

Separation of Variables in Cylindrical Coordinates

Overview and Motivation: Today we look at separable solutions to the wave equation in cylindrical coordinates. Three of the resulting ordinary differential equations are again harmonic-oscillator equations, but the fourth equation is our first foray into the world of special functions, in this case Bessel functions. We then graphically look at some of these separable solutions.

Key Mathematics: More separation of variables; Bessel functions.

I. Cylindrical-Coordinates Separable Solutions

Last time we assumed a product solution $q(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t)$ to the cylindrical-coordinate wave equation

$$\frac{1}{c^2} \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial q}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 q}{\partial \phi^2} + \frac{\partial^2 q}{\partial z^2}, \quad (1)$$

which allowed us to transform Eq. (1) into

$$\frac{1}{c^2} \frac{T''}{T} = \left(R'' + \frac{1}{\rho} R' \right) \frac{1}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z}. \quad (2)$$

We now go through a separation-of-variable procedure similar to that which we carried out using Cartesian coordinates in Lecture 19. The procedure here is a bit more complicated than with Cartesian coordinates because the variable ρ appears in the Φ''/Φ term. However, as we shall see, the equation is still separable.

A. Dependence on Time

As with Cartesian coordinates, we can again make the argument that the rhs of Eq. (2) is independent of t , and so the lhs of this equation must be constant. Cognizant of the fact that we are interested in solutions that oscillate in time, we call this constant $-k^2$ (where we are thinking of k as real) and so we have

$$\frac{1}{c^2} \frac{T''}{T} = -k^2, \quad (3)$$

which can be rearranged as

$$T'' + c^2 k^2 T = 0. \quad (4)$$

By now we should recognize this as the harmonic oscillator equation, which has two linearly independent solutions,

$$T_k^\pm(t) = T_0 e^{\pm i k c t}. \quad (5)$$

And so again we have harmonic time dependence to the separable solution.¹ For the present case it is simplest to assume that k can be positive or negative; thus we really only need one of these solutions. So we simply go with

$$T_k(t) = T_0 e^{i k c t}. \quad (6)$$

B. Dependence on z

Using Eq. (3), the lhs of Eq. (2) can be replaced with $-k^2$, which gives, after a small bit of rearranging,

$$-k^2 - \frac{Z''}{Z} = \left(R'' + \frac{1}{\rho} R' \right) \frac{1}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi}. \quad (7)$$

Notice that the rhs of Eq. (7) is independent of z . Thus the lhs is constant, which we call $-a^2$.² We thus have

$$-k^2 - \frac{Z''}{Z} = -a^2, \quad (8)$$

which can be rearranged as

$$Z'' + (k^2 - a^2)Z = 0, \quad (9)$$

which, yet again, is the harmonic oscillator equation! The solutions are

$$Z_{k,a}^\pm = Z_0 e^{\pm i \sqrt{k^2 - a^2} z}. \quad (10)$$

Although k and a can be any complex numbers, let's consider the case where both k^2 and a^2 are real and positive. Then these solutions oscillate if $k^2 > a^2$, exponentially grow and decay if $k^2 < a^2$, and are constant if $k^2 = a^2$. Note that here, because the square root is always taken as positive, we need to keep both the positive-sign and negative-sign solutions. We can, however, assume that $a \geq 0$.

¹ It should perhaps be obvious that this will always be the case for separable solutions to the wave equation.

² Why not? We can call the constant anything we want. It just so happens that $-a^2$ is convenient.

C. Dependence on ϕ

Using Eq. (8), the lhs of Eq. (7) can be replaced with $-a^2$, which gives us

$$-a^2 = \left(R'' + \frac{1}{\rho} R' \right) \frac{1}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi}. \quad (11)$$

To separate the ρ and ϕ dependence this equation can be rearranged as

$$-\frac{\Phi''}{\Phi} = \left(R'' + \frac{1}{\rho} R' \right) \frac{\rho^2}{R} + \rho^2 a^2. \quad (12)$$

Because each side only depends on one independent variable, both sides of this equation must be constant. This gives us our third separation constant, which we call n^2 . The equation for Φ we can then write as

$$\Phi'' + n^2 \Phi = 0, \quad (13)$$

which is again the harmonic oscillator equation. The solutions to Eq. (13) are

$$\Phi_n^\pm = \Phi_0 e^{\pm i n \phi}. \quad (14)$$

Now we need to use a little physics. Because we expect any physical solution to have the same value for $\phi = 0$ and $\phi = 2\pi$ we must have (for the Φ_n^+ solution)

$$e^0 = e^{i n 2\pi}, \quad (15)$$

or

$$1 = e^{i n 2\pi} \quad (16)$$

Using Euler's relation, it is easy to see that Eq. (16) requires that n be an integer. Now because n can be a positive or negative integer, we do not need to explicitly keep up with the Φ_n^- solution, and so we write for the ϕ dependence

$$\Phi_n = \Phi_0 e^{i n \phi}, \quad n = 0, \pm 1, \pm 2, \dots \quad (17)$$

D. Dependence on ρ

We are now left with one last equation. Also setting the rhs of Eq. (12) equal to n^2 and doing a bit of rearranging yields

$$\rho^2 R'' + \rho R' + (a^2 \rho^2 - n^2)R = 0 \quad (18)$$

This is definitely *not* the harmonic oscillator equation! Its solutions are well known functions, but to see what the solutions to Eq. (18) are we need to put it in "standard" form, which is a form that we can look up in a book such as *Handbook of Mathematical Functions* by Abramowitz and Stegun (the definitive, concise book on special functions, which has been updated and is available online as the *NIST Digital Library of Mathematical Functions*).

To put Eq. (18) in standard form, we make the substitution $s = a\rho$. Now the substitution would be trivial, except that because Eq. (18) is a differential equation in the independent variable ρ , we need to change the derivatives in Eq. (18) to derivatives in s . As usual we do this using the chain rule. If we now think of $R(\rho)$ as a function of ρ through the new independent variable s , i.e., as $R[s(\rho)]$, then using

$$\frac{dR}{d\rho} = \frac{dR}{ds} \frac{ds}{d\rho} = \frac{dR}{ds} a \quad (19a)$$

and

$$\frac{d^2 R}{d\rho^2} = \frac{d}{d\rho} \left(\frac{dR}{ds} \frac{ds}{d\rho} \right) = \frac{d^2 R}{ds^2} \left(\frac{ds}{d\rho} \right)^2 + \frac{dR}{ds} \frac{d^2 s}{d\rho^2} = \frac{d^2 R}{ds^2} a^2. \quad (19b)$$

we can rewrite Eq. (18) as

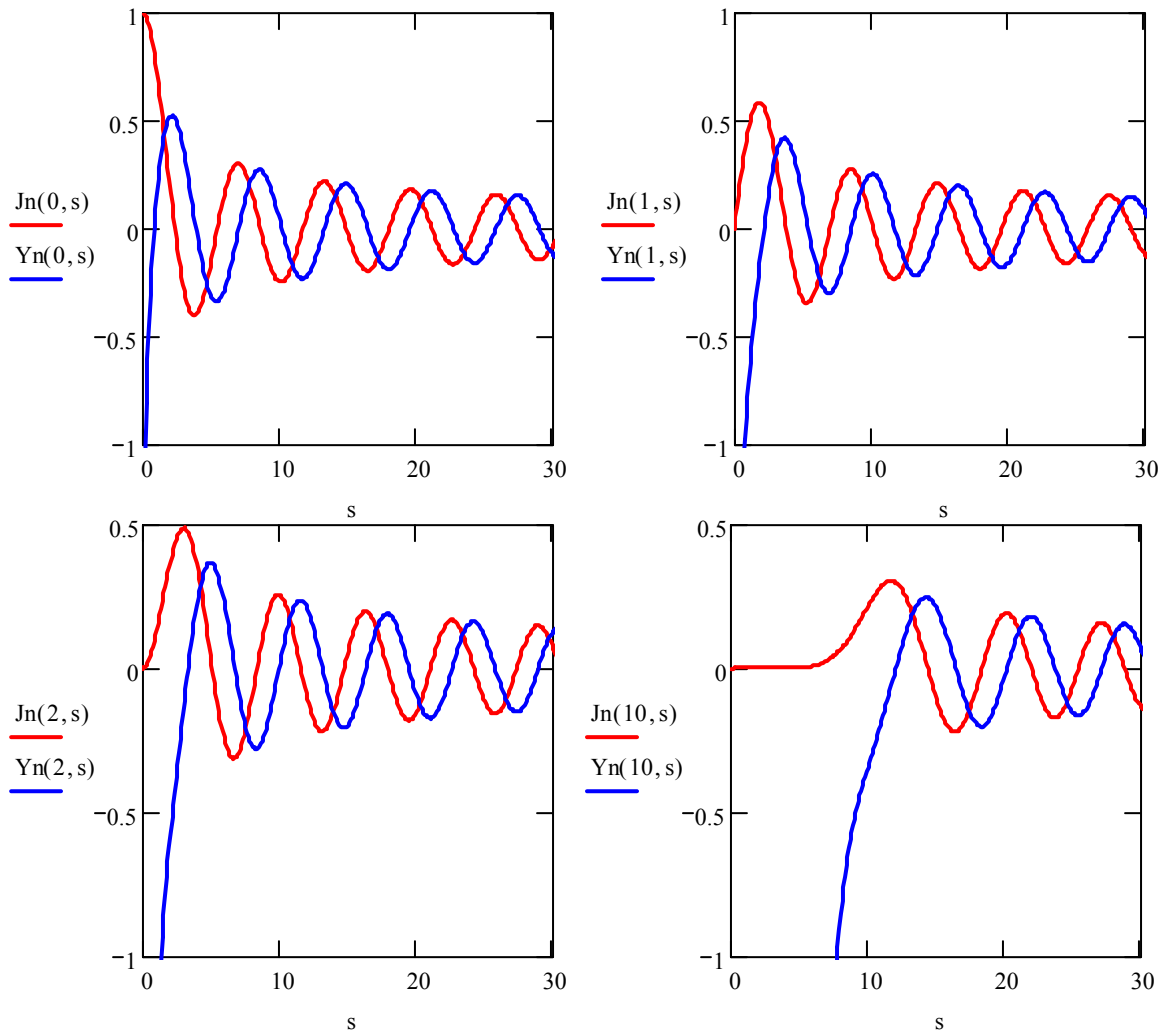
$$\left(\frac{s}{a} \right)^2 R''(s) a^2 + \frac{s}{a} R'(s) a + (s^2 - n^2) R(s) = 0. \quad (20)$$

where we now emphasize that $R = R(s)$. Equation (20) obviously simplifies to

$$s^2 R''(s) + s R'(s) + (s^2 - n^2) R(s) = 0. \quad (21)$$

This equation is known as **Bessel's equation**. Its two linearly independent solutions are known as **Bessel functions** $J_n(s)$ and **Neumann functions** $Y_n(s)$. Sometimes the functions $J_n(s)$ and $Y_n(s)$ are called **Bessel functions of the first and second kind**, respectively. The subscript n is known as the **order** of the Bessel function. Although one can define Bessel functions of non-integer order, one outcome of the Φ equation is that n is an integer, so we only need deal with integer-order Bessel functions for this problem.

The following figure plots both $J_n(s)$ and $Y_n(s)$ for several values of (positive) n .



There are several facts about Bessel functions that are worth noting, some of which can be discerned from these graphs:

(i) $J_0(0)=1$; $J_n(0)=0$, $n \neq 0$. $Y_n(s)$ diverges as $s \rightarrow 0$. Also notice the behavior of these functions as $s \rightarrow 0$ for increasing n : the functions $J_n(s)$ converge more rapidly while the functions $Y_n(s)$ diverge more rapidly.

(ii) $J_n(s)$ and $Y_n(s)$ oscillate with decreasing amplitude as $s \rightarrow \infty$.

(iii) It is customary to define $J_{-n} = (-1)^n J_n$ and $Y_{-n} = (-1)^n Y_n$. (Notice that the Bessel equation only depends upon n^2 , so the $-n$ solution must be essentially the same as the $+n$ solution.)

(ii) Bessel functions have the series representation

$$J_n(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \Gamma(m+n+1)} \left(\frac{s}{2}\right)^{2m+n}, \quad (22)$$

where

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (23)$$

is the **Gamma** function. Note that for integer x , $\Gamma(x+1) = x! = x(x-1)(x-2)\cdots 1$.

$$(y) \quad Y_n(s) = \lim_{n' \rightarrow n} \frac{J_{n'}(s) \cos(n'\pi) - J_{-n'}(s)}{\sin(n'\pi)} \quad (24)$$

More entertaining facts about Bessel functions can be found in the *NIST Digital Library of Mathematical Functions*.

Now because $s = a\rho$, the solutions to Eq. (18) are thus simply $J_n(a\rho)$ and $Y_n(a\rho)$. That is, we have the two linearly independent results

$$R_{n,a}^J(\rho) = R_J J_n(a\rho) \quad \text{and} \quad R_{n,a}^Y(\rho) = R_Y Y_n(a\rho). \quad (25a) - (25b)$$

II. Separable Solutions

Let's put all of this analysis together and write down our separable solutions. Let's further assume that (1) we want the z dependence of the solution to oscillate (or at least not either grow or decay exponentially) and (2) we are only interested in solutions that remain finite as $\rho \rightarrow 0$. We then have the following pieces to our separable solution

$$T_k(t) = T_0 e^{ikct}, \quad -\infty < k < \infty \quad (25a)$$

$$Z_{k,a}^{\pm} = Z_0 e^{\pm i\sqrt{k^2 - a^2} z}, \quad a \leq |k| \quad (25b)$$

$$\Phi_n = \Phi_0 e^{in\phi}, \quad n = 0, \pm 1, \pm 2, \dots \quad (25c)$$

$$R_{a,n}(\rho) = R_0 J_n(a\rho). \quad (25d)$$

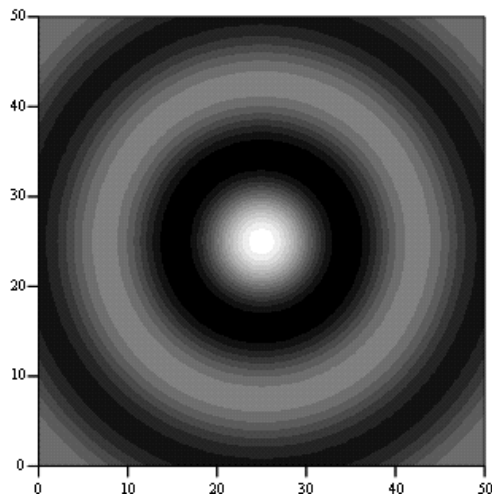
Putting all of Eq. (24) together we can write our separable solutions as

$$q_{n,a,k}^{\pm}(\rho, \phi, z, t) = C_{n,a,k}^{\pm} J_n(a\rho) e^{in\phi} e^{\pm i\sqrt{k^2 - a^2}z} e^{ikct}. \tag{26}$$

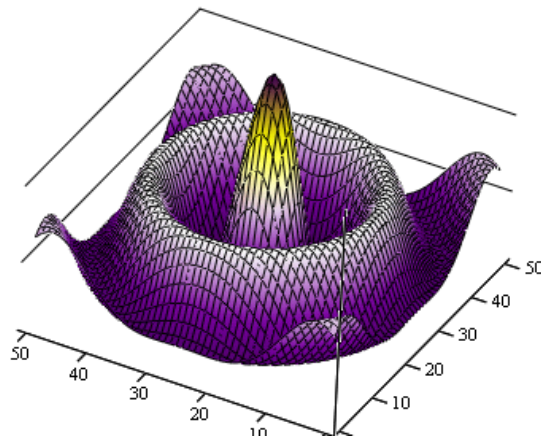
Let's look at some graphs of some of these functions. For simplicity we look at functions that have no z dependence. From Eq. (26) we see that these are solutions where $k = \pm a$. With this constraint we can write Eq. (6) as

$$q_{n,a,\pm a}(\rho, \phi, z, t) = C_{n,a,\pm a} J_n(a\rho) e^{in\phi} e^{\pm iact}. \tag{27}$$

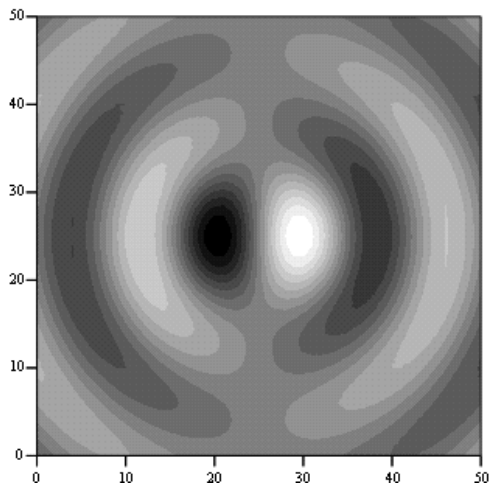
The following pictures show graphs (with $a = 1$) of $\text{Re}[q_{0,a,a}(\rho, \phi, z, 0)] = J_0(a\rho)$, $\text{Re}[q_{1,a,a}(\rho, \phi, z, 0)] = J_1(a\rho)\cos(\phi)$, and $\text{Re}[q_{2,a,a}(\rho, \phi, z, 0)] = J_2(a\rho)\cos(2\phi)$.



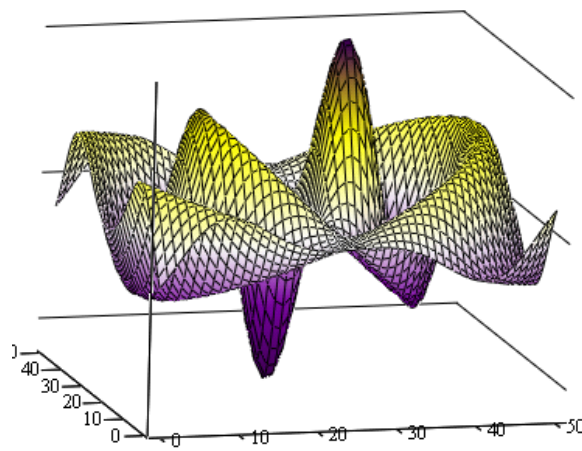
Re(q₀)



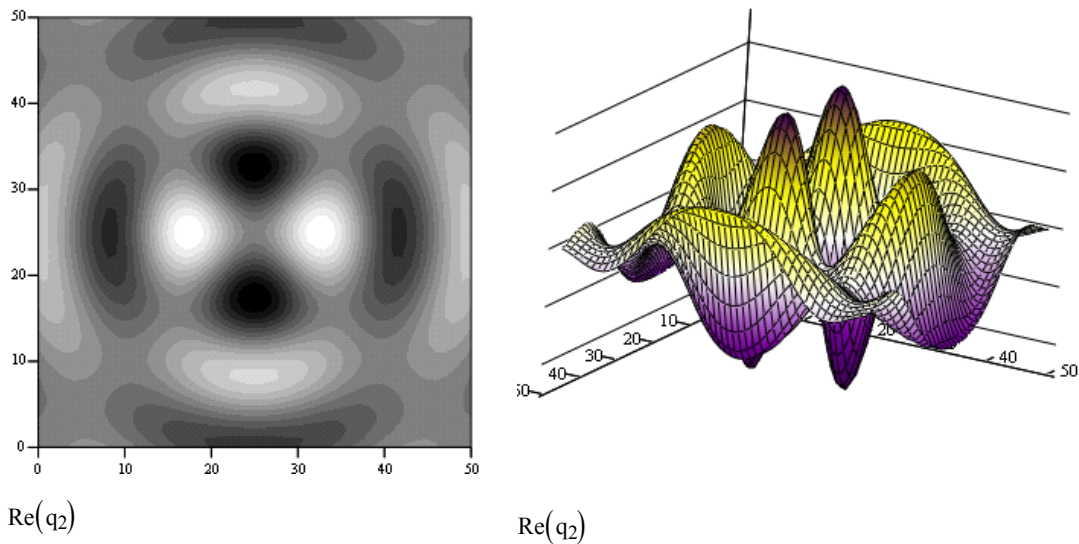
Re(q₀)



Re(q₁)



Re(q₁)



For the moment, this ends our discussion of cylindrical coordinates. In the next lecture we move on to studying the wave equation in spherical-polar coordinates.

Exercises

*21.1 Dispersion Relation.

Consider the solution $q_{n,a,k}^{\pm}(\rho, \phi, z, t) = C_{n,a,k}^{\pm} [J_n(a\rho) + iY(a\rho)] \cos(n\phi) e^{\pm i\sqrt{k^2 - a^2} z} e^{-ikct}$.

From this equation it is clear that the frequency ω is equal to ck . This looks like there is a dispersion relation, but it is not really clear what k represents in this case.

Here we show that if positive, it is essentially the magnitude of a wave vector $\mathbf{k} = k_{\rho} \hat{\rho} + k_z \hat{z}$ (where the meanings of the unit vectors should be obvious). To do this we take advantage of the fact that for large s , the Bessel functions have the asymptotic expansions

$$J_n(s) \approx \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) \quad \text{and} \quad Y_n(s) \approx \sqrt{\frac{2}{\pi s}} \sin\left(s - \frac{1}{2}n\pi - \frac{1}{4}\pi\right).$$

- (a) Using these equations show that component of the wave vector in the ρ direction is $k_{\rho} = a$.
- (b) Identify the z component k_z of the wave vector.
- (c) Then, show that $k_{\rho}^2 + k_z^2 = k^2$.

***21.2 Linear Combinations of Bessel Functions.** The Bessel functions of the *third* kind (also known as *Hankel* functions) are defined as

$$H_n^{(1)}(s) = J_n(s) + iY_n(s) \quad \text{and} \quad H_n^{(2)}(s) = J_n(s) - iY_n(s).$$

(a) Using the asymptotic expansions given in Exercise 21.1, show for large s that

$$H_n^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}} e^{i(s - \frac{1}{2}n\pi - \frac{1}{4}\pi)} \quad \text{and} \quad H_n^{(2)}(s) \approx \sqrt{\frac{2}{\pi s}} e^{-i(s - \frac{1}{2}n\pi - \frac{1}{4}\pi)}.$$

(b) Using the results of (a) describe the behavior of the waves

$$q_{0,a,a}^{(1)}(\rho, \phi, z, t) = H_0^{(1)}(a\rho) e^{-iact} \quad \text{and} \quad q_{0,a,a}^{(2)}(\rho, \phi, z, t) = H_0^{(2)}(a\rho) e^{-iact}.$$

for large ρ (assuming that $a > 0$) (That is, are they standing or traveling waves, etc).