Separation of Variables in Cartesian Coordinates

Overview and Motivation: Today we begin a more in-depth look at the 3D wave equation. We introduce a technique for finding solutions to partial differential equations that is known as separation of variables. We first do this for the wave equation written in Cartesian coordinates. In subsequent lectures we will do the same using cylindrical and spherical-polar coordinates. The technique is applicable not only to the wave equation, but to a wide variety of partial differential equations that are important in physics.

Key Mathematics: The technique of separation of variables!

I. Separable Solutions

Last time we introduced the 3D wave equation, which can be written in Cartesian coordinates as

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}=\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}+\frac{\partial^2 q}{\partial z^2}, \tag{1}$$

and we spent some time looking at the plane-wave solutions

$$q^+_{k_x,k_y,k_z}(x,y,z,t)=A^+e^{i(k_x x+k_y y+k_z z-\omega t)}, \tag{2a}$$

$$q^-_{k_x,k_y,k_z}(x,y,z,t)=A^-e^{i(k_x x+k_y y+k_z z+\omega t)}, \tag{2b}$$

where $\omega=c|\mathbf{k}|$ is the familiar dispersion relation. Equation (2a) describes a wave that travels in the $\mathbf{k}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}}$ direction with wavelength $\lambda=2\pi/|\mathbf{k}|$, while Eq. (2b) describes a similar wave that travels in the $-\mathbf{k}$ direction.

Today we introduce **separation of variables**, a technique that leads to **separable** (also known as **product**) solutions. A separable solution is of the form

$$q(x,y,z,t)=X(x)Y(y)Z(z)T(t). \tag{3}$$

That is, the function $q(x,y,z,t)$ is a product of the functions $X(x)$, $Y(y)$, $Z(z)$, and $T(t)$.

So how do we find these solutions? Well, let's just substitute the rhs of Eq. (3) into Eq. (1) and see what happens. With this substitution Eq. (1) becomes

$$\frac{1}{c^2} X\,Y\,Z\,T'' = X''YZT + XY''ZT + XYZ''T\ , \tag{4}$$

where we have suppressed the independent variables and the double prime indicates the second derivative of the function with respect to its argument. Note that because each of these functions is a function of only one variable, all derivatives are now ordinary derivatives. Equation (4) looks kind of ugly, but notice what happens if we divide Eq. (4) by $q(x,y,z,t) = X(x)Y(y)Z(z)T(t)$. We then get

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{5}$$

Now here is where the magic happens. Notice that each term is only a function of its associated independent variable. So for example, if we vary t only the term on the lhs of Eq. (5) can vary. But, because none of the other terms depends upon t, the rhs cannot vary, which means that the lhs cannot vary, which means that T''/T is *independent* of t! Following the same logic for each of the other terms means that each term is constant. So we can write

$$\frac{1}{c^2}\frac{T''}{T} = \alpha\ , \qquad \frac{X''}{X} = \beta\ , \qquad \frac{Y''}{Y} = \gamma\ , \qquad \frac{Z''}{Z} = \delta \qquad\qquad \text{(6a) – (6d)}$$

where the constants α, β, γ, and δ are known as **separation constants**. However, only three of them are independent because Equation (5) tells us that

$$\alpha = \beta + \gamma + \delta\ . \tag{7}$$

Notice, by demanding a product solution the partial differential wave equation has transformed into four ordinary differential equations. We can easily solve all of these equations. Let's start with the equation for $T(t)$,

$$T''(t) - \alpha\, c^2 T(t) = 0\,. \tag{8}$$

This is essentially the harmonic oscillator equation (if α is real and <0), and so it has the two, linearly independent solutions

$$T(t) = T_0 e^{\pm c\sqrt{\alpha}\, t}\ , \tag{9a}$$

where T_0 is some undetermined constant. Similarly, the X, Y, and Z equations have solutions

$$X(t) = X_0 e^{\pm\sqrt{\beta}\,x}, \tag{9b}$$

$$Y(t) = Y_0 e^{\pm\sqrt{\gamma}\,y}, \tag{9c}$$

$$Z(t) = Z_0 e^{\pm\sqrt{\delta}\,z}, \tag{9d}$$

Putting this all together gives us a solution of the form

$$q(x,y,z,t) = X_0\,Y_0\,Z_0\,T_0\;e^{\pm\sqrt{\beta}\,x}\,e^{\pm\sqrt{\gamma}\,y}\,e^{\pm\sqrt{\delta}\,z}\,e^{\pm c\sqrt{\alpha}\,t}. \tag{10}$$

II. Some Physics Added In

Notice that the solution in Eq. (10) can have all sorts of behavior. For example, if the constant β is real and positive, then the x dependent part of the solution can either exponentially increase or decrease with increasing x (depending upon the sign in the exponent). This points out the fact that separation-of-variables solutions often give you more than you really need in any given situation. For example, if we are looking for solutions to the wave equation that are physically meaningful as $x \to \infty$, we are (probably) not going to be interested in a solution that exponentially increases with increasing x.

With that in mind, let's see what the constraints on the separation constants must be if we are only interested in purely oscillatory solutions (in all independent variables). Let's consider the x dependent part of the solution. If $e^{\pm\sqrt{\beta}\,x}$ is to only oscillate then it must be of the form $e^{\pm ik_x x}$, where k_x is real. Then, if we write a generally complex β as $\beta = \beta_1 + i\beta_2$ we have $\sqrt{\beta_1 + i\beta_2} = ik_x$ which implies $\beta_1 + i\beta_2 = -k_x^2$. Thus $\beta_2 = 0$ and $\beta_1 = -k_x^2$. That is, β is real and negative. Similarly, γ and δ must be real and negative. Eq. (7) then implies that α is also real and negative. So if we rename the constants as

$$\beta = -k_x^2, \qquad \gamma = -k_y^2, \qquad \delta = -k_z^2, \qquad \alpha = -\omega^2/c^2, \tag{11a) – (11d}$$

(where k_x, k_y, k_z, and ω are all real) then Eq. (10) can be rewritten as

$$q(x,y,z,t) = A\,e^{\pm ik_x x}\,e^{\pm ik_y y}\,e^{\pm ik_z z}\,e^{\pm i\omega t} \tag{12}$$

where $A = X_0\,Y_0\,Z_0\,T_0$ is an arbitrary constant. Eq. (7) can now be written as $\omega^2/c^2 = k_x^2 + k_y^2 + k_z^2$, which is the well known dispersion relation! Now k_x, k_y, and k_z

can be positive or negative, so we don't need the $\pm$ on the spatial-function exponents, and so we are left with two linearly independent solutions

$$q^{+}_{k_x,k_y,k_z}(x,y,z,t) = A^{+}e^{i(k_x x + k_y y + k_z z - \omega t)}, \tag{13a}$$

$$q^{-}_{k_x,k_y,k_z}(x,y,z,t) = A^{-}e^{i(k_x x + k_y y + k_z z + \omega t)}, \tag{13b}$$

which are the plane-wave solutions of Eq. (2)! So, by demanding that the separable solutions to the wave equation oscillate, we have ended up with the plane-wave solutions that we discussed in the last lecture.

III. Utility of the Separable Solutions

Because they are product solutions, the separable solutions are pretty specialized. That is, there are many solutions to the wave equation that *cannot* be written as a product solution. And so you may ask, what is the usefulness of these solutions? There are two parts to the answer. The first is that sometimes these solutions have intrinsic interest. For example, in the case of the time-dependent Schrödinger equation the separable solutions (with some appropriate physics thrown in) are the energy eigenstates of the system. The second, and perhaps more important, part of the answer is that a basis can be constructed from the set of separable solutions that can be used to represent any solution.

Let's think about this statement with regards to the wave equation. Let's first consider the 1D wave equation. Back in the Lecture 17 notes we solved the initial-value problem on the interval $-\infty < x < \infty$. The solution to that problem can be written as

$$q(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-\infty}^{\infty} dk \left\{ \left[\hat{a}(k) + \frac{\hat{b}(k)}{i|k|c} \right] e^{i(kx+\omega t)} + \left[\hat{a}(k) - \frac{\hat{b}(k)}{i|k|c} \right] e^{i(kx-\omega t)} \right\}, \tag{14}$$

where $\hat{a}(k)$ and $\hat{b}(k)$ are the Fourier transforms of the initial conditions, and $\omega = c|k|$ is the dispersion relation. Notice that in Eq. (14) we have written the general solution $q(x,t)$ as a linear combination of the functions $e^{i(kx+\omega t)}$ and $e^{i(kx-\omega t)}$. Although we did not go through the separation of variables procedure for the 1D wave equation to produce these solutions, by writing these functions as $e^{ikx}e^{i\omega t}$ and $e^{ikx}e^{-i\omega t}$, we readily see that they are indeed product solutions.

Similarly, for the 3D wave equation we can write the solution to the initial value problem as a linear combination of the separable (plane-wave) solutions [Eq. (13)]

$$q(\mathbf{r},t)=\frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int d^3k\left\{\left[\hat{a}(\mathbf{k})+\frac{\hat{b}(\mathbf{k})}{i|\mathbf{k}|c}\right]e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}+\left[\hat{a}(\mathbf{k})-\frac{\hat{b}(\mathbf{k})}{i|\mathbf{k}|c}\right]e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\}, \tag{15}$$

where $\mathbf{k}\cdot\mathbf{r}=k_x x+k_y y+k_z z$, $\int d^3k=\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z$, $\omega=c\sqrt{k_x^2+k_y^2+k_z^2}$, and $\hat{a}(\mathbf{k})$ and $\hat{b}(\mathbf{k})$ are the 3D Fourier transforms of the initial conditions.

In the coming lectures we will look at separable solutions in cylindrical and spherical-polar coordinates. While we will not do a lot with the general solution written as a linear combination of these solutions, we should keep in mind that, in principle, it can be done.

Exercises

****19.1 Heat Equation**. A partial differential equation known as the heat equation, which is used to describe heat or temperature flow in an object, is given by
$\nabla^2 q=\frac{1}{\lambda}\frac{\partial q}{\partial t}$, where $\lambda>0$.

(**a**) If there is no y or z dependence to the problem, write down a simplified version of this equation.

(**b**) Use separation of variables to find two ordinary differential equations, one in x and one in t. What are the orders of these two equations? Are they linear or nonlinear?

(**c**) Find the general solutions to the two ordinary differential equations. Thus write down the general separable solutions to the heat equation. [Note: there should be *two* linearly independent solutions.]

(**d**) In solving the heat equation, you should have found that $X''(x)/X(x)=C$, where C is some constant. Assume that this constant is real. If $C>0$, describe the behavior of the solutions vs x and t. In what ways are these solutions like solutions to the wave equation? In what ways are they different?

(**e**) If $C<0$, describe the behavior of the solutions vs x and t. In what ways are these solutions like solutions to the wave equation? In what ways are they different?

19.2 Normal Modes Inside a Rectangular Room: Here we consider sound waves inside a rectangular room with one corner located at the origin and occupying space as indicated: $0 < x < L_x$, $0 < y < L_y$, $0 < z < L_z$. While we could deal with displacement (a vector field), it is easier to think about the pressure which is also governed by the wave equation, but is a scalar field. The boundary conditions on the pressure $p(x, y, z, t)$ are $\frac{\partial p}{\partial x} = 0$ at $x = 0$ and $x = L_x$, $\frac{\partial p}{\partial y} = 0$ at $y = 0$ and $y = L_y$, and $\frac{\partial p}{\partial z} = 0$ at $z = 0$ and $z = L_z$.

(**a**) As discussed in the notes, the separable *traveling-wave* solutions to the wave equation can be written (with q replaced by p) as $p^{\pm}_{k_x,k_y,k_z}(x, y, z, t) = A^{\pm}\, e^{i(k_x x + k_y y + k_z z \mp \omega t)}$ where $\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. However, we could have expressed the separable solutions as the *standing-wave* solutions

$$
\begin{aligned}
p^{\pm}_{k_x k_y k_z}(x, y, z, t) = &\left[A_x \sin(k_x x) + B_x \cos(k_x x)\right]\left[A_y \sin(k_y y) + B_y \cos(k_y y)\right] \\
&\times\left[A_z \sin(k_z z) + B_z \cos(k_z z)\right] e^{\pm i\omega t}
\end{aligned}
$$

Starting with this standing-wave form, find the specific solutions that satisfy the above boundary conditions. Note, these solutions should be labeled with three integers; call these integers n_x, n_y, and n_z. Express the allowed frequencies ω as a function of n_x, n_y, and n_z, the wave velocity c, and the dimensions of the room.

(**b**) The solutions that you found in (a) describe the *normal modes* for sound waves in a typical room where L_x and L_y are the widths and L_z is the height of the room. Consider a room of dimensions 10 ft × 11 ft × 8 ft. Calculate the frequencies ($f(n_x, n_y, n_z) = \omega(n_x, n_y, n_z)/2\pi$) of the 10 lowest-frequency room modes, and order them from lowest to highest frequency. For the speed of sound you may use 330 m/s.

(**c**) These lowest-frequency modes often wreak havoc with sound reproduction because they serve to amplify, through resonance, reproduced frequencies near their (resonance) frequencies. This is especially troublesome if two (or more) modes have frequencies that are close together (degenerate). For example a 10' × 10' × 10' room would have its three lowest modes at exactly the same frequency. Are any of the modes you calculated in (b) degenerate or nearly so?