

The Wave Equation in Cylindrical Coordinates

Overview and Motivation: While Cartesian coordinates are attractive because of their simplicity, there are many problems whose symmetry makes it easier to use a different system of coordinates. For example, there are times when a problem has cylindrical symmetry (the fields produced by an infinitely long, straight wire, for example). In this case it is easier to use cylindrical coordinates. So today we begin our discussion of the wave equation in cylindrical coordinates.

Key Mathematics: Cylindrical coordinates and the chain rule for calculating derivatives.

I. Transforming the Wave Equation

As previously mentioned the (spatial) coordinate independent wave equation

$$\frac{1}{c^2} \frac{\partial^2 q}{\partial t^2} = \nabla^2 q \quad (1)$$

can take on different forms, depending upon the coordinate system in use. In Cartesian coordinates the Laplacian ∇^2 is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

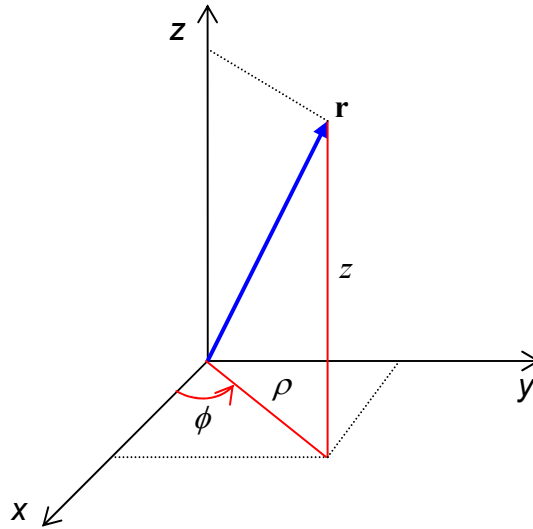
Our first goal is to re-express ∇^2 in terms of cylindrical coordinates (ρ, ϕ, z) , which are defined in terms of the Cartesian coordinates (x, y, z) as

$$\rho = (x^2 + y^2)^{1/2}, \quad (2a)$$

$$\phi = \arctan\left(\frac{y}{x}\right), \quad (2b)$$

$$z = z. \quad (2c)$$

The following picture illustrates the relationships expressed by Eq. (2). For the point given by the vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, the coordinate ρ is the distance of that point from the z axis, the coordinate ϕ is the angle of the projection of the vector onto the x - y plane from the x axis toward the y axis, and z is the (signed) distance of the point from the x - y plane. Note that $\rho \geq 0$ and we can restrict $0 \leq \phi < 2\pi$.



In order to express ∇^2 in terms of these new coordinates we start with a function $f(\rho, \phi, z)$ and consider it to be function of x , y , and z *through* the variables ρ , ϕ , and z by writing

$$f = f[\rho(x, y, z), \phi(x, y, z), z(x, y, z)]. \quad (3)$$

Then, for example, using the chain rule we can write

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} \quad (4)$$

Notice that this equation (as well as some later equations) have two types of terms. The first type is a derivative of the function f , while the second type is a derivative of a new coordinate with respect to an old coordinate. The goal here is to use the relationship between the two coordinate systems [Eq. (2)] to write the second type of term as a function of the new set of coordinates ρ , ϕ , and z . Then equations such as Eq. (4) will be entirely expressed in terms of the new coordinate system.

For the particular case at hand the transformation is a bit simpler than the general case because [see Eq. (2)] $\rho = \rho(x, y)$, $\phi = \phi(x, y)$, and $z = z(z)$. Thus Eq. (3) simplifies to

$$f = f[\rho(x, y), \phi(x, y), z(z)], \quad (5)$$

so that Eq. (4) reduces to

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x}. \quad (6)$$

Let's now express the first term, $\partial^2 f / \partial x^2$, as a function of cylindrical coordinates. We do this by calculating another derivative of Eq. (6) with respect to x .

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x} \right). \quad (7)$$

Now we must be a bit careful here. We must use the chain rule on the functions $\partial f / \partial \rho$ and $\partial f / \partial \phi$ because they are functions of ρ and ϕ [as is f , see Eq. (5)]. However, we do not need to use the chain rule on the terms $\partial \rho / \partial x$ and $\partial \phi / \partial x$ because they are both functions of x and y [as are ρ and ϕ , see Eq. (2)]. With this in mind Eq. (7) becomes

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial \rho^2} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \phi \partial \rho} \frac{\partial \phi}{\partial x} \right) \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial x^2} + \left(\frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \phi^2} \frac{\partial \phi}{\partial x} \right) \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial x^2}, \quad (8)$$

which expands to

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \rho^2} \left(\frac{\partial \rho}{\partial x} \right)^2 + \frac{\partial^2 f}{\partial \phi \partial \rho} \frac{\partial \phi}{\partial x} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial^2 f}{\partial \phi^2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial x^2}. \quad (9)$$

You can see that this process is rather tedious! Now to eliminate the old variables x and y from Eq. (9) we must calculate the quantities

$$\frac{\partial \rho}{\partial x}, \quad \frac{\partial \phi}{\partial x}, \quad \frac{\partial^2 \rho}{\partial x^2}, \quad \frac{\partial^2 \phi}{\partial x^2} \quad (10a) - (10d)$$

and express them in terms of ρ and ϕ . Using Eq. (2a) we calculate the first of these terms as

$$\frac{\partial \rho}{\partial x} = \frac{\partial [(x^2 + y^2)^{1/2}]}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-1/2} 2x = \frac{x}{(x^2 + y^2)^{1/2}}, \quad (11)$$

Now $x = \rho \cos(\phi)$ and $\rho^2 = x^2 + y^2$ so Eq. (11) becomes

$$\frac{\partial \rho}{\partial x} = \frac{\rho \cos(\phi)}{\rho} = \cos(\phi). \quad (12)$$

Similarly, using

$$\frac{\partial[\arctan(u)]}{\partial x} = \frac{1}{1+u^2} \frac{\partial u}{\partial x} \quad (13)$$

and Eq. (2b) we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial[\arctan(y/x)]}{\partial x} = \frac{1}{1+(y/x)^2} \frac{-y}{x^2} = \frac{-y}{x^2+y^2}. \quad (14)$$

Using $y = \rho \sin(\phi)$ and $\rho^2 = x^2 + y^2$, we now re-express Eq. (14) in terms of the new coordinates as

$$\frac{\partial \phi}{\partial x} = \frac{-\rho \sin(\phi)}{\rho^2} = \frac{-\sin(\phi)}{\rho}. \quad (15)$$

In similar fashion one can express the second derivatives [Eq. (9c) and Eq. (9d)], in terms of ρ and ϕ as

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{\sin^2(\phi)}{\rho} \quad (16)$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{2 \cos(\phi) \sin(\phi)}{\rho^2}. \quad (17)$$

If we now insert Eqs. (12), (15), (16) and (17) into Eq. (9) we obtain

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} = & \frac{\partial^2 f}{\partial \rho^2} \cos^2(\phi) - \frac{\partial^2 f}{\partial \phi \partial \rho} \frac{\sin(\phi) \cos(\phi)}{\rho} + \frac{\partial f}{\partial \rho} \frac{\sin^2(\phi)}{\rho} \\ & - \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\sin(\phi) \cos(\phi)}{\rho} + \frac{\partial^2 f}{\partial \phi^2} \frac{\sin^2(\phi)}{\rho^2} + \frac{\partial f}{\partial \phi} \frac{2 \cos(\phi) \sin(\phi)}{\rho^2}. \end{aligned} \quad (18)$$

So we have now expressed the first term of the Laplacian (acting on a function f) $\partial^2 f / \partial x^2$ in terms of cylindrical coordinates.

In a manner analogous to the procedure that we have just carried out one can also derive the result¹

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} = & \frac{\partial^2 f}{\partial \rho^2} \sin^2(\phi) + \frac{\partial^2 f}{\partial \phi \partial \rho} \frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial f}{\partial \rho} \frac{\cos^2(\phi)}{\rho} \\ & + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial^2 f}{\partial \phi^2} \frac{\cos^2(\phi)}{\rho^2} - \frac{\partial f}{\partial \phi} \frac{2\cos(\phi)\sin(\phi)}{\rho^2}. \end{aligned} \quad (19)$$

And, of course, we also trivially have

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z^2}. \quad (20)$$

Putting Eqs. (18) – (20) together then gives us the fairly simple result

$$\begin{aligned} \nabla^2 f = & \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ = & \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \quad (21)$$

Thus, in cylindrical coordinates the wave equation becomes

$$\frac{1}{c^2} \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial q}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 q}{\partial \phi^2} + \frac{\partial^2 q}{\partial z^2} \quad (22)$$

where now $q = q(\rho, \phi, z, t)$.

II. Separation of Variables

To look for separable solutions to the wave equation in cylindrical coordinates we posit a product solution

$$q(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t). \quad (23)$$

Substituting this into Eq. (22) produces

$$\frac{1}{c^2} R\Phi Z T'' = R''\Phi Z T + \frac{1}{\rho} R'\Phi Z T + \frac{1}{\rho^2} R\Phi'' Z T + R\Phi Z'' T. \quad (24)$$

¹ Notice that Eq. (19) is the same as Eq. (18) with $\sin(\phi) \rightarrow \cos(\phi)$ and $\cos(\phi) \rightarrow -\sin(\phi)$.

If, as in the case of Cartesian coordinates we now divide by q , which is now written as $R\Phi ZT$, this reduces to

$$\frac{1}{c^2} \frac{T''}{T} = \left(R'' + \frac{1}{\rho} R' \right) \frac{1}{R} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z}. \quad (25)$$

Next time we shall look at the four ordinary differential equations that are equivalent to Eq. (25). Three of these equations will be rather simple (again, essentially harmonic-oscillator equations), but one of them [for $R(\rho)$] will be a new equation. Its solutions are known as Bessel Functions.

Exercises

***20.1** Derive Eqs. (16) and (17). That is, using the chain rule show that $\frac{\partial^2 \rho}{\partial x^2} = \frac{\sin^2(\phi)}{\rho}$ and $\frac{\partial^2 \phi}{\partial x^2} = \frac{2 \cos(\phi) \sin(\phi)}{\rho^2}$.

***20.2** In these lecture notes we have derived the wave equation in cylindrical coordinates by thinking of a function f as a function of x , y , and z through the variables ρ , ϕ , and z , specifically $f = f[\rho(x, y), \phi(x, y), z(z)]$. We can just as easily think of the reverse situation where $f = f[x(\rho, \phi), y(\rho, \phi), z(z)]$. Consider such a function where f is only a function of the distance from the z axis, $\rho = \sqrt{x^2 + y^2}$. That is $f = f(\rho) = f(\sqrt{x^2 + y^2})$. For such a function show that $\partial f / \partial \phi = 0$

(a) directly in cylindrical coordinates $f = f(\rho)$ and

(b) using the chain rule starting with $f = f(\sqrt{x^2 + y^2})$.