Energy Density / Energy Flux / Total Energy in 1D

Overview and Motivation: From your study of waves in introductory physics you should be aware that waves can transport energy from one place to another – consider the generation and detection of radio waves, for example. In the next two lectures we consider some of the details of the energy associated with wave phenomena. To keep it initially simple, we start out with one dimensional waves. In the next lecture we generalize the concepts discussed here to three dimensions.

Key Mathematics: density, flux, and the continuity equation.

I. Density, Flux, and the Continuity Equation

Let's start by considering some quantity Q that has associated with it a density ρ. Because we are interested in only one (spatial) dimension, the density associated with Q will be so much Q per unit length. If we then integrate that density between two points in space, x_1 and x_2, then we will get the total amount of Q between the points x_1 and x_2. Mathematically, we write this as

$$Q(x_1, x_2, t) = \int_{x_1}^{x_2} \rho(x,t)\, dx \tag{1}$$

We have explicitly included time because Q may be a dynamic quantity.

Often, Q is a conserved quantity. That is, it cannot be either created or destroyed. If that is the case then the change in Q within the region from x_1 to x_2

$$\frac{\partial Q(x_1, x_2, t)}{\partial t} = \int_{x_1}^{x_2} \frac{\partial \rho(x,t)}{\partial t}\, dx \tag{2}$$

must be equal to the net flow of Q into the region. That is,

$$\frac{\partial Q(x_1, x_2, t)}{\partial t} = j(x_1, t) - j(x_2 t), \tag{3}$$

where j is the Q **current density** (or **flux**). Convention is that if $j > 0$, then the flow is in the positive x direction, and if $j < 0$, then the flow is the negative x direction. Note that the units of ρ are $[Q]/\mathrm{m}$ and the units of j are $[\mathrm{Q}]/\mathrm{s} = [\rho]\mathrm{m/s}$. For example, if Q represents charge, then $[\rho]$ = Coulomb/m and $[j]$ = Coulomb/s.

If we now equate the rhs's of Eqs. (2) and (3) we get

$$\int_{x_1}^{x_2} \frac{\partial \rho(x,t)}{\partial t}\, dx = j(x_1,t) - j(x_2 t). \tag{4}$$

Now the rhs of Eq. (4) can be written as

$$j(x_1,t) - j(x_2 t) = -\int_{x_1}^{x_2} \frac{\partial j(x,t)}{\partial x}\, dx \tag{5}$$

and so we can rewrite Eq. (4) as

$$\int_{x_1}^{x_2} \left[\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} \right] dx = 0. \tag{6}$$

Now, because the limits x_1 and x_2 are arbitrary, the integrand must vanish. This gives us an important relationship between the density and flux,

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0. \tag{7}$$

Equation (7) is known as the (1D) **continuity equation**. Because the density and flux are *local* quantities (which means that they can be defined at each point in space, as opposed to Q, which is a *global* quantity), Eq. (7) is a local statement about the conservation of Q.

II. Energy Density and Flux for 1D Waves

Let's now apply this discussion to the energy associated with 1D waves. That is, we let Q be the total energy associated with 1D waves between two points x_1 and x_2. To be specific, let's think about transverse waves on a string. For this particular physical system, where the wave speed c is given by $c = \sqrt{\tau/\mu}$, where τ is the tension in the string and μ is the mass density (mass per unit length), the energy density can be written as[1]

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[1] We do not prove this result here. For its derivation we refer you to an intermediate mechanics text, such *Classical Dynamics* by Marion and Thornton.

$$\rho(x,t) = \frac{\mu}{2}\left(\frac{\partial q}{\partial t}\right)^2 + \frac{\tau}{2}\left(\frac{\partial q}{\partial x}\right)^2 \tag{8}$$

The first term on the rhs is the **kinetic energy density** ρ_T while the second is the **potential energy density** ρ_V.

So, we have the energy density, but what about the energy flux j? Well, whatever it is it must satisfy Eq. (7), the continuity equation. Let's thus calculate $\partial\rho/\partial t$ and see what happens. Using Eq. (8) we have

$$\frac{\partial \rho}{\partial t} - \left[\mu \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial t^2} + \tau \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial t \partial x}\right] = 0 \tag{9}$$

Comparing this with Eq. (7) we see that we would like to be able to write

$$-\left[\mu \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial t^2} + \tau \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial t \partial x}\right] \tag{10}$$

as $\partial/\partial x$ of some quantity, which we could then identify as the flux j. To do this we can get some help from the wave equation (here we use $c^2 = \tau/\mu$ for string waves),

$$\frac{\partial^2 q}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 q}{\partial x^2}, \tag{11}$$

and the equality of mixed partial derivatives,

$$\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial q}{\partial t}\right), \tag{12}$$

to rewrite Eq. (9) as

$$\frac{\partial \rho}{\partial t} - \tau\left[\frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial x^2} + \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x \partial t}\right] = 0, \tag{13}$$

which can be compacted as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left\{-\tau \frac{\partial q}{\partial t}\frac{\partial q}{\partial x}\right\} = 0. \tag{14}$$

We can thus identify the energy flux as

$$j(x,t) = -\tau \frac{\partial q}{\partial t}\frac{\partial q}{\partial x}. \tag{15}$$

III. Several Examples

Let's look at three examples: a traveling wave, a standing wave, and two colliding wave packets. In each example we consider the two energy densities ρ_T and ρ_V and the energy flux j.

A. Traveling Wave

To follow along at this point you will need to go to the class web site and bring up the video file *Energy in 1D Traveling Wave.avi*, which shows these time dependent quantities for the traveling wave

$$q(x,t) = q_0 \cos(kx - ckt). \tag{16}$$

Notice that, as perhaps expected, that all quantities move to the right at the wave speed c. Furthermore, the "wavelength" of the energy densities and flux is half that of the displacement q. For the energy densities, this should be obvious from their definitions. Furthermore, because for this traveling-wave example it is not hard to show that $j = c\rho$, j looks essentially the same as the total energy density ρ.

B. Standing Wave

The next video on the web site, *Energy in 1D Standing Wave.avi*, shows these same time dependent quantities for the standing wave

$$q(x,t) = q_0 \sin(kx)\cos(ckt). \tag{17}$$

This example is a bit more interesting. Notice that now the energy oscillates back and forth between kinetic and potential, which now have their (stationary) maxima at different spatial points. A careful examination of the flux shows that the energy at a given point flows one direction and then the other as it is converted between potential and kinetic.

C. Colliding wave packets

The last example can be found in the video *Energy in 1D Colliding Pulses.avi*. The displacement for this wave is given by

$$q(x,t) = q_0 \cos(kx - ckt)\, e^{-[(kx-ckt)/a_0]^2} - \left[q_0 \cos(kx + ckt)\, e^{-[(kx+ckt)/a_0]^2}\right] . \tag{18}$$

For most of the time the wave looks like two noninteracting wave packets (or pulses), each moving at the speed c but in different directions. Notice that when the pulses are far apart the behavior of the density and flux is similar to that for a traveling wave, but as the pulses overlap the density and flux behave in a manner similar to a standing wave (which can, of course, be described as the superposition of two traveling waves).

IV. Total Energy

Now that we have seen some examples illustrating the local quantities $\rho(x,t)$ and $j(x,t)$, let's consider the total energy associated with wave motion. In particular, let's consider transverse waves on a string on the interval $0 \le x \le L$ with the standard boundary conditions $q(0,t) = q(L,t) = 0$. In that case we can write any wave on the string as a linear superposition of standing waves as[2]

$$q(x,t) = \sum_{n=1}^{\infty} \sin\left(\tfrac{n\pi}{L} x\right)\left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right], \tag{19}$$

where $\omega_n = n\pi\, c/L$, and the coefficients A_n and B_n depend upon the initial conditions.

Using Eq. (1) the total kinetic and potential energies can be expressed in terms of their densities as

$$T(t) = \int_0^L \rho_T(x,t)\, dx = \frac{\mu}{2} \int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx , \tag{20a}$$

$$V(t) = \int_0^L \rho_V(x,t)\, dx = \frac{\tau}{2} \int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx , \tag{20b}$$

Let's now substitute the general form of the displacement on the rhs of in Eq. (19) into Eq. (20) and calculate the total kinetic and potential energies. For the kinetic energy we have

[2] See Lecture 10 notes.

$$T(t)=\frac{\mu}{2}\int_0^L\left\{\sum_{n=1}^{\infty}\sin\left(\tfrac{n\pi}{L}x\right)\omega_n\left[-A_n\sin(\omega_n t)+B_n\cos(\omega_n t)\right]\right\}\\ \times\left\{\sum_{m=1}^{\infty}\sin\left(\tfrac{m\pi}{L}x\right)\omega_m\left[-A_m\sin(\omega_m t)+B_m\cos(\omega_m t)\right]\right\}dx \tag{21}$$

Notice that we use different indices on the two sums so that we know which quantities go with each sum. Switching the orders of integration and summation we can rewrite Eq. (21) as

$$T(t)=\frac{\mu}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\omega_n\omega_m\left[-A_n\sin(\omega_n t)+B_n\cos(\omega_n t)\right]\left[-A_m\sin(\omega_m t)+B_m\cos(\omega_m t)\right]\\ \times\int_0^L\sin\left(\tfrac{n\pi}{L}x\right)\sin\left(\tfrac{m\pi}{L}x\right)dx\} \tag{22}$$

We can now simplify this considerably because of the orthogonality of the sine functions. That is, using

$$\int_0^L\sin\left(\tfrac{n\pi}{L}x\right)\sin\left(\tfrac{m\pi}{L}x\right)dx\}=\frac{L}{2}\delta_{nm}, \tag{23}$$

Eq. (22) becomes

$$T(t)=\frac{\mu L}{4}\sum_{n=1}^{\infty}\omega_n^2\left[A_n\sin(\omega_n t)-B_n\cos(\omega_n t)\right]^2. \tag{24}$$

Now

$$T_n(t)=\frac{\mu L}{4}\omega_n^2\left[A_n\sin(\omega_n t)-B_n\cos(\omega_n t)\right]^2 \tag{25}$$

is simply the kinetic energy contained in the nth normal mode. Thus Eq. (24) can be simply viewed as the sum of kinetic energies contained in all of the normal modes,

$$T(t) = \sum_{n=1}^{\infty} T_n(t) \tag{26}$$

Similarly, starting with Eq. (20b), one can show that the total potential energy can be written as

$$V(t) = \sum_{n=1}^{\infty} V_n(t), \tag{27}$$

where

$$V_n(t) = \frac{\mu L}{4} \omega_n^2 \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right]^2 \tag{28}$$

is the potential energy contained in each normal mode. Further, using Eqs. (25) and (28) it is not difficult to show that the total energy contained in each mode, $E_n(t) = T_n(t) + V_n(t)$, is equal to

$$E_n(t) = \frac{\mu L}{4} \omega_n^2 \left(A_n^2 + B_n^2 \right), \tag{29}$$

and is thus constant. That the energy in each normal mode is constant is due to the fact that the normal modes do not interact. Why? Because the equation of motion for each normal mode is independent of the other normal modes. Notice also that the energy is each normal mode is proportional the square of the amplitude $\left(A_n^2 + B_n^2 \right)$. Finally, because $E_n(t)$ is constant, the total energy

$$E(t) = \sum_{n=1}^{\infty} E_n(t) \tag{30}$$

is also constant.

Exercises

*__24.1__. **Energy Density and Current for a Traveling Wave**. Consider the traveling wave solution to the 1D wave equation $q(x,t) = q_0 \cos(kx - ckt)$.
(**a**) Calculate the kinetic, potential and total energy densities $\rho_T(x,t)$, $\rho_V(x,t)$, and $\rho(x,t)$, respectively, and the energy current density $j(x,t)$. Show that $\rho_T(x,t) = \rho_V(x,t)$. Further show that $j(x,t) = c\rho(x,t)$.
(**b**) Does the energy current flow in the direction that you expect? Explain.
(**c**) Show that the 1D continuity equation is satisfied by your expressions for $\rho(x,t)$ and $j(x,t)$.

__24.2__. **Energy Density and Current for a Standing Wave. Consider the standing-wave solution to the wave equation for transverse waves on a string $q(x,t) = q_0 \sin(kx)\sin(\omega t)$.
(**a**) Calculate the kinetic and potential energy densities $\rho_T(x,t)$ and $\rho_V(x,t)$, respectively, and the energy current density $j(x,t)$. Express your answers using the parameters μ, c, and ω.
(**b**) Show that the 1D continuity equation [Eq. (7)] is satisfied for this wave.
(**c**) For this wave find the total kinetic energy $T(t)$ and potential energy $V(t)$ in a string of length L on the interval $0 \le x \le L$, assuming that $k = n\pi/L$, where n is some positive integer. Here, use the parameters μ, L, and ω to express your answers.
(**d**) Show that the total energy $E(t) = T(t) + V(t)$ is independent of time.

*__24.3__ **Total Energy in Vibrating String**.
(**a**) As was done for the kinetic energy in Sec. IV, show that the total potential energy contained in the vibrating string is given by

$$V(t) = \frac{\mu L}{4} \sum_{n=1}^{\infty} \omega_n^2 \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right]^2$$

(**b**) Using Eqs. (25) and (28) show that the total energy $E_n(t)$ in a normal mode is constant.

*__24.4__ Consider the generic traveling wave $q(x,t) = f(x - ct)$. Show that for this wave
(**a**) $\rho_T = \rho_V$ and
(**b**) $j = c\rho$.