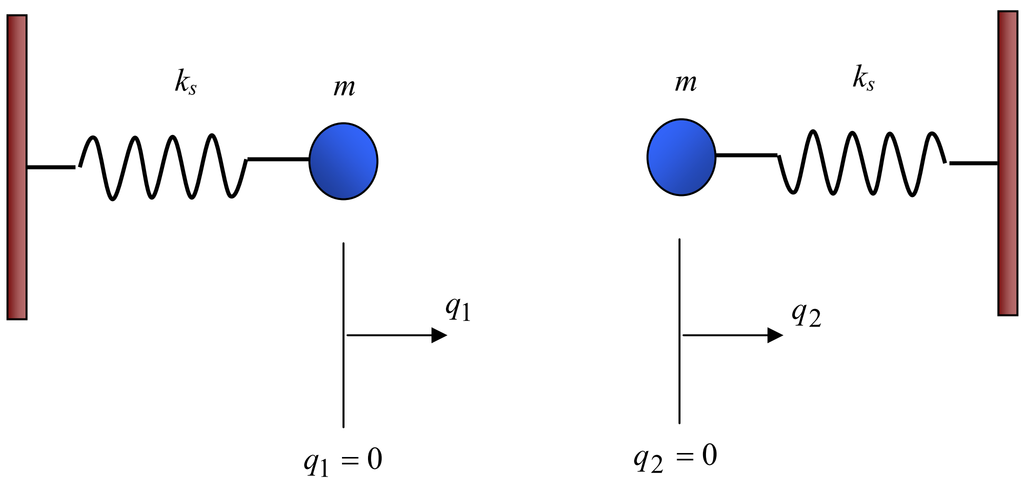
Two Coupled Oscillators / Normal Modes

Overview and Motivation: Today we take a small, but significant, step towards wave motion. We will not yet observe waves, but this step is important in its own right. The step is the coupling together of two oscillators via a spring that is attached to both oscillating objects.

Key Mathematics: We gain some experience with coupled, linear ordinary differential equations. In particular we find special solutions to these equations, known as normal modes, by solving an eigenvalue problem.

I. Two Coupled Oscillators

Let's consider the diagram shown below, which is nothing more than 2 copies of an harmonic oscillator, the system that we discussed last time. We assume that both oscillators have the same mass m and spring constant k_s. Notice, however, that because there are two oscillators each has it own displacement, either q_1 or q_2.



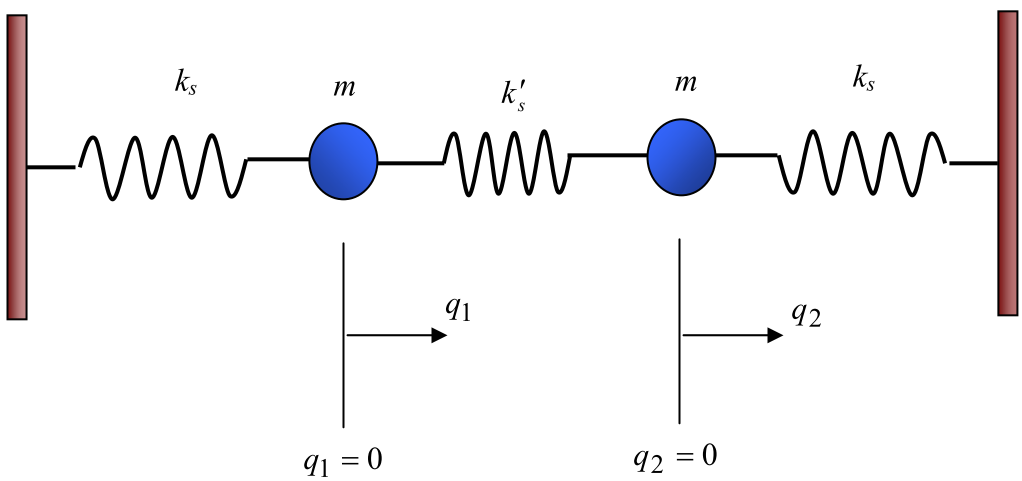
Based on the discussion last time you should be able to immediately write down the equations of motion (one for each oscillating object) as

$$\ddot{q}_1 + \tilde{\omega}^2 q_1 = 0\text{, and} \tag{1a}$$

$$\ddot{q}_2 + \tilde{\omega}^2 q_2 = 0\,, \tag{1b}$$

where $\tilde{\omega}^2 = k_s/m$. As we saw last time, the solution to each of theses equations is harmonic motion at the (angular) frequency $\tilde{\omega}$. As should be obvious from the picture, the motion of each oscillator is independent of the other oscillator. This is also reflected in the equation of motion for each oscillator, which has nothing to do with the other oscillator.

Let's now make things a bit more interesting by adding in another spring that connects the two oscillating objects together, as illustrated in the following picture. To make things even more interesting we assume that this new spring has a different constant k_s'. However, to keep things simple we assume that the middle spring provides no force if $q_1 - q_2 = 0$. That is, this spring is neither stretched or compressed if its length is equal to the its length when two objects are at equilibrium.



Thinking about this picture we should realize that the two equations of motion will no longer be independent. That is, the equation of motion for the first object will depend (somehow) upon what the second object is doing, and vice versa.

Let's use Newton's second law to write down the equation motion for each object. Recall that Newton's second law for either object ($i = 1,2$) can be written as

$$\ddot{q}_i = \frac{F_i}{m}, \tag{2}$$

where F_i is the net force on object i. The tricky part, if there is a tricky part, is to determine the sum F_i on each object. The net force on the first object comes from the spring on the left and the spring in the middle. With a little thought you should realize that this net force F_1 is

$$F_1 = -k_s q_1 - k_s'(q_1 - q_2). \tag{3a}$$

Make sure that you understand the signs of all the term on the rhs of this equation. Notice that the force provided by the middle spring depends not only on the first object's displacement but also on the second object's displacement. Similarly, the net force on the second object is

$$F_2 = -k_s q_2 - k_s'(q_2 - q_1). \tag{3b}$$

Substituting these two forces into Eq. (2), once for each object, we obtain the two equations of motion,

$$\ddot{q}_1 + \tilde{\omega}^2 q_1 + \tilde{\omega}'^2(q_1 - q_2) = 0 \tag{4a}$$

for the first object and

$$\ddot{q}_2 + \tilde{\omega}^2 q_2 + \tilde{\omega}'^2(q_2 - q_1) = 0 \tag{4b}$$

for the second. Here $\tilde{\omega}'^2 = k'/m$. Given the symmetry of the problem, it might not surprise you that you can obtain one equation of motion from the other with the transformation $1 \leftrightarrow 2$ in the subscripts that label the objects.

So now we have a considerably more complicated problem: as expected from looking at the drawing above, the equation of motion for each object depends upon what the other object is doing. Specifically, each equation of motion depends upon the displacement of the other object.

II. Normal Modes

A. Harmonic Ansatz

So what are the solutions to these differential equations? Well, we will eventually write down the general solution (next lecture). But right now we are going to look at a special class of solutions known as normal-mode solutions, or simply, normal modes. A **normal mode** is a solution in which both masses harmonically oscillate at the same frequency. We state why these special solutions are extremely useful at the end of the lecture. For now let's see if we can find them. We use the complex form of harmonic motion and write

$$q_1(t) = q_{01}e^{i\Omega t} \text{ and} \tag{5a}$$

$$q_2(t) = q_{02} e^{i\Omega t} . \tag{5b}$$

Notice that the (unknown) **frequency of oscillation** Ω of both oscillators is the same, a key feature of a normal mode. Also, because we are using the complex form of harmonic motion, the amplitudes q_{01} and q_{02} may be complex, but they too are unknown at this point. Keep in mind that Eq. (5) is only the form of a normal-mode solution: it is only a solution if it satisfies the equations of motion. In other words, we now need to find values of the frequency Ω and amplitudes q_{01} and q_{02} that satisfy Eqs. (4a) and (4b), the equations of motion.

So let's substitute Eq. (5) into Eq. (4) and see what that tells us about Ω, q_{01}, and q_{02}. Carrying out the substitution and calculating the derivatives yields

$$-\Omega^2 q_{01} e^{i\Omega t} + \tilde{\omega}^2 q_{01} e^{i\Omega t} + \tilde{\omega}'^2 \left(q_{01} e^{i\Omega t} - q_{02} e^{i\Omega t} \right) = 0 \text{, and} \tag{6a}$$

$$-\Omega^2 q_{02} e^{i\Omega t} + \tilde{\omega}^2 q_{02} e^{i\Omega t} + \tilde{\omega}'^2 \left(q_{02} e^{i\Omega t} - q_{01} e^{i\Omega t} \right) = 0 . \tag{6b}$$

Dividing by $e^{i\Omega t}$ (Is this legal?) and rearranging some terms gives

$$\left(\tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 \right) q_{01} - \tilde{\omega}'^2 q_{02} = 0 \text{, and} \tag{7a}$$

$$-\tilde{\omega}'^2 q_{01} + \left(\tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 \right) q_{02} = 0 . \tag{7b}$$

So what do we have here? We have two algebraic equations and three unknowns Ω, q_{01}, and q_{02}. The problem seems a bit underspecified, and it is: as we shall see below, we will only be able to solve for Ω and the ratio q_{01}/q_{02}.

B. Eigenvalue Problem

If you have previously studied differential equations and linear algebra you may be inclined to write Eq. (7) in matrix notation as

$$\begin{pmatrix} \tilde{\omega}^2 + \tilde{\omega}'^2 & -\tilde{\omega}'^2 \\ -\tilde{\omega}'^2 & \tilde{\omega}^2 + \tilde{\omega}'^2 \end{pmatrix} \begin{pmatrix} q_{01} \\ q_{02} \end{pmatrix} = \Omega^2 \begin{pmatrix} q_{01} \\ q_{02} \end{pmatrix}, \tag{8}$$

which you would recognize as an **eigenvalue problem**. Generally, an eigenvalue problem is one where some **linear operator** (in this case a matrix) operating on some object (in this case a 2D column **vector**) produces a constant (in this case Ω^2) times the original object. Generally, an N-dimensional linear-algebra eigenvalue problem has N solutions [which consist of special values of the constant (or **eigenvalue**) Ω^2 and amplitudes $q_{01} \ldots q_{0n}$ (or **eigenvector** $\begin{pmatrix} q_{01} \\ \vdots \\ q_{0N} \end{pmatrix}$)].

C. Eigenvalues

Well, that was a lot of terminology, but what about the solution? Well, let's rewrite Eq. (8) as

$$\begin{pmatrix} \tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 & -\tilde{\omega}'^2 \\ -\tilde{\omega}'^2 & \tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 \end{pmatrix} \begin{pmatrix} q_{01} \\ q_{02} \end{pmatrix} = 0 . \tag{9}$$

Now this is interesting. Expressed in this way, we have the product of two quantities equal to zero. There are two ways that Eq. (9) can be true. The first, which is the trivial (i.e., uninteresting) solution, is $q_{01} = q_{02} = 0$. Physically, this corresponds to no motion of the system – pretty uninteresting! The nontrivial way that Eq. (9) can be satisfied is if the **determinant** of the 2 × 2 matrix is zero. That is

$$\begin{vmatrix} \tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 & -\tilde{\omega}'^2 \\ -\tilde{\omega}'^2 & \tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 \end{vmatrix} = 0 . \tag{10}$$

For a 2×2 matrix the determinant is easily calculated, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC$, so in this case Eq. (10) can be expressed as

$$\left(\tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2\right)^2 - \tilde{\omega}'^4 = 0 . \tag{11}$$

Eq. (11) [or Eq. (10)] is known as the **characteristic equation** for the eigenvalue problem. This is great! We now have an equation for the eigenvalue Ω^2 and thus the normal-mode frequency Ω,

$$\tilde{\omega}^2 + \tilde{\omega}'^2 - \Omega^2 = \pm \tilde{\omega}'^2 . \tag{12}$$

Solving Eq. (12) for Ω^2 produces the two eigenvalues

$$\Omega^2 = \tilde{\omega}^2 , \tilde{\omega}^2 + 2\tilde{\omega}'^2 , \tag{13}$$

which gives us four solutions for the normal-mode frequency

$$\Omega = \pm\tilde{\omega}, \pm\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2} \ . \tag{14}$$

D. Eigenvectors and Normal Modes

So now that we have the eigenvalues $\tilde{\omega}^2$ and $\tilde{\omega}^2 + 2\tilde{\omega}'^2$, we need to find the eigenvector associated with each eigenvalue. To do this we substitute each eigenvalue into either Eq. (7a) or (7b) (It doesn't matter which, you get the same equation in either case).

1. first normal mode

For the first eigenvalue, $\Omega^2 = \tilde{\omega}^2$, this substitution produces

$$\tilde{\omega}'^2 q_{01} - \tilde{\omega}'^2 q_{02} = 0 , \tag{15}$$

which gives us the result for the amplitudes

$$q_{01} = q_{02} , \tag{16}$$

and so the eigenvector associated with the first normal mode is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This result tells us that both oscillators oscillate identically [check out Eq. (5) with the result of Eq. (16)] if this normal mode is excited. That is, the objects oscillate with exactly the same amplitude and the same phase.

Now because the eigenvalue $\Omega^2 = \tilde{\omega}^2$ corresponds to two normal-mode frequencies $\Omega = \pm\tilde{\omega}$, this first eigensolution of the linear algebra problem gives us to two linearly-independent solutions to the equations of motion.

$$q_1(t) = A_1 e^{i\tilde{\omega}t} \text{ and } q_2(t) = A_1 e^{i\tilde{\omega}t} \qquad \text{(17a),(17b)}$$

is the first solution, and

$$q_1(t) = B_1 e^{-i\tilde{\omega}t} \text{ and } q_2(t) = B_1 e^{-i\tilde{\omega}t} \qquad \text{(18a),(18b)}$$

is the second, where the amplitudes A_1 and B_1 are arbitrary. Equations (17) and (18) can be written in linear-algebra inspired notation as

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{1+} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\tilde{\omega}t} \tag{19}$$

and

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{1-} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\tilde{\omega}t}, \tag{20}$$

respectively. The $1+$ and $1-$ denote the $+\tilde{\omega}$ and $-\tilde{\omega}$ solutions. However, because oscillations at frequency $-\tilde{\omega}$ are really just oscillations at $\tilde{\omega}$, any linear combination of these two solutions really just oscillates at the frequency $\tilde{\omega}$, and so any linear combination of Eq. (19) and (20) can be thought of as the first normal-mode solution at frequency $\tilde{\omega}$. That is, the most general way we can write the first normal-mode solution is

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{1} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{1+} + \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{1-} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(A_1 e^{i\tilde{\omega}t} + B_1 e^{-i\tilde{\omega}t} \right), \tag{21}$$

where A_1 and B_1 are unspecified constants. Note that arbitrariness of A_1 and B_1 is consistent with our knowledge that for an harmonic oscillator the frequency is independent of the amplitude. Note also that (if we wish to at this point) we can specify the solution to be real, in which case we would set $B_1 = A_1^*$.

2. second normal mode

Let's now look at the second normal-mode solution, which corresponds to the second eigenvalue $\tilde{\omega}^2 + 2\tilde{\omega}'^2$. As before, we substitute this eigenvalue into Eq. (7a), which gives us

$$-\tilde{\omega}'^2 q_{01} - \tilde{\omega}'^2 q_{02} = 0 \tag{22}$$

or

$$q_{01} = -q_{02}, \tag{23}$$

and so the eigenvector associated with the second normal mode is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. So if this normal mode is excited the two oscillators oscillate with the same amplitude, but with opposite phase (or a phase difference of $\pi/2$). That is, when the first oscillator is moving to the left the second is moving to the right with the same magnitude in its displacement, and vice versa.

As in the case of the first eigenvalue, there are two linearly-independent solutions corresponding to the two frequencies $\pm\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}$. The first solution is

$$q_2(t) = A_2 e^{i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} \text{ and } q_2(t) = -A_2 e^{i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} \qquad \text{(24a),(24b)}$$

and the second is

$$q_1(t) = B_2 e^{-i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} \text{ and } q_2(t) = -B_2 e^{-i\sqrt{\omega^2 + 2\omega'^2}\,t}, \qquad \text{(25a),(25b)}$$

or in linear-algebra notation

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{2-} = A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} \qquad (26)$$

and

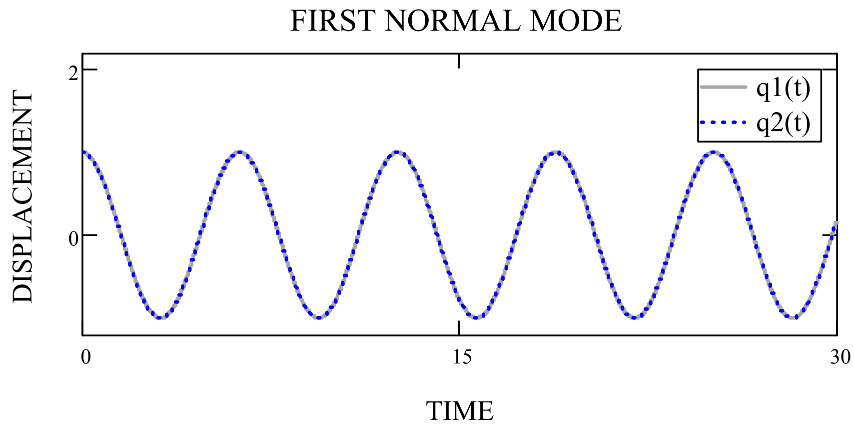
$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{2-} = B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t}. \qquad (27)$$

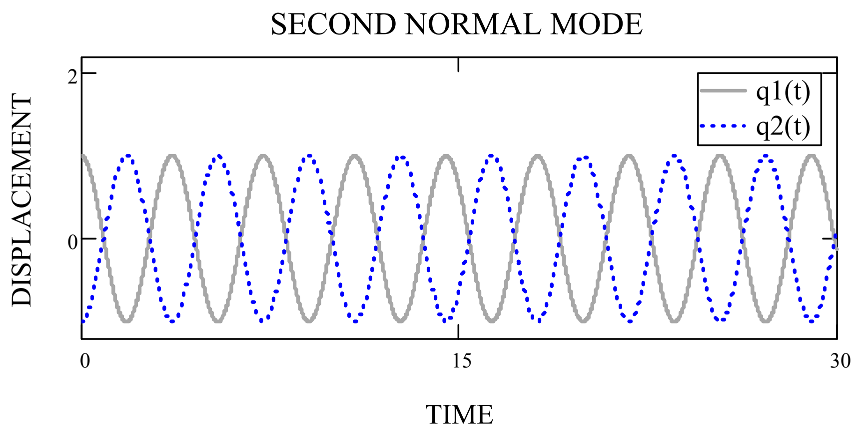
As before, the general form of this normal-mode solution is

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{2} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{2+} + \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_{2-} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(A_2 e^{i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} + B_2 e^{-i\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2}\,t} \right) \qquad (28)$$

The graphs on the following page plot the time-dependent amplitudes $q_1(t)$ and $q_2(t)$ for the two normal modes for the following values of the arbitrary constants: $A_1 = B_1 = 1/2$ for the first normal mode and $A_2 = B_2 = 1/2$ for the second normal mode. (With these choices the solutions are real.) For these graphs we have also set m, k_s, and k_s' equal to 1, so that $\tilde{\omega} = 1$ and $\sqrt{\tilde{\omega}^2 + 2\tilde{\omega}'^2} = \sqrt{3}$. (Admittedly, we have not specified units here, but if standard SI units are used for the mass, spring constants, and time, then the unit for displacement is meters.) As the graphs show, in the first normal mode the two objects oscillate identically, while in the second normal mode they oscillate exactly oppositely.

As we will see in the next lecture, a great usefulness of the normal mode solutions is that ANY solution of Eqs. (4a) and (4b), the equations of motion for this coupled oscillator system, can be written as a linear combination of these two normal-mode solutions. Indeed, this property of normal-mode solutions is so important that it will be a theme throughout the course.





Exercises

***3.1** Starting with the Euler formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ (and its complex conjugate), write $\cos(\theta)$ and $\sin(\theta)$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

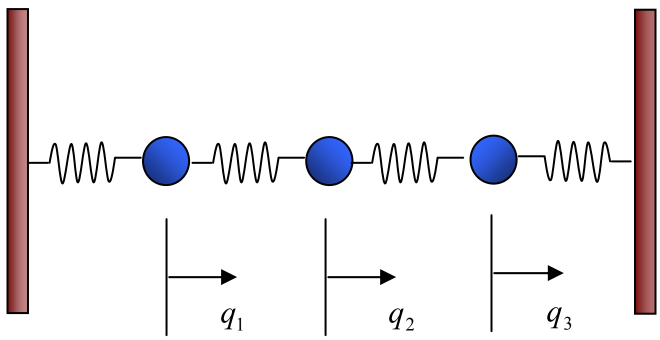
***3.2** Write the expression $Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}$ in the form $C\cos(\tilde{\omega}t) + D\sin(\tilde{\omega}t)$. That is, find C and D in terms of A and B. From this result show that if $B = A^*$ then C and D are both real (which means that $Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}$ is real).

***3.3** In the graph of the first normal-mode solution, $q_1(t)$ and $q_2(t)$ both look like cosine functions. Show for $A_1 = B_1 = 1/2$, that the solution

$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(A_1 e^{i\tilde{\omega}t} + B_1 e^{-i\tilde{\omega}t}\right)$ can indeed be written as $\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\tilde{\omega}t)$.

*__3.4__ If we take the limit $k_s' \to 0$ in the coupled oscillator problem, what does this correspond to physically? What happens to the normal mode frequencies? Does this make sense? (Note: if two normal modes have the same frequency, then they are said to be degenerate, and any linear combination of those two normal modes is also a normal mode.)

**__3.5 Three coupled oscillators__. In this problem you will find the normal modes of three coupled oscillators, as illustrated below. Assume that each object has mass m and each spring constant is k_s.



(The following steps lead you through the same procedure as is used in the notes to solve the two-oscillator problem in order to solve this problem. It will be most helpful if you carefully study that procedure before tackling this problem.)

(**a**) Using Newton's second law, write down the equation of motion for each object [in the form of Eq. (4) in the notes].
(**b**) Assume a normal-mode type solution and find the three algebraic equations [equivalent to Eq. (7) in the notes] that govern Ω^2 and the amplitudes q_{01}, q_{02}, and q_{03}.
(**c**) Write your equations in (**b**) in matrix form [equivalent to Eq. (9) in the notes].
(**d**) Find the characteristic equation [equivalent to Eq. (11) in the notes] that determines the three eigenvalues. (Hint: you will need to calculate the determinant of a 3×3 matrix.)
(**e**) Solve the characteristic equation and show that the three eigenvalues for this problem are $\Omega^2 = \left(2-\sqrt{2}\right)\tilde{\omega}^2, 2\tilde{\omega}^2, \left(2+\sqrt{2}\right)\tilde{\omega}^2$.
(**f**) For each eigenvalue, find the eigenvector associated with that eigenvalue.
(**g**) Write down the 3 normal mode solutions in the form of Eqs. (21) and (28) in the notes.
(**h**) As precisely as possible, describe the motion of the three objects for each of the normal modes.