Spherical Coordinates II / A Boundary Value Problem / Separation of Variables Summary

Overview and Motivation: We look at the fourth differential equation that arises in the separable solution to the spherical-coordinates wave equation. We then review the separable solutions in all three coordinate systems – Cartesian, cylindrical, and spherical. Finally, we use a separable solution to find the normal modes of a drumhead.

Key Mathematics: More separation of variables, spherical Bessel functions, and normal modes in polar coordinates.

I. Separation of Variables in Spherical Coordinates (continued)

Last time we began our search for separable solutions to the wave equation in spherical coordinates,

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial q}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial q}{\partial\theta}\right] + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 q}{\partial\phi^2}. \tag{1}$$

We assumed a solution of the form $q(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$ and then solved three ordinary differential equations, which gave us the three functions $T(t)$, $\Phi(\phi)$, and $\Theta(\theta)$. Because each of these differential equations is second-order, linear, and homogeneous, there are two linearly independent solutions that can be (linearly) combined to produce the most general form of each solution. For the functions $T(t)$, $\Phi(\phi)$, and $\Theta(\theta)$ the most general forms can be written as

$$T_k(t) = A_k e^{ikct} + B_k e^{-ikct}, \tag{2}$$

$$\Phi_m(\phi) = C_m e^{im\phi} + D_m e^{-im\phi}, \qquad m = 0, 1, 2, \ldots l, \tag{3}$$

$$\Theta_{l,m}(\theta) = E_{l,m}\, P_l^m(\cos(\theta)) + F_{l,m} Q_l^m(\cos(\theta)), \qquad l = 0, 1, 2, \ldots \tag{4}$$

Here A_k, C_m, and $F_{l,m}$, etc. are undetermined constants, and P_l^m and Q_l^m are associated Legendre functions of the first and second kind, respectively.

Let's now look at the $R(r)$ part of $q(r,\theta,\phi,t)$. Looking back at p. 4 of the Lecture 22 notes we see that the ordinary differential equation for $R(r)$ is

$$-l(l+1) = -\frac{1}{R}\left(r^2 R'\right)' - k^2 r^2, \tag{5}$$

which can be rewritten as

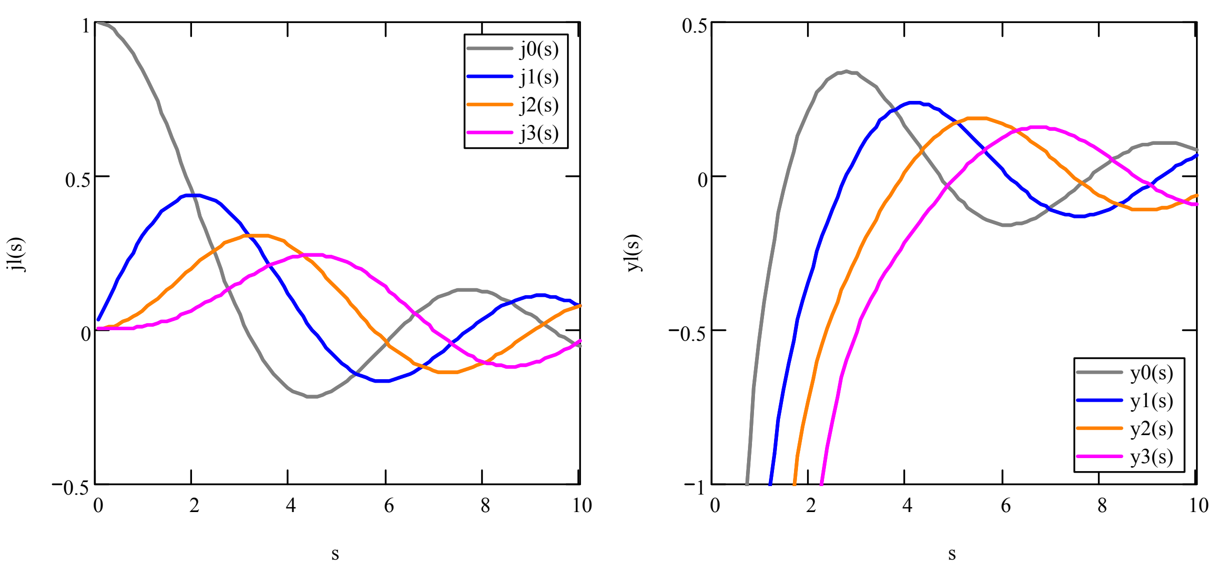
$$r^2 R'' + 2rR' + \left[k^2 r^2 - l(l+1)\right]R = 0 . \qquad (6)$$

As with some of the other equations that we have looked at, this not quite in standard form. Equation (6) can be put in standard form (in a manner similar to the Bessel equation that arose in cylindrical coordinates) by defining the new independent variable $s = kr$. With this definition (and the chain rule) Eq. (6) can be transformed into

$$s^2 R''(s) + 2sR'(s) + \left[s^2 - l(l+1)\right]R(s) = 0 , \qquad (7)$$

where we emphasize that R is now a function of the new variable s. If you look back at Eq. (21) of the Lecture 21 notes (Bessel's equation), you will see that Eq. (7) is quite similar to that equation.

The solutions to Eq. (7) (which are indeed similar to Bessel functions) are known as **spherical Bessel functions**. The spherical Bessel functions of the **first and second kind** are denoted $j_l(s)$ and $y_l(s)$. The following figure plots these functions (for $l = 0,1,2,3$).



The following facts about these functions, some of which can be discerned from the graphs, are worth noting.

(*i*) $j_l(s)$ are finite everywhere; $y_l(s)$ diverge as $s \to 0$. Also notice the behavior of these functions as $s \to 0$ for increasing l: the functions $j_l(s)$ converge more rapidly while the functions $y_l(s)$ diverge more rapidly.

(*ii*) The functions oscillate with decreasing amplitude as $s \to \infty$.

(*iii*) The spherical Bessel functions can be written in terms of (standard) Bessel functions of *noninteger* order as

$$j_l(s) = \sqrt{\tfrac{1}{2}\pi/s}\; J_{l+\frac{1}{2}}(s), \qquad y_l(s) = \sqrt{\tfrac{1}{2}\pi/s}\; Y_{l+\frac{1}{2}}(s) \tag{8a), (8b}$$

(*iv*) The $j_l(s)$ functions have the convenient integral representation[1]

$$j_l(s) = \frac{s^l}{2^{l+1}\Gamma(l+1)} \int_0^\pi \cos[s\cos(\theta)] \sin^{2l+1}(\theta)\, d\theta\,. \tag{9}$$

(*v*) The first three $j_l(s)$'s and $y_l(s)$'s can be represented in terms of sine and cosine functions as

$$j_0(s) = \frac{\sin(s)}{s}, \qquad y_0(s) = -\frac{\cos(s)}{s}, \tag{10a), (10b}$$

$$j_1(s) = \frac{\sin(s)}{s^2} - \frac{\cos(s)}{s}, \qquad y_1(s) = -\frac{\cos(s)}{s^2} - \frac{\sin(s)}{s}, \tag{10c), (10d}$$

$$j_2(s) = \left(\frac{3}{s^3} - \frac{1}{s}\right)\sin(s) - \frac{3}{s^2}\cos(s), \qquad y_2(s) = -\left(\frac{3}{s^3} - \frac{1}{s}\right)\cos(s) - \frac{3}{s^2}\sin(s). \tag{10e), (10f}$$

Similar, although increasingly more complicated, formulae can be derived for higher-order $j_l(s)$'s and $y_l(s)$'s. Notice that the formulae for $y_l(s)$ are identical to the formula for $j_l(s)$ with the changes $\sin(s) \to -\cos(s)$ and $\cos(s) \to \sin(s)$.

Let's now get back to our solution to the wave equation. Because $s = kr$, the solution to Eq. (6) [the equation for $R(r)$], can be generally written as

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[1] At least Eq. (9) can be convenient when using a computer mathematic program such as Mathcad. In fact, you can use Eq. (9) and Mathcad to generate the formulae for $j_l(s)$ in Eq. (10).

$$R_{k,l}(r) = G_{k,l} j_l(kr) + H_{k,l} y_l(kr). \tag{11}$$

So putting Eqs. (2), (3), (4), and (11) together we now have the general form of the separable solution in spherical coordinates

$$\begin{aligned} q_{k,l,m}(r,\theta,\phi,t) = &\left[G_{k,l} j_l(kr) + H_{k,l} y_l(kr)\right]\left[E_{l,m} P_l^m(\cos(\theta)) + F_{l,m} Q_l^m(\cos(\theta))\right] \\ &\times\left(C_m e^{im\phi} + D_m e^{-im\phi}\right)\left(A_k e^{ikct} + B_k e^{-ikct}\right) \end{aligned}, \tag{12}$$

where the parameters l and m are specified by $l = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots l$. The parameter k is unspecified.

II. Summary of Separable Solutions

We previously wrote down separable solutions in the other coordinate systems, but never in as general a form as Eq. (12) for spherical coordinates. Let's now do this for the previous coordinate systems. For cylindrical coordinates we can write the general separable solution as

$$\begin{aligned} q_{n,a,k}(\rho,\phi,z,t) = &\left[G_{n,a} J_n(a\rho) + H_{n,a} Y_n(a\rho)\right]\left(E_n e^{in\phi} + F_n e^{-in\phi}\right) \\ &\times\left(C_{k,a} e^{i\sqrt{k^2-a^2}\,z} + D_{k,a} e^{-i\sqrt{k^2-a^2}\,z}\right)\left(A_k e^{ikct} + B_k e^{-ikct}\right) \end{aligned}, \tag{13}$$

where J_n and Y_n are Bessel functions. The parameter n is specified by $n = 0, 1, 2, \ldots$, but the parameters a and k are unspecified. Similarly, for Cartesian coordinates we have[2]

$$\begin{aligned} q_{k_x,k_y,k_z}(x,y,z,t) = &\left(G_{k_x} e^{ik_x x} + H_{k_x} e^{-ik_x x}\right)\left(E_{k_y} e^{ik_x y} + F_{k_y} e^{-ik_y y}\right) \\ &\times\left(C_{k_z} e^{ik_x z} + D_{k_z} e^{-ik_y z}\right)\left(A_{k_x,k_y,k_z} e^{ic\sqrt{k_x^2+k_y^2+k_x^2}\,t} + B_{k_x,k_y,k_z} e^{-ic\sqrt{k_x^2+k_y^2+k_x^2}\,t}\right) \end{aligned}. \tag{14}$$

In this equation none of the parameters k_x, k_y, or k_z are specified.

Some general remarks concerning Eqs. (12) – (14) are in order. First, notice that all three solutions depend upon three parameters, which are (essentially) the three independent separation constants that arise in the separation-of-variables process. Second, the discrete parameters (l,m in spherical coordinates, n in cylindrical coordinates) are discrete because of mathematical considerations related to the coordinate system being used. The other parameters (k in spherical coordinates, k,a

[2] You have seen an almost general form for Cartesian coordinates in Exercise 19.2.

in cylindrical coordinates, k_x, k_y, k_z in Cartesian coordinates) can take on any values and the solutions will still satisfy the wave equation.

For a given problem, the physics of the situation may dictate that these parameters take on only certain values. For example, in Exercise 19.2, where you found the normal-mode standing waves in a rectangular room, you found that because of the boundary conditions the parameters k_x, k_y, and k_z could only take on the values

$$k_x = \frac{n_x \pi}{L_x}, \qquad k_y = \frac{n_y \pi}{L_y}, \qquad k_z = \frac{n_z \pi}{L_z}, \tag{15a) – (15c}$$

where the n_i's are integers and the L_i's are the dimensions of the room. Physical considerations, such as boundary conditions, may also place constraints on the multiplicative factors that appear in Eqs. (12) – (14).

III. A Vibrating Circular Drumhead

Let's look at another example where physics constrains some of these unspecified parameters. In particular, let's find the normal modes of vibration of a circular drumhead. First, we must recognize that this is a two dimensional, rather than a three dimensional problem. If we were working in Cartesian coordinates we would only need x and y, but because we are interested in a circular drumhead we should recognize that it might be better to work in polar coordinates (with the origin at the center of the drum head). Well, if we were to write the 2D wave equation in polar coordinates and do separation of variables, the solutions would be the same as the $k^2 = a^2$ solutions for cylindrical coordinates in Eq. (13). That is, the solutions would be

$$q_{n,a}(\rho,\phi,t) = \left[G_{n,a} J_n(a\rho) + H_{n,a} Y_n(a\rho)\right] \left(E_n e^{in\phi} + F_n e^{-in\phi}\right)\left(A_a e^{iact} + B_a e^{-iact}\right). \tag{16}$$

Let's now apply some physics. First we know that we want the displacement q to be everywhere finite. So because the functions $Y_n(s)$ diverge for $s \to 0$ we must have $H_{n,a} = 0$. Also, we want the displacement to be a real quantity. With that in mind we explicitly write the ϕ and t parts of Eq. (16) in terms of sine and cosine functions, rather then the complex exponential functions.[3] We then have

$$q_{n,a}(\rho,\phi,t) = G_{n,a} J_n(a\rho)\left[\tilde{E}_n \cos(n\phi) + \tilde{F}_n \sin(n\phi)\right]\left[\tilde{A}_a \cos(act) + \tilde{B}_a \sin(act)\right], \tag{17}$$

[3] There are, of course, other ways to make sure that the solution is real. See the discussion on p. 8 of the Lecture 2 notes.

where the quantities with tildes in Eq. (17) are real multiplicative factors [that can be related to the quantities without tildes in Eq. (16), if necessary].

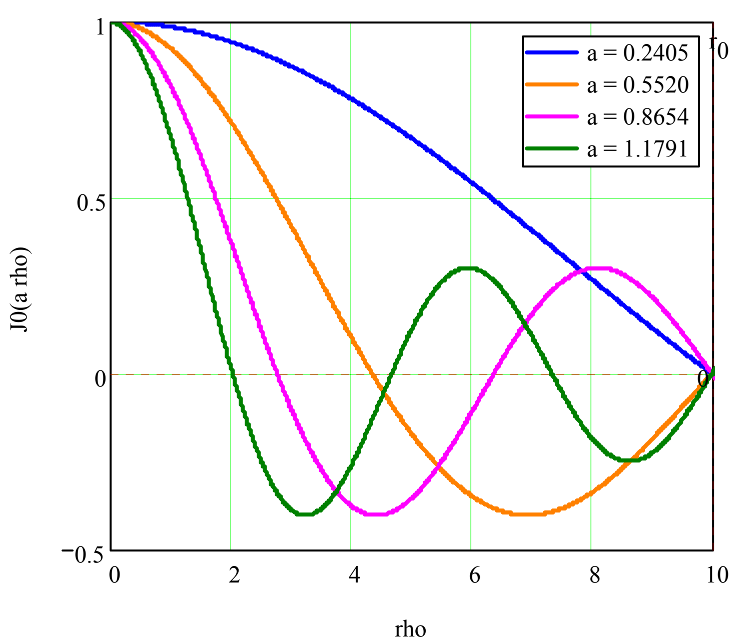
There is now one more bit of physics to consider: the boundary condition at the edge of the drum head. For simplicity, we assume that this bc is $q_{n,a}(\rho_0,\phi,t)=0$, where ρ_0, is the radius of the drumhead. Applying this bc to Eq. (17) then gives us

$$0 = G_{n,a}J_n(a\rho_0)\left[\tilde{E}_n\cos(n\phi)+\tilde{F}_n\sin(n\phi)\right]\left[\tilde{A}_a\cos(act)+\tilde{B}_a\sin(act)\right] . \qquad (18)$$

So how can Eq. (18) be satisfied? The only nontrivial way is to require that a be such that $J_n(a\rho_0)=0$. Now, each Bessel function J_n has an infinite number of discrete zeros (see the figure on p. 5 of Lecture 27 notes) so that there are an infinite number of discrete values of a (which are different for each n) that will satisfy Eq. (18). Unfortunately, in contrast to the Cartesian coordinate example specified by Eq. (15), there are no nice, simple formulae for the zeros of the Bessel functions. Nonetheless, the zeros can be found numerically.

A. $n=0$ Normal Modes

We first look at normal mode solutions specified by $n=0$. For these solutions the Bessel function $J_0(s)$ comes into play. The zeros of this function are equal to 2.405, 5.520, 8.654, 11.791, …[4] Thus, the bc will be satisfied for



[4] The zeros are tabulated in *Handbook of Mathematical Functions* by Abramowitz and Stegun (where else?).

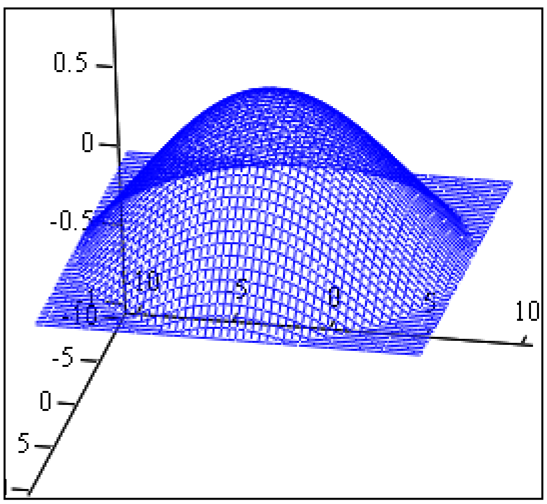
$$a\rho_0 = 2.405,\ 5.520,\ 8.654,\ 11.791,\ \ldots \tag{20}$$

This is illustrated in the graph on the previous page, where we have set $\rho_0 = 10$. Notice that $J_0(a\rho)$ is indeed equal to zero at $\rho = \rho_0$ for each value of a given by Eq. (20).

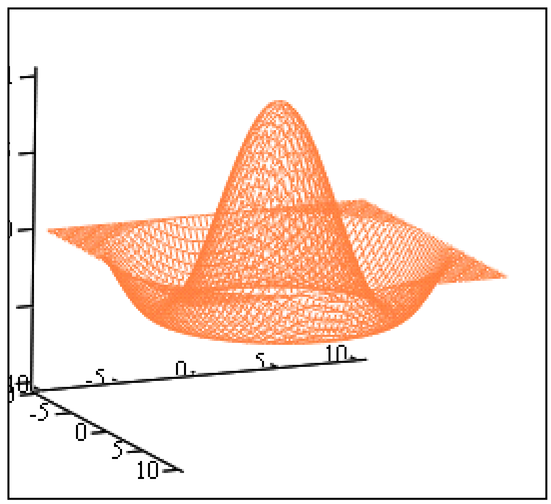
Let's now look at the complete solution for $n = 0$. Simplifying Eq. (17) we can write the $n = 0$ normal modes as

$$q_{0,i}(\rho,\phi,t) = G_{0i} J_0(a_{0i}\rho)\, \widetilde{E}_0 \left[\widetilde{A}_{0i}\cos(a_{0i}ct) + \widetilde{B}_{0i}\sin(a_{0i}ct)\right], \tag{21}$$

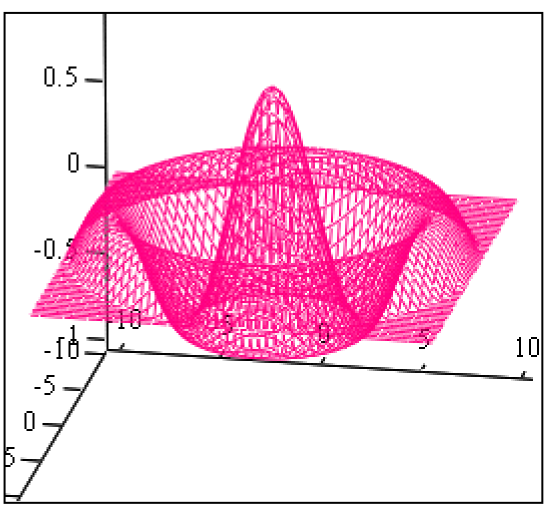
where $a_{0i}\rho_0$ is the i'th zero of $J_0(s)$. Notice that in Eq. (21) we have changed the labeling of the solution; we now label the normal modes with two integers: the first



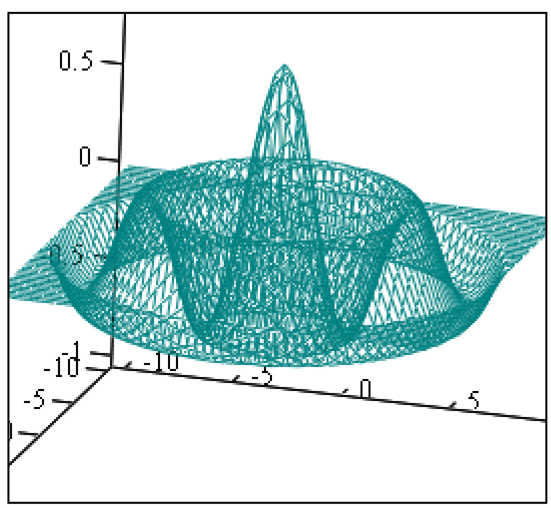
z1



z2



z3



z4

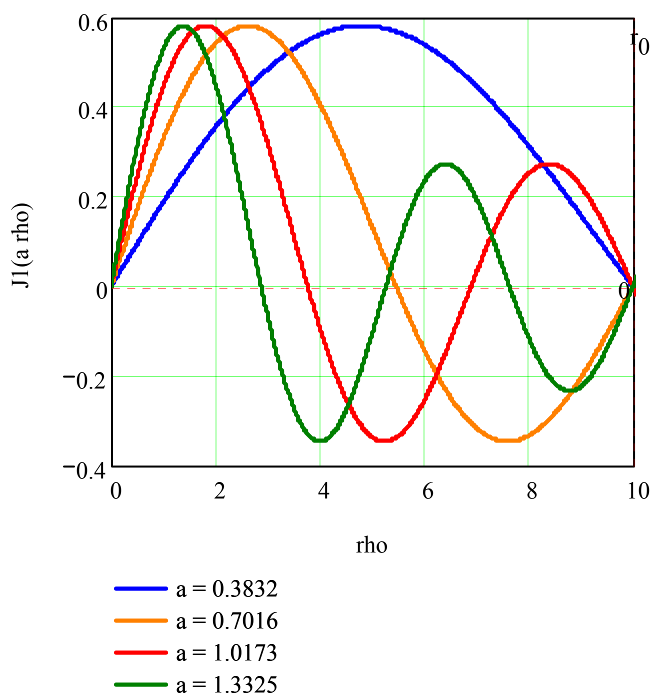
$(=0)$ corresponds to n, and the second (i) labels the value of a that makes the solution vanish at the boundary. Notice that the only spatial variable in Eq. (21) is ρ: the dependence on ϕ is gone. Snapshots of first four normal-mode solutions are shown on the previous page. The class web site has animated (time dependent) versions of these solutions. As evident in the animations [and should be evident from Eq. (21)], these solutions are radially symmetric versions of standing waves.

B. $n=1$ Normal Modes

For any other value of n, the normal modes are found in essentially the same manner. We must again satisfy the bc at the edge of the drum head, which determines the values of a. For $n=1$ the zeros of $J_1(s)$ are 3.832, 7.016, 10.173, 13.325, … Thus, for $n=1$ the boundary condition is satisfied for

$$a\rho_0 = 3.832,\ 7.016,\ 10.175,\ 13.325,\ \ldots \tag{22}$$

This is illustrated in the following figure.



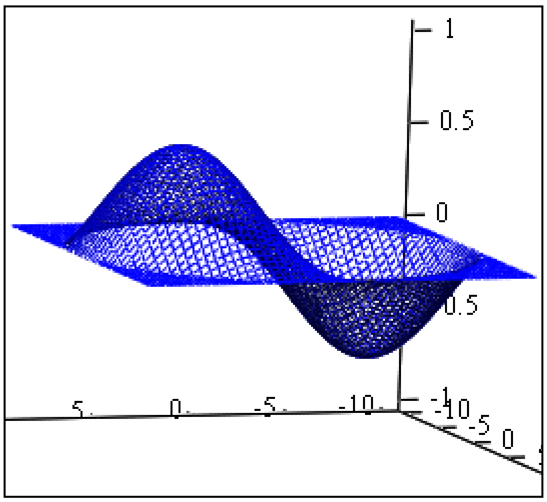
For the complete solution we thus have

$$q_{1,i}(\rho,\phi,t) = G_{1i} J_1(a_{1i}\rho)\left[\tilde{E}_1 \cos(\phi) + \tilde{F}_1 \sin(\phi)\right]\left[\tilde{A}_{1i} \cos(a_{1i}ct) + \tilde{B}_{1i} \sin(a_{1i}ct)\right]. \tag{23}$$

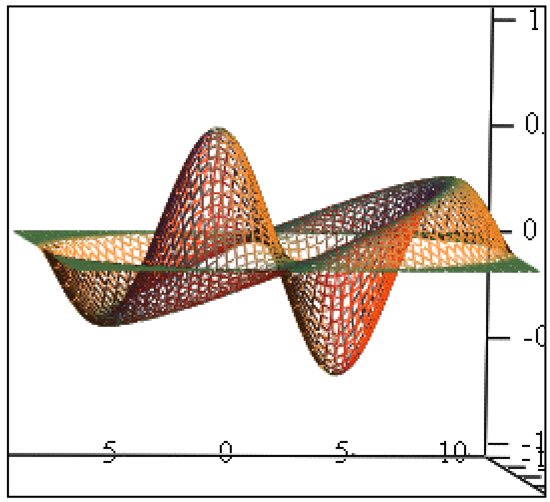
Because the ϕ dependence is no longer trivial, this solution is a bit more complicated than that for $n = 0$. For $n = 0$ the disappearance of ϕ resulted in only one linearly independent solution (in terms of spatial variables – there are still two linearly independent solutions if one considers time). Here, however, the $\cos(\phi)$ and $\sin(\phi)$ solutions are linearly independent. For simplicity, let's just consider the solution with $\widetilde{F}_1 = 0$, which leaves only the $\cos(\phi)$ term. Then we have

$$q_{1,i}(\rho,\phi,t) = G_{1,i} J_1(a_{1i}\rho)\, \widetilde{E}_1 \cos(\phi)\left[\widetilde{A}_i \cos(a_{1i}ct) + \widetilde{B}_i \sin(a_{1i}ct)\right] . \tag{24}$$

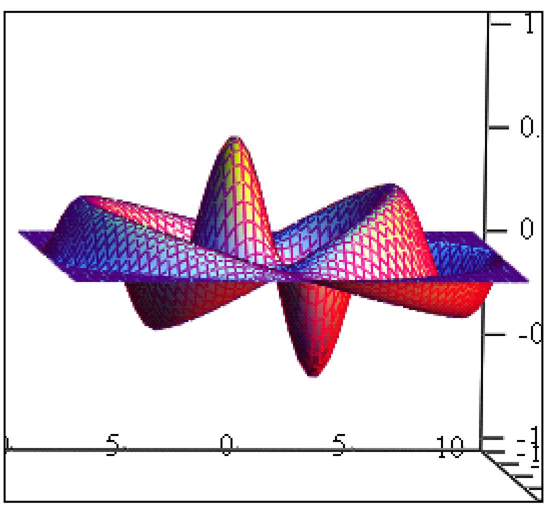
The following figure shows snapshots of this solution. Animated versions are again available on the class web site.



z1



z2



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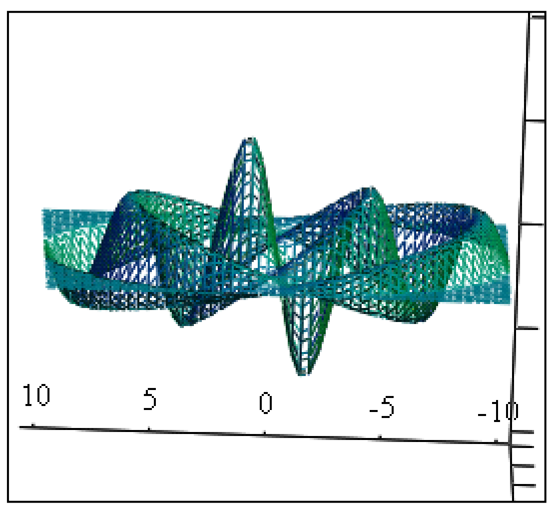

z4

IV. Concluding Remarks

That is it for separation of variables (until you run into them again in some later course, or perhaps while doing some research!). Summarizing, we have seen that we can obtain solutions to the wave equation in three different coordinate systems using this technique. Sometimes these solutions are simple, sometime they are not so simple, at least in terms of functions with which you may be familiar. Each solution is labeled by three independent separation constants, which may or may not be constrained to certain values. Because the ordinary differential equations that give us these solutions are homogeneous and linear, there are also undetermined multiplicative factors associated with each part of the solution. Often (at least) some of the separation constants and the multiplicative factors are determined by physical considerations, such as imposed boundary conditions. In the example that we just did we saw that the parameter a must take on discrete values in order for the boundary condition at the edge of the drumhead to be satisfied.

While separation-of-variables solutions can be interesting in their own right (as in the case of the drumhead modes), I'll again remind you that they are also quite useful because they can be used to construct a basis for any solution to the wave equation, much as we previously discussed for the 1D wave equation. This point has been previously discussed in Sec. III of the Lecture 19 notes.

Exercises

***23.1** Using the definition of s and the chain rule, derive Eq. (7) from Eq. (6). (Hint: you may wish to review the Lecture 21 notes.)

***23.2** The figure on p. 2 indicates that $j_2(0)=0$. Using Eq. (10e) show that this is indeed the case.

***23.3** The two linearly independent combinations of the spherical Bessel functions

$$h_l^{(1)}(s)=j_l(s)+iy_l(s) \text{ and } h_l^{(2)}(s)=j_l(s)-iy_l(s)$$

are know as spherical Bessel functions of the *third kind*. Using these definitions and Eq. (10), express $h_1^{(1)}(s)$ and $h_1^{(2)}(s)$ in terms of the functions e^{is} and e^{-is}.

***23.4 Normal Modes Inside a Sphere**.
Starting with Eq. (12) find all normal-mode solutions to the wave equation inside a sphere that satisfy **all** of the following conditions: the solutions are (1) real, (2) spherically symmetric, (3) finite everywhere, and (4) vanish on the boundary of the sphere, which has a radius of r_0.