{r}) = \frac{1}{dl^3} \sum_{\substack{6\ faces \\ (n)}} V_{\perp,n}(\mathbf{r}_S) dl^2 \,. \tag{7}$$

The quantity $V_{\perp,n}(\mathbf{r}_S) dl^2$ is usually interpreted as the **flux** of $\mathbf{V}$ (or total amount of $\mathbf{V}$) passing through the surface n with area dl^2. So what does Eq. (7) tell us? It tells us that the divergence of $\mathbf{V}(\mathbf{r})$ is equal to the *net* flux of $\mathbf{V}$ leaving the volume / unit volume. Thus, if the divergence of $\mathbf{V}$ is zero, then there is as much $\mathbf{V}$ pointing into the infinitesimal volume as is pointing out of the volume, or the net flux of $\mathbf{V}$ through the surface is zero. If the divergence of $\mathbf{V}$ is positive, then we say that there is a **source** of $\mathbf{V}$ within the volume. Conversely, if the divergence of $\mathbf{V}$ is negative, we say that there is a **sink** of $\mathbf{V}$ within the volume.

As we shall discuss in more detail shortly, one of Maxwell's equations is

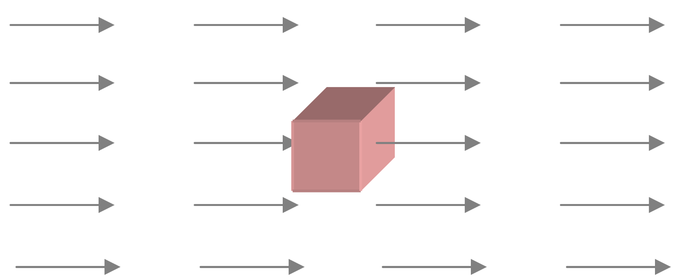
$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{8}$$

where $\mathbf{E}(\mathbf{r})$ is the electric field and $\rho(\mathbf{r})$ is the charge density. With our discussion above we see that the charge density is either a source (if positive) or sink (if negative) of electric field. If we are in a region of space without any charge density, then $\nabla \cdot \mathbf{E}(\mathbf{r}) = 0$, which tells us that for any volume there is as much $\mathbf{E}$ pointing inwards as outwards.

Example 1 To get a feeling for the divergence we apply Eq. (7) to a vector field to find its divergence. Let's consider a simple vector field,

$$\mathbf{V}(x, y, z) = V_0\,\hat{\mathbf{x}}, \tag{9}$$

a constant vector field that points in the x direction. At each point in space this vector field is exactly the same, as the following drawing shows (positive x is to the right).

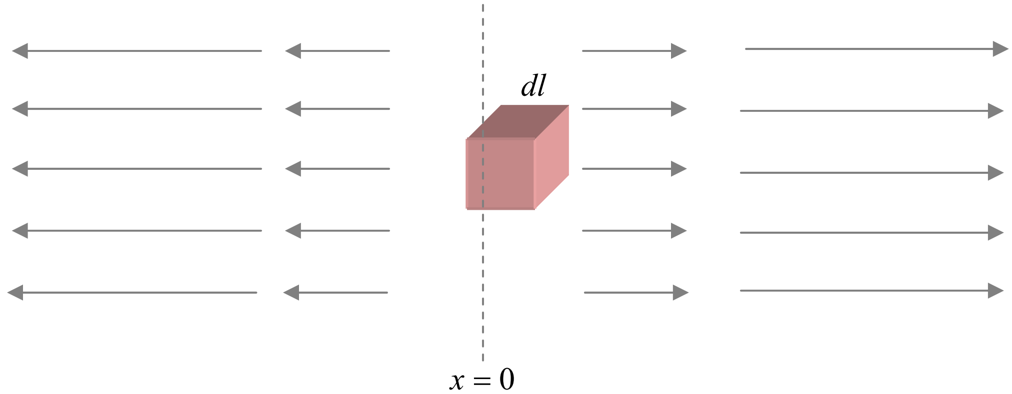


We now consider the flux coming out of each side of our infinitesimal volume dl^3. For the four sides that are parallel to the field there is no flux. One of the remaining sides has a flux equal to $V_0\, dl^2$ and other has $-V_0\, dl^2$, making the net flux emerging from the surface equal to zero. Via Eq. (7) we thus see that the divergence of this field is zero. We could, of course, have used Eq. (3) to directly calculate $\nabla \cdot \mathbf{V}(x, y, z)$, obtaining $\nabla \cdot \mathbf{V}(x, y, z) = 0$, in agreement with our analysis using Eq. (7).

Example 2 Let's look at a slightly more complicated field,

$$\mathbf{V}(x, y, z) = V_0 x\,\hat{\mathbf{x}}. \tag{10}$$

A graphical representation of this field would look something like the following picture.



Again, let's use Eq. (7) to calculate $\nabla\cdot\mathbf{V}(x,y,z)$. As we shall see below, the divergence is the same everywhere for this field, so for simplicity we consider a cube oriented as shown with one side at $x=0$. In this case only the right-hand-side (at $x=dl$) has any net flux, which is equal to $V_0\,dl^3$, and so via Eq. (7) we have that the divergence is simply V_0. As in Example 1, we could have directly used Eq. (3), also producing $\nabla\cdot\mathbf{V}(x,y,z)=V_0$, which is indeed independent of position.

III. The Curl and its Interpretation

Another useful first-derivative function of a vector field $\mathbf{V}(\mathbf{r})$ is known as the **curl** of $\mathbf{V}(\mathbf{r})$, usually denoted $\nabla\times\mathbf{V}(\mathbf{r})$. It has this notation due to its similarity to the cross product of two vectors. Indeed, in Cartesian coordinates it is expressed as

$$\nabla\times\mathbf{V}(x,y,z)=\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{\mathbf{x}}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{\mathbf{y}}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{\mathbf{z}} \tag{11}$$

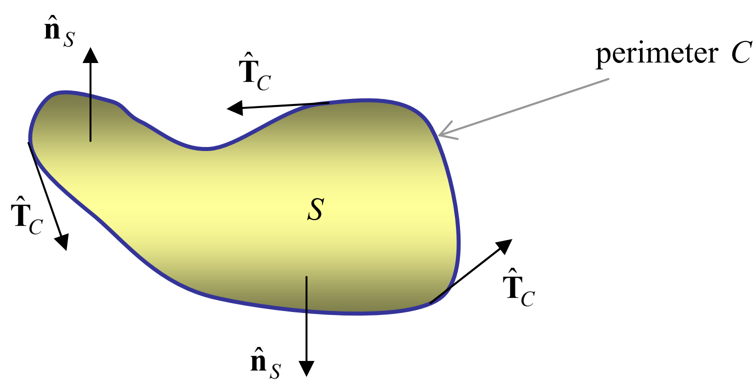
Notice that $\nabla\times\mathbf{V}(\mathbf{r})$ is itself a vector function of $\mathbf{r}$. As with the gradient and divergence, the curl is an object that is independent of the coordinate system that is used to calculate it. An easy way to remember $\nabla\times\mathbf{V}(\mathbf{r})$ in Cartesian coordinates is to see that Eq. (10) can be written as a "determinant"

$$\nabla\times\mathbf{V}(x,y,z)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}. \tag{12}$$

So what is the interpretation of $\nabla \times \mathbf{V}(\mathbf{r})$? Again, we appeal to a theorem of multivariable calculus, **Stoke's theorem** in this case, which can be written as

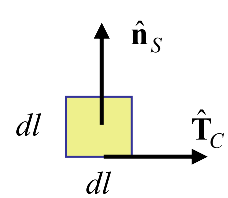
$$\int_S [\nabla \times \mathbf{V}(\mathbf{r}_S)] \cdot \hat{\mathbf{n}}_S(\mathbf{r}_S) dS = \oint_C \mathbf{V}(\mathbf{r}_C) \cdot \hat{\mathbf{T}}_C(\mathbf{r}_C) dC \qquad (13)$$

In words this says that surface integral of the normal component of the curl of a vector field $\mathbf{V}(\mathbf{r}_S)$ equals the line integral around the perimeter (of that surface) of the tangential component of that same vector field $\mathbf{V}(\mathbf{r})$.



As illustrated above, $\hat{\mathbf{n}}_S(\mathbf{r}_S)$ is again a field of unit vectors that is only defined on the surface S, but this time the surface is an *open* surface so there is no outward direction. Because of this lack of outwardness, one must arbitrarily pick one side of the surface for the field $\hat{\mathbf{n}}_S(\mathbf{r})$ to point from. Similarly, $\hat{\mathbf{T}}_C(\mathbf{r}_C)$ is a unit-vector field that is only defined on the perimeter curve C. It is **tangent** to the curve C at all points along C. Its direction is found by the "right-hand rule" whereby one's thumb points along the unit vector field $\hat{\mathbf{n}}_S(\mathbf{r}_S)$ and one's fingers curl (hum…) in the direction of the field $\hat{\mathbf{T}}_C(\mathbf{r}_C)$.

To use Stoke's theorem to gain some insight into the curl, lets consider the surface to be an infinitesimal, flat, square surface with sides of length dl, as shown. One normal vector $\hat{\mathbf{n}}_S(\mathbf{r}_S)$ and one tangential vector $\hat{\mathbf{T}}_C(\mathbf{r}_C)$ associated with this surface are also shown.



Because of the infinitesimal nature of the surface we can take $\nabla\times\mathbf{V}(\mathbf{r}_S)$ to be constant on this surface, and so we can rewrite Eq. (13) as

$$[\nabla\times\mathbf{V}(\mathbf{r}_S)]_\perp\, dl^2 = \sum_{\substack{4\,sides\\(n)}} V_{\parallel,n}(\mathbf{r}_C)dl \tag{14}$$

or

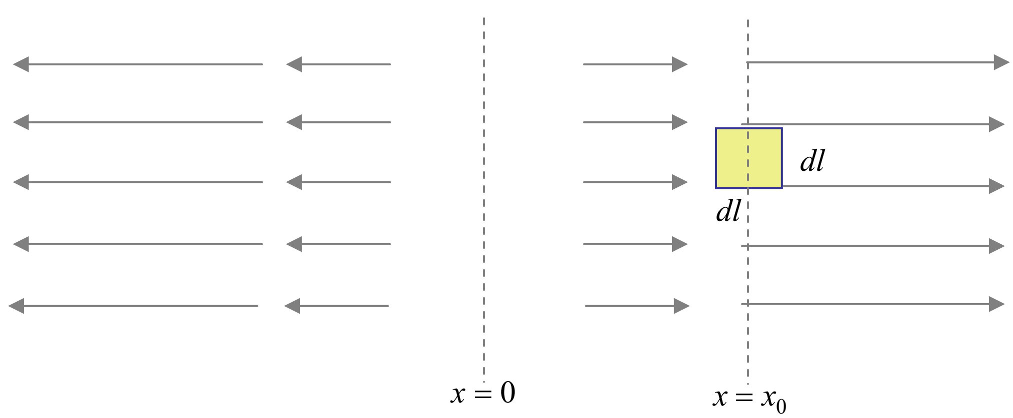
$$[\nabla\times\mathbf{V}(\mathbf{r})]_\perp = \frac{1}{dl^2}\sum_{\substack{4\,sides\\(n)}} V_{\parallel,n}(\mathbf{r})dl\,. \tag{15}$$

The quantity on the rhs of Eq. (15) is know as the **circulation** of $\mathbf{V}(\mathbf{r})$ around the curve C. So what does Eq. (15) tell us? This equation tells us that component of $\nabla\times\mathbf{V}(\mathbf{r})$ normal to an infinitesimal area is equal to the circulation of $\mathbf{V}(\mathbf{r})$ around the perimeter of that area.

Example 1 Well, maybe that last statement is just words to you. So let's do some examples to see if we can get a feel for the curl. Let's first consider the vector field that we discussed above in connection with the divergence,

$$\mathbf{V}(x,y,z)=V_0 x\,\hat{\mathbf{x}}\,. \tag{16}$$

We now consider an infinitesimal, square loop lying in the $x-y$ plane, with its center at $x=x_0$, and each side parallel to either x or y, as illustrated. (The z direction is perpendicular to the page.)



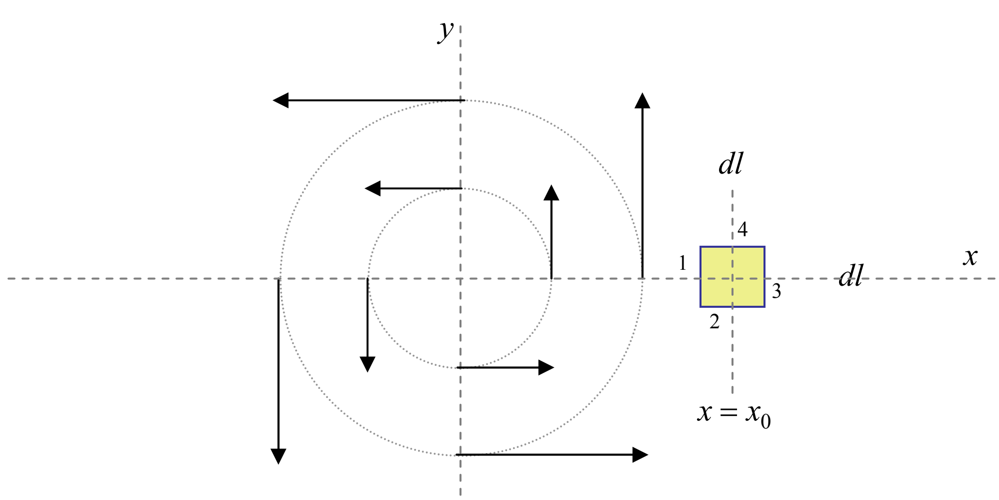
With the loop lying in the $x-y$ plane, we can use Eq. (15) to find the z component of the curl of $\mathbf{V}$ (because then the surface normal can be taken to be in the $+\hat{z}$ direction). Because $\mathbf{V}$ points in the x direction the two sides parallel to y do not contribute to the circulation. And because the top and bottom sides have tangential vectors $\hat{\mathbf{T}}_C(\mathbf{r}_C)$ pointing in opposite directions, the contributions to the circulation from these two sides cancel each other, resulting in zero net circulation for this loop. Thus $[\nabla\times\mathbf{V}(x,y,z)]_z$ is zero for this vector field.

In a similar fashion you should be able to determine the other components of $\nabla\times\mathbf{V}(\mathbf{r})$. How would you orient your loop to enable you to use Eq. (15) to calculate these other components? Do you get zero for these components as well? You can confirm that, indeed, $\nabla\times\mathbf{V}(x,y,z)=0$ by using Eq. (11) or Eq. (12). That is, all components of the curl of $\mathbf{V}(\mathbf{r})$ are, indeed, zero.

Example 2 How about the vector field

$$\mathbf{V}(x,y,z)=V_0(-y\,\hat{\mathbf{x}}+x\,\hat{\mathbf{y}})? \tag{17}$$

Let's determine whether this vector field has a nonzero curl.



The above sketch illustrates some vectors associated with this field (again drawn in the x-y plane). Does the field seem to circulate around? Because it does, we might expect the curl to be nonzero. Here we consider a small, square area sitting on the x axis at $x = x_0$ and use Eq. (15) to see what that curl is like. Here we must consider all fours sides because each side has some contribution to the curl. Assuming that the surface normal vector is again the $\hat{z}$ direction, the fours sides contribute the following amounts to the circulation:

side 1: $-V_0(x_0 - dl/2)dl$,
side 2: $V_0(dl/2)dl$,
side 3: $V_0(x_0 + dl/2)dl$,
side 4: $V_0(dl/2)dl$.

The total circulation is thus $2V_0\, dl^2$, which tells us that $[\nabla \times \mathbf{V}(x,y,z)]_z = 2V_0$ for this field. Indeed, it is nonzero (and the same everywhere). You must think about orienting the square in three orthogonal direction in order to obtain all three components of $\nabla \times \mathbf{V}(\mathbf{r})$. You should be able to convince yourself that the other two components of $\nabla \times \mathbf{V}(\mathbf{r})$ are zero. If we so desire, we can resort to Eq. (11) or (12) to calculate the curl more directly, which produces $\nabla \times \mathbf{V}(x,y,z) = 2V_0\,\hat{\mathbf{z}}$, confirming that there is a net curl in the z direction that is the same everywhere.

IV. Some Useful Identities

We will not prove them here, but there are three identities that will be useful when we discuss Maxwell's equations. They are

$$\nabla \times (\nabla f(\mathbf{r})) = 0, \tag{18}$$

$$\nabla \cdot (\nabla \times \mathbf{V}(\mathbf{r})) = 0, \tag{19}$$

$$\nabla \times (\nabla \times \mathbf{V}(\mathbf{r})) = \nabla(\nabla \cdot \mathbf{V}(\mathbf{r})) - \nabla^2 \mathbf{V}(\mathbf{r}). \tag{20}$$

We have not previously considered the Laplacian ∇^2 operating on a vector field, but looking at Eq. (4) we see that in Cartesian coordinates

$$\nabla^2 \mathbf{V}(x,y,z) = \frac{\partial^2 \mathbf{V}}{\partial x^2} + \frac{\partial^2 \mathbf{V}}{\partial y^2} + \frac{\partial^2 \mathbf{V}}{\partial z^2}, \tag{21}$$

and so $\nabla^2 \mathbf{V}(\mathbf{r})$ is also a vector field.

Exercises

***30.1** Using Cartesian coordinates, show that
(**a**) $\nabla \times (\nabla f(x, y, z)) = 0$ [for any sufficiently differentiable function $f(x, y, z)$], and
(**b**) $\nabla \cdot (\nabla \times \mathbf{V}(x, y, z)) = 0$ [for any sufficiently differentiable vector field $\mathbf{V}(x, y, z)$].

****30.2** Using Cartesian coordinates, show that $\nabla \times (\nabla \times \mathbf{V}(\mathbf{r})) = \nabla(\nabla \cdot \mathbf{V}(\mathbf{r})) - \nabla^2 \mathbf{V}(\mathbf{r})$ [for any sufficiently differentiable vector field $\mathbf{V}(x, y, z)$].

***30.3** Is Stoke's theorem applicable to a Mobius strip? Why or why not?