Energy Density / Energy Flux / Total Energy in 3D

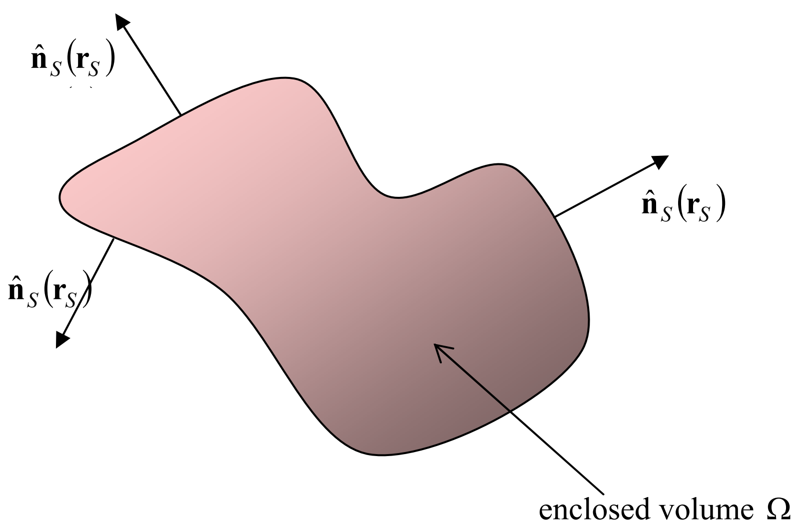
Overview and Motivation: In this lecture we extend the discussion of the energy associated with wave motion to waves described by the 3D wave equation. In fact, the first part of the discussion is exactly the same as the 1D case, just extended to 3D. In the examples we look at the energy associated with spherically symmetric waves.

Key Mathematics: Some 3D calculus, especially the divergence theorem and the spherical-coordinates version of the gradient.

I. Density, Flux, and the Continuity Equation

As in the 1D case let's assume that we are interested in some quantity $Q(t)$ that has an associated density $\rho(x,t)$. Since we are dealing with a density that lives in a 3D space, the units of density will be the units of Q divided by m^3. That is, $[\rho]=[Q]/\text{m}^3$.

Let's consider a volume Ω enclosed by a surface S, as illustrated in the following figure. At each point on the surface we define a perpendicular, outward-pointing unit vector $\hat{\mathbf{n}}_S(\mathbf{r}_S)$ associated with each point $\mathbf{r}_S$ on the surface.



The amount of Q contained in Ω can be written as

$$Q_\Omega(t)=\int_\Omega \rho(\mathbf{r},t)\,d^3r \tag{1}$$

As in the 1D case, if Q is a conserved quantity, then the change in Q inside Ω,

$$\frac{dQ_\Omega(t)}{dt} = \int_\Omega \frac{\partial \rho(\mathbf{r},t)}{\partial t}\, d^3r \,, \tag{2}$$

must be equal to the net flow of Q into Ω,

$$\frac{dQ_\Omega(t)}{dt} = -\oint_S \mathbf{j}(\mathbf{r}_S,t)\cdot \hat{n}_S(\mathbf{r}_S)\, dS \,. \tag{3}$$

The (vector) quantity $\mathbf{j}$ is again known as the Q **current density** or the Q **flux**. The dimensions of $\mathbf{j}$ are the dimensions of ρ times a velocity, so $[\mathbf{j}] = [\rho]\text{m/s}$. Thus also $[\mathbf{j}] = [Q]/(\text{m}^2\text{s})$. Note that the rhs of Eq. (3) can be interpreted as the **total** Q **current** flowing out through the surface S. Equating the rhs's of Eqs. (2) and (3) gives us

$$\int_\Omega \frac{\partial \rho(\mathbf{r},t)}{\partial t}\, d^3r = -\oint_S \mathbf{j}(\mathbf{r}_S,t)\cdot \hat{\mathbf{n}}_S(\mathbf{r})\, dS \,. \tag{4}$$

We can now use the **divergence theorem** (which is one of several 3D extensions of the fundamental theorem of calculus),

$$\int_\Omega \nabla\cdot \mathbf{A}(\mathbf{r}_S)\, d^3r = \oint_S \mathbf{A}(\mathbf{r})\cdot \hat{\mathbf{n}}_S(\mathbf{r}_S)\, dS \,, \tag{5}$$

to rewrite Eq. (4) as

$$\int_\Omega \left[\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla\cdot \mathbf{j}(\mathbf{r},t) \right] d^3r = 0 \tag{6}$$

Now, because the volume Ω is arbitrary, the integrand must vanish. Thus

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla\cdot \mathbf{j}(\mathbf{r},t) = 0 \,. \tag{7}$$

Equation (7) is the 3D version of the **continuity equation**, which again is a local statement of the conservation of Q.

II. Energy Density and Flux for 3D Waves

We now apply this discussion to the energy associated with 3D waves. In this case Q represents the energy associated with a wave (within some volume). For waves described by the 3D wave equation the energy density can be written as

$$\rho(\mathbf{r},t) = \frac{\mu}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 + c^2(\nabla q)^2\right], \tag{8}$$

where $q(\mathbf{r},t)$ is the variable that is governed by the wave equation. The first term on the rhs of Eq. (8) is the kinetic energy density ρ_T, and the second is the potential energy density ρ_V. Now Eq. (8) is fairly general as long as μ is suitably interpreted. If q is a true displacement, then μ will be a parameter with the units of mass density. If q represents something else, say an electric field, then it will have some other units. From Eq. (8) it is fairly easy to see that the units of μ are generally given by $[\mu] = (\text{Joule s}^2)/(\text{m}^3\,[q]^2)$. It is not hard to show that the energy flux, which can be written as

$$\mathbf{j}(\mathbf{r},t) = -\mu c^2 \frac{\partial q}{\partial t}\nabla q\,, \tag{9}$$

together with the energy density in Eq. (8) satisfy Eq. (7), the continuity equation.

III. Several Examples

Let's look at some examples that involve spherically symmetric waves.

A. Spherical Standing Wave

Let's look at a standing-wave example. You may recall that a spherical-coordinates separable solution that is finite everywhere is of the form

$$q_{k,l,m}(r,\theta,\phi,t) = C_{k,l,m}\; j_l(kr)\, P_l^m(\cos(\theta))\left(C_m e^{im\phi} + D_m e^{-im\phi}\right)\left(A_k e^{ikct} + B_k e^{-ikct}\right), \tag{10}$$

where j_l is a spherical Bessel function (of the first kind), and P_l^m is an associated Legendre function (of the first kind). The parameter m is an integer whose absolute value can be no larger than the nonnegative integer l. If we want a solution with spherical symmetry, then there can be no θ or ϕ dependence. This means that both l and m must be zero because the only associated Legendre function independent of θ is $P_0^0(\cos(\theta)) = 1$. Thus, the spherical Bessel function in Eq. (10) must be $j_0(kr) = \sin(kr)/(kr)$, and so Eq. (10) simplifies to

$$q_{k,0,0}(r,\theta,\phi,t) = C_{k,0,0}\,\frac{\sin(kr)}{kr}\left(A_k e^{ikct} + B_k e^{-ikct}\right). \tag{11}$$

If we simplify this further by letting A_k be a real number and let $B_k = A_k$ (making the solution explicitly real) then we have

$$q_{k,0,0}(r,\theta,\phi,t) = 2A_k C_{k,0,0}\,\frac{\sin(kr)}{kr}\cos(kct). \tag{12}$$

Because all parts of the system oscillate with the same phase, this is a spherically symmetric version of a standing wave.

Using Eqs. (8) and (9) we can calculate the kinetic and potential energy densities and the energy flux associated with the wave in Eq. (12). To do this in a fairly simple manner we can use the spherical-coordinates version of the gradient

$$\nabla f(r,\theta,\phi) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\varphi}}, \tag{13}$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ are unit vectors in the r, θ, and ϕ directions, respectively. The nice thing about spherically symmetric solutions is that only the first term on the rhs of Eq. (13) contributes to the gradient.

A video of q, ρ_T, ρ_V, and j for the wave in Eq. (12), *Energy in 3D Standing Wave.avi*, is available on the class web site. As the video shows, the displacement is indeed a standing wave. Unfortunately, the energy densities and flux fall off with the radial distance r so fast that it is hard to really see their behavior.

Given this, we have made another video, *Energy in 3D Standing Wave 2.avi*, which plots the surface integrated density and flux,[1]

$$D(r) = \oint_S \rho(\mathbf{r}_S)\,dS, \tag{14}$$

and

$$I(r) = \oint_S \mathbf{j}(\mathbf{r})\cdot\hat{\mathbf{n}}_S(\mathbf{r}_S)\,dS, \tag{15}$$

[1] The video separately shows the kinetic and potential contributions to $D(r)$.

where the surface S is of radius r centered at the origin. Now because a spherically symmetric solution is independent of the two angles θ and ϕ, this amounts to multiplying the density ρ and flux $\mathbf{j}$ by the factor $4\pi r^2$, which is the surface area of the sphere S. The quantity $D(r)$ (which has units of Joule/m) can be though of as a linear energy density (i.e., the energy per unit length along the radial direction), while the quantity $I(r)$ (which has units of Joule/s) is the total (energy) current flowing through S. Notice that this new video is very similar to the 1D standing wave video that we looked at in the last lecture.

B. Spherical Traveling Wave

Let's also look at a spherically symmetric traveling wave. If we are thinking about sound waves, this is the sort of wave that would result from a pulsating sphere centered at the origin. We can construct a traveling wave solution from a linear combination of 2 linearly independent standing waves. We thus need to use both kinds of spherical Bessel functions. The linear combination that produces a spherically symmetric, outgoing, traveling wave is[2]

$$q_{k,0,0}(r,\theta,\phi,t) = C_{k,0,0}\left[j_0(kr)\cos(kct) - y_0(kr)\sin(kct)\right]. \tag{16}$$

which can be written in terms of sine and cosine functions as

$$q_{k,0,0}(r,\theta,\phi,t) = C_{k,0,0}\left[\frac{\sin(kr)}{kr}\cos(kct) - \frac{\cos(kr)}{kr}\sin(kct)\right]. \tag{17}$$

The video, *Energy in 3D Traveling Wave.avi*, shows $D(r)$ and $I(r)$ for this wave. Indeed, away from the origin the wave appears to be an outgoing traveling wave. However, at the origin something rather different seems to be happening – something with some standing-wave character, perhaps?

Well, as it turns out, the current density $\mathbf{j}$ has terms with two types of behavior. The first type has a $1/r^2$ dependence. These terms describe the **radiative** part of the wave, which carries energy off to infinity. Because the radiative part of $\mathbf{j}$ varies as $1/r^2$ the radiative part of $I(r)$ does not vanish as $r \to \infty$. However, there are also **nonradiative** terms, which vary as $1/r^3$. These terms act more like a standing wave: the energy associated with these terms just oscillates back and forth and never really goes anywhere. Because of the $1/r^3$ behavior to the nonradiative part of $\mathbf{j}$, the current $I(r)$ associated with these terms vanishes as $1/r$ as $r \to \infty$. These terms are

[2] Note that this solution is only valid in the region of space outside the source. In the video you may think of the source as being infinitesimally small, so that the solution is valid infinitesimally close to the origin.

thus sometimes called the **local fields** associated with the source. In the video *Energy in 3D Traveling Wave 2.avi* we separately show the current $I(r)$ associated with each type of term. Notice that the radiative piece looks essentially like a 1D traveling wave while the nonradiative piece is really only important in the vicinity of the origin.

Exercises

***25.1** Show that the expressions for the density ρ and $\mathbf{j}$ in Eqs. (8) and (9), respectively, satisfy Eq. (7), the continuity equation.

****25.2 Spherical Traveling Wave**
(**a**) Write the wave in Eq. (17),

$$q_{k,0,0}(r,\theta,\phi,t)=C_{k,0,0}\left[\frac{\sin(kr)}{kr}\cos(kct)-\frac{\cos(kr)}{kr}\sin(kct)\right],$$

as an explicit function of $(r-ct)$, thus showing that it is a traveling wave moving outward from the origin.
(**b**) Using your result from part (a), show that the radiative and nonradiative components of the current density $\mathbf{j}$ can be written, respectively, as

$$\mathbf{j}_R(r,t)=\frac{\mu c^3 q_0^2}{r^2}\cos^2(kr-kct)\,\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{j}_{NR}(r,t)=-\frac{\mu c^3 q_0^2}{kr^3}\cos(kr-kct)\sin(kr-kct)\,\hat{\mathbf{r}}\,.$$

(**c**) Calculate the time average of each of these current-density components (defined as $\frac{1}{T}\int_0^T \mathbf{j}(r,t)\,dt$, where T is one period of oscillation) and show that the average of the radiative part points in the positive $\hat{\mathbf{r}}$ direction, while the time average of the nonradiative part is zero. (Note: neither answer should have any dependence on T.)

***25.3 Plane Wave Energy Density**. Consider the plane-wave solution to the 3D wave equation $q(x,y,z,t)=q_0\exp\{i(k_x x+k_y y+k_z z-kct)\}$.
(**a**) Calculate the kinetic, potential, and total energy densities $\rho_T(x,y,z,t)$, $\rho_V(x,y,z,t)$, and $\rho(x,y,z,t)$, respectively and the energy current density $\mathbf{j}(x,y,z,t)$.
(**b**) Show that the 3D continuity equation is satisfied by your expressions for ρ and $\mathbf{j}$.

25.4 Spherical Standing Wave Energy Density. Consider the spherically symmetric standing wave solution to the 3D wave equation
$q(r,\theta,\phi) = q_0 \dfrac{\sin(kr)}{kr}\cos(kct)$.

(**a**) Calculate the kinetic, potential, and total energy densities $\rho_T(r,\theta,\phi,t)$, $\rho_V(r,\theta,\phi,,t)$, and $\rho(r,\theta,\phi,,t)$, respectively.

(**b**) Show for large distances from the origin ($kr >> 1$) that the total energy density for this wave is approximately $\rho(r,\theta,\phi,,t) = \dfrac{\mu}{2} q_0^2 (kc)^2 \left\{ \left[\dfrac{\sin(kr)}{kr}\sin(kct)\right]^2 + \left[\dfrac{\cos(kr)}{kr}\cos(kct)\right]^2 \right\}$.