## Harmonic Oscillations / Complex Numbers

Overview and Motivation: Probably the single most important problem in all of physics is the simple harmonic oscillator. It can be studied classically or quantum mechanically, with or without damping, and with or without a driving force. As we shall shortly see, an array of coupled oscillators is the physical basis of wave phenomena, the overarching subject of this course. In this lecture we will see how the differential equation that describes the simple harmonic oscillator naturally arises in a classical-mechanics setting. We will then look at several (equivalent) ways to write down the solutions to this differential equation.

Key Mathematics: We will gain some experience with the equation of motion of a classical harmonic oscillator, see a physics application of Taylor-series expansion, and review complex numbers.

I. Harmonic Oscillations

The freshman-physics concept of an (undamped, undriven) harmonic oscillator (HO) is something like the following picture, an object with mass m attached to an (immovable) wall with a spring with spring constant k_s. (There is no gravity here; only the spring provides any force on the object.)
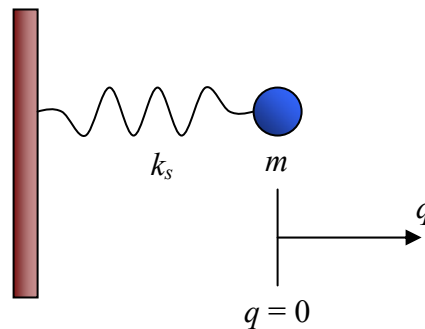
Note that the oscillator has only two parameters, the **mass** m and **spring constant** k_s. Assuming that the mass is constrained to move in the horizontal direction, its **displacement** q (away from equilibrium) as a function of **time** t can be written as

$$q(t) = B\sin(\widetilde{\omega}t + \phi), \tag{1}$$

where the **angular frequency** $\widetilde{\omega}$ depends upon the oscillator parameters m and k_s via the relation $\widetilde{\omega} = \sqrt{k_s/m}$. The **amplitude** B and the **phase** ϕ do not depend upon m and k_s, but rather depend upon the initial conditions (the initial displacement $q(0)$

and initial velocity $\frac{dq}{dt}(0) = \dot{q}(0)$ of the object). Note that the term **initial conditions** is a technical term that generally refers to the minimal specifications needed to describe the state of the system at time $t = 0$. You should remember that the angular frequency is related to the (plain old) **frequency** ν via $\tilde{\omega} = 2\pi\nu$ and that the frequency ν and **period** T are related via $\nu = 1/T$.

Now because the sine and cosine functions are really the same function (but with just a shift in their argument by $\pm\pi/2$) we can also write Eq. (1) as

$$q(t) = B\cos(\tilde{\omega}t + \psi), \tag{2}$$

where the amplitude B is the same, but the phase $\psi = \phi - \pi/2$ (for the same initial conditions). It probably is not obvious (yet), but we can also write Eq. (1) as

$$q(t) = D\cos(\tilde{\omega}t) + E\sin(\tilde{\omega}t), \tag{3}$$

where the amplitudes D and E depend upon the initial conditions of the oscillator. Note that the term **harmonic function** simply means a sine or cosine function. Note also that all three forms of the displacement each have two parameters that depend upon the initial conditions.

II. Classical Origin of Harmonic Oscillations
A. The Harmonic Potential
The harmonic motion of the classical oscillator illustrated above comes about because of the nature of the spring force (which is the only force and thus the net force) on the mass, which can be written as

$$F_s(q) = -k_s q. \tag{4}$$

Because the spring force is conservative, F_s can be derived from a **potential energy function** $V(q)$ via the general (in 1D) relationship

$$F(q) = -\frac{dV(q)}{dq}. \tag{5}$$

A potential energy function for the spring that gives rise to Eq. (4) for the spring force is

$$V_s(q) = \frac{1}{2} k_s q^2 \tag{6}$$

(do the math!).

Let's see how the potential energy represented by Eq. (6) arises in a rather general way. Let's consider an object constrained to move in one dimension (like the oscillator above). However, in this case all we know is that the potential energy function has at least one local minimum, as illustrated in the following graph of $V(q)$ vs q.
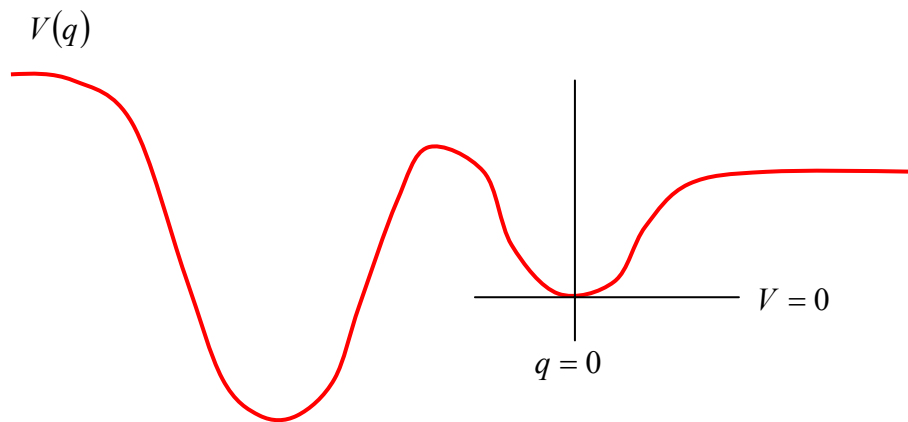
Let's now assume that the mass is located near the potential energy minimum on the right side of the graph and that its energy is such that it does not move very far away from this minimum. Just because we can, let's also assume that this minimum defines where $q = 0$ and $V = 0$, as shown in the picture.

Now here is where some math comes in. If the mass does not move very far away from $q = 0$ then we are only interested in motion for small q. Let's see what the potential energy function looks like in this case. For small q it makes sense to expand the function $V(q)$ in a **Taylor series**

$$V(q) = V(0) + V'(0)q + \frac{1}{2}V''(0)q^2 + \frac{1}{6}V'''(0)q^3 + \dots, \tag{7}$$

where, e.g., $V'(0)$ is the first derivative of V evaluated at $q = 0$. Now the rhs of Eq. (7) has an infinite number of terms and so is generally quite complicated, but often only one of these terms is important. Let's look at each term in order. The first term $V(0)$ is zero because we defined the potential energy at this minimum to be zero. So far so good. The next term is also zero because the slope of the function $V(q)$ is also zero at

the minimum. The next term is not zero and neither, in general, are any of the others. However, if q is small enough, these other terms are negligible compared to the third (quadratic) term.[1] Thus, if the object's motion is sufficiently close to the minimum in potential energy, then we have

$$V(q) \approx \frac{1}{2}V''(0)q^2 .$$ (8)

So this is pretty cool. Even though we have no idea what the potential energy function is like, except that it has a minimum somewhere, we see that if the object is moving sufficiently close to that minimum, then the potential energy is the same as for a mass attached to a spring where the effective spring constant k_s is simply the curvature V'' evaluated at the potential energy minimum ($q = 0$), $V''(0)$.[2] Equation (8) is known as the **harmonic approximation** to the potential $V(q)$ (near the minimum).

B. Harmonic Oscillator Equation of Motion
OK, so we see that the potential energy near a minimum is equivalent to the potential energy of an harmonic oscillator. If we happen to know how an harmonic oscillator behaves, then we know how our mass will behave near the minimum. But let's assume for the moment that we know nothing about the specifics of a harmonic oscillator. Where do we go from here to determine the motion of the mass near the minimum? Well, as in most classical mechanics problems we use **Newton's second law**, which is generally written (for 1D motion) as

$$a = \frac{F_{net}}{m} ,$$ (9)

where a is the acceleration of the object and F_{net} is the **net force** (i.e., sum of all the forces) on the object.

In the case at hand, in which the object's acceleration is $d^2q/dt^2 = \ddot{q}$ and the net force comes only from the potential energy (near the minimum) Eq. (9) becomes

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[1] In physics we are often interested in comparing terms in expressions such as Eq. (7), and we often use the comparators $\gg$ and $\ll$ to compare these terms. These two comparators actually mean $< \frac{1}{10} \times$ and $> 10\times$, respectively. For example, $a \gg b$ indicates that $a > 10b$.

[2] Exceptions can occur. If the potential minimum is so flat that $V''(0) = 0$ then the third term will be zero and it will be some higher-order term(s) that determine(s) the motion near the potential-energy minimum. The object will oscillate, but it will not oscillate harmonically.

$$\ddot{q} = -\frac{V''(0)}{m}q ,$$ (10)

or, identifying $V''(0)$ as the spring constant k_s we can write Eq. (10) as

$$\ddot{q} + \frac{k_s}{m}q = 0 .$$ (11)

Equation (11) is known as the **equation of motion** for an harmonic oscillator. Generally, the equation of motion for an object is the specific application of Newton's second law to that object. Also quite generally, the classical equation of motion is a differential equation such as Eq. (11). As we shall shortly see, Eq. (11) along with the initial conditions $q(0)$ and $\dot{q}(0)$ completely specify the motion of the object near the potential energy minimum. Note that two initial conditions are needed because Eq. (11) is a second-order equation.

Let's take few seconds to classify this differential equation. It is **second order** because the highest derivative is second order. It is **ordinary** because the derivatives are only with respect to one variable (t). It is **homogeneous** because q or its derivatives appear in every term, and it is **linear** because q and its derivatives appear separately and linearly in each term (where they appear). An major consequence of the homogeneity and linearity is that linear combinations of solutions to Eq. (11) are also solutions. This fact will be utilized extensively throughout this course.

C. HO Initial Value Problem
The solution to Eq. (11) can be written most generally as either Eq. (1), Eq. (2), or Eq. (3) (where $\tilde{\omega} = \sqrt{k_s/m}$). Let's consider Eq. (3)

$$q(t) = D\cos(\tilde{\omega}t) + E\sin(\tilde{\omega}t),$$ (3)

and see that indeed the constants D and E are determined by the initial conditions. By applying the initial conditions we are solving the **initial value problem** (IVP) for the HO. Setting $t = 0$ in Eq. (3) we have

$$q(0) = D$$ (12)

Similarly, taking the time derivative of Eq. (3)

$$\dot{q}(t) = -\tilde{\omega}D\sin(\tilde{\omega}t) + \tilde{\omega}E\cos(\tilde{\omega}t)$$ (13)

and again setting $t = 0$ gives us

$$\dot{q}(0) = \tilde{\omega} E . \tag{14}$$

Hence, the general solution to the (undamped, undriven) harmonic oscillator problem can be written as

$$q(t) = q(0)\cos(\tilde{\omega}t) + \frac{\dot{q}(0)}{\tilde{\omega}}\sin(\tilde{\omega}t). \tag{15}$$

To summarize, for a given set of initial conditions $q(0)$ and $\dot{q}(0)$, Eq. (15) is the solution to Eq. (11), the harmonic oscillator equation of motion.

III. Complex Numbers
In our discussion so far, all quantities are real number (with possibly some units, such as $q = 3$ cm). However, when dealing with harmonic oscillators and wave phenomena, it is often useful to make use of complex numbers, so let's briefly review some facts regarding complex numbers.

The key definition associated with complex numbers is the square root of -1, known as i. That is, $i = \sqrt{-1}$. From this all else follows.

Any complex number z can always be represented in the form

$$z = x + iy , \tag{16}$$

where x and y are both *real* numbers. Common notations for the real and imaginary parts of z are $x = \text{Re}(z)$ and $y = \text{Im}(z)$. It is also often convenient to represent a complex number as a point in the complex plane, in which the x coordinate is the real part of z and the y coordinate is the imaginary part of z, as illustrated by the picture on the following page.

As this can be inferred from this picture, we can also use polar coordinates r and θ to represent a complex number as

$$z = r\cos(\theta) + ir\sin(\theta) = r[\cos(\theta) + i\sin(\theta)]. \tag{17}$$

Using the infamous **Euler relation** (which you should never forget!)

Imaginary Axis (y)

$z = x + i\,y$

r

θ

y

x

Real Axis (x)
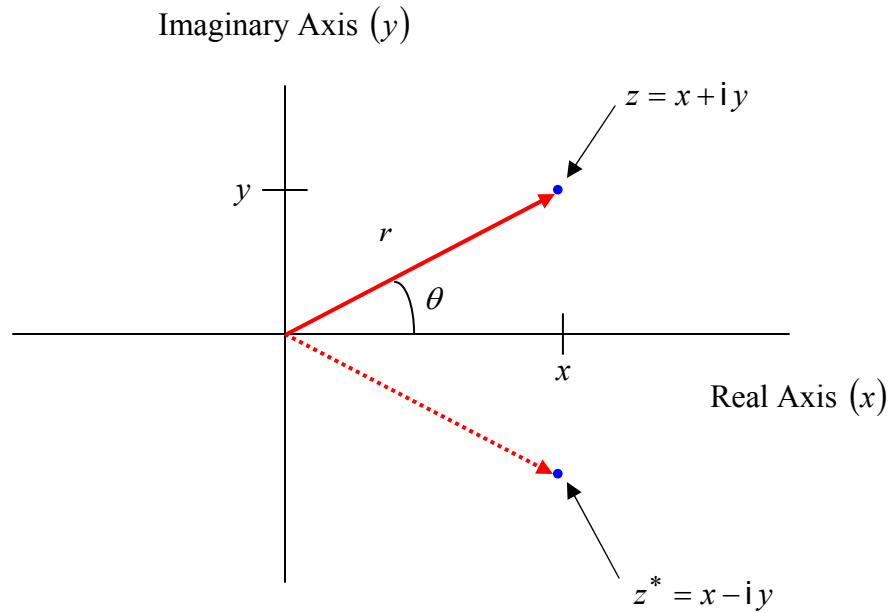
$z^* = x - i\,y$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{18}$$

we see that a complex number can also be written as

$$z = re^{i\theta}. \tag{19}$$

The last important definition associated with complex numbers is the complex conjugate of z defined as

$$z^* = x - iy. \tag{20}$$

As is apparent in the diagram above, this amounts to a reflection about the real (x) axis. Note the following relationships:

$$\frac{z + z^*}{2} = x = \mathrm{Re}(z), \tag{21}$$

$$\frac{z - z^*}{2i} = y = \mathrm{Im}(z),$$

and

$$zz^* = x^2 + y^2 = r^2 \,. \tag{22}$$

Note also the absolute value of a complex number, which is equal to r, is given by

$$|z| = r = \sqrt{zz^*} \,. \tag{23}$$

IV. Complex Representations of Harmonic Oscillator Solutions

Because the HO equation of motion [Eq. (11)] is linear and homogeneous, linear combinations of solutions are also solutions. These linear combinations can be complex combinations. For example, because $\cos(\tilde\omega t)$ and $\sin(\tilde\omega t)$ are both solutions to Eq. (11) (we are not worrying about any particular initial conditions at the moment), another solution to Eq. (11) is the complex linear combination

$$q(t) = \alpha\cos(\tilde\omega t) + i\alpha\sin(\tilde\omega t) = \alpha e^{i\tilde\omega t} \,, \tag{24}$$

where α is some complex number.

So what is the point here? Well, as we shall see as we go along, it is often convenient to work with complex representations of solutions to the harmonic oscillator equation of motion (or to the wave equation that we will be dealing with later). So what does it mean to have a complex displacement? Nothing, really – a displacement cannot be complex, it is indeed a real quantity. So if we are dealing with a complex solution, what do we do to get a physical (real) answer? There are at least three approaches:

(1) The first approach is to, up front, make the solution manifestly real. For example, let's say you want to work with the general complex solution

$$q(t) = \alpha e^{i\tilde\omega t} + \beta e^{-i\tilde\omega t} \,, \tag{25}$$

where α and β are complex numbers. You can impose the condition $\beta = \alpha^*$, which results in $q(t)$ being real.

(2) Another approach is to simply work with the complex solution until you need to doing something such as impose the initial conditions. Then, for example, if we are working with the form in Eq. (25), the initial conditions might be something like $\mathrm{Re}[q(0)] = A$, $\mathrm{Im}[q(0)] = 0$, $\mathrm{Re}[\dot q(0)] = B$, and $\mathrm{Im}[\dot q(0)] = 0$. These four conditions would then determine the four unknowns, the real and imaginary parts of α and β (and again would result in $q(t)$ being real).

(3) A third approach is to simply use the real part of a solution when something like the initial conditions are needed, and then impose the initial conditions. (Alternatively, because the imaginary part of the solution is also a real number, you could use the imaginary part of the solution if you were so inclined.)

Note that these different approaches work because the harmonic oscillator equation of motion is real and linear. We will not worry about the details of this at the moment, but if you are interested you can read more about this on pp. 9-10 of Dr. Torre's text, *Foundations of Wave Phenomena* (FWP).

Exercises

*2.1 Show that Eqs. (1) and (2) are solutions to Eq. (11), the equation of motion for the harmonic oscillator.

*2.2 Assuming the form of Eq. (1) for the solution to Eq. (11), find the values of B and ϕ in terms of the initial conditions $q(0)$ and $\dot{q}(0)$.

*2.3 Assuming the form of Eq. (2) for the solution to Eq. (11), find the values of B and ψ in terms of the initial conditions $q(0)$ and $\dot{q}(0)$. [This exercise along with Exercise 2.2 shows that Eqs. (1) and (2) are equivalent.]

2.4 **Quadratic approximation to a particular potential energy. Consider the potential energy function $V(q) = 1 - \cos(\kappa q)$, where κ is some positive constant.
(a) Carefully (!) plot (using a computer math program such as Mathcad, Maple, etc.) this function for $-\pi < \kappa q < \pi$ (For simplicity in making this and the following graph you may set κ to 1) **Make sure that your axes are carefully labeled on this and all other graphs**.
(b) Find the force $F(q)$ associated with this potential-energy function (for arbitrary positive κ).
(c) Taylor series expand the potential-energy function about the point $q = 0$; keep all terms up to the q^4 term.
(d) What is the effective spring constant k_s associated with this potential-energy function?
(e) What condition on $|q|$ is necessary so that the q^4 term (in the Taylor series expansion) is much smaller ($<<$) that the (harmonic) q^2 term? (See footnote 1 on p. 4.)
(f) Replot the original function $V(q) = 1 - \cos(\kappa q)$ and the **harmonic** approximation to this function from $-\pi < \kappa q < \pi$. Based on this graph, for what values of κq do you expect the harmonic approximation to be valid? How does this compare to your answer in (e)?

**2.5 The Euler relation. The Euler relation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ can be shown to be true by comparing, term by term, the Taylor series of both sides of the relation. Calculate the first five Taylor series terms (about $\theta = 0$, so these will be the θ^0, θ^1, θ^2, θ^3, θ^4, and θ^5 terms) of each side of the equation and show that they are equivalent.

*2.6 Write the following complex expressions in the form $x + iy$
(a) $5e^{i(\pi/4)}$
(b) $(5 - 4i)(3 + 2i)$
(c) $\dfrac{1}{6 - 7i}$
(d) $2e^{i\pi} \cdot 16e^{-i(\pi/2)}$

*2.7 Write the expressions in Exercise 2.6 in the form $Ae^{i\theta}$, where A and θ are both real.

*2.8 Find the real part of $q(t) = \alpha e^{i\widetilde{\omega}t} + \alpha^* e^{-i\widetilde{\omega}t}$. Here α is complex. Write your solution in terms of $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, $\sin(\widetilde{\omega}t)$, and $\cos(\widetilde{\omega}t)$.

*2.9 Complex solution and initial conditions
(a) Show that $q(t) = \alpha e^{i\widetilde{\omega}t} + \beta e^{-i\widetilde{\omega}t}$ is a solution to Eq. (11), the harmonic oscillator equation of motion.
(b) Assuming that α and β are complex, find α and β in terms of the initial conditions $q(0)$ and $\dot{q}(0)$. (Do this using one of the three methods discussed on p. 8.)