3D Wave Equation and Plane Waves / 3D Differential Operators

Overview and Motivation: We now extend the wave equation to three-dimensional space and look at some basic solutions to the 3D wave equation, which are known as plane waves. Although we will not discuss it, plane waves can be used as a basis for any solutions to the 3D wave equation, much as harmonic traveling waves can be used as a basis for solutions to the 1D wave equation. We then look at the gradient and Laplacian, which are linear differential operators that act on a scalar field. We also touch on the divergence, which operates on a vector field.

Key Mathematics: The 3D wave equation, plane waves, fields, and several 3D differential operators.

I. The 3D Wave Equation and Plane Waves

Before we introduce the 3D wave equation, let's think a bit about the 1D wave equation,

$$\frac{\partial^2 q}{\partial t^2} = c^2 \frac{\partial^2 q}{\partial x^2} . \tag{1}$$

Some of the simplest solutions to Eq. (1) are the harmonic, traveling-wave solutions

$$q_k^+(x,t) = A\, e^{i(kx-\omega t)}, \tag{2a}$$

$$q_k^-(x,t) = B\, e^{i(kx+\omega t)}, \tag{2b}$$

where, without loss of generality, we can assume that $\omega = |ck| > 0$.[1] Let's think about these solutions as a function of the wave vector k. First, we should remember that k is related to the wavelength via $k = 2\pi/\lambda$. Let's now specifically think about the solution $q_k^+(x,t)$. For this solution, if $k > 0$ then the wave propagates in the $+x$ direction, and if $k < 0$, then the wave propagates in the $-x$ direction. Thus, in either case, the wave propagates *in the direction of* k. Similarly, for the solution $q_k^-(x,t)$ the wave propagates in the direction opposite to the direction of k.

We now introduce the 3D wave equation and discuss solutions that are analogous to those in Eq. (2) for the 1D equation. The 3D extension of Eq. (1) can be obtained by adding two more spatial-derivative terms, yielding

[1] If we assume $\omega < 0$, then the two $\omega > 0$ solutions just map into each other.

$$\frac{\partial^2 q}{\partial t^2} = c^2\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}\right) \tag{3}$$

where now $q = q(x, y, z, t)$ and x, y, and z are standard Cartesian coordinates. This equation can be used to describe, for example, the propagation of sound waves in a fluid. In that case q represents the longitudinal displacement of the fluid as the wave propagates through it.

The 3D solutions to Eq. (3) that are analogous to the 1D solutions expressed by Eq. (2) can be written as

$$q^+_{k_x,k_y,k_z}(x, y, z, t) = A\,e^{i(k_x x + k_y y + k_z z - \omega t)}, \tag{4a}$$

$$q^-_{k_x,k_y,k_z}(x, y, z, t) = B\,e^{i(k_x x + k_y y + k_z z + \omega t)} \tag{4b}$$

As you may suspect, the wave equation determines a relationship between the set $\{k_x, k_y, k_z\}$ and the frequency ω. Substituting either Eq. (4a) or (4b) into Eq. (3) yields

$$\omega^2 = c^2\left(k_x^2 + k_y^2 + k_z^2\right). \tag{5}$$

As above, we can assume $\omega > 0$, which gives

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}\,,$$

the dispersion relation for the Eq. (4) solutions to the 3D wave equation.

The solutions in Eq. (4) can be also written in a more elegant form. If we define the 3D wave vector

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}, \tag{6}$$

and use the Cartesian-coordinate form of the position vector

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}, \tag{7}$$

then we see that we can rewrite Eq. (4) as

$$q^+_{\mathbf{k}}(\mathbf{r}, t) = A\,e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{8a}$$

$$q_{\mathbf{k}}^{-}(\mathbf{r},t) = B\,e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}, \tag{8b}$$

where $\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$ is the standard dot product of two vectors. The dispersion relation can then also be written more compactly as

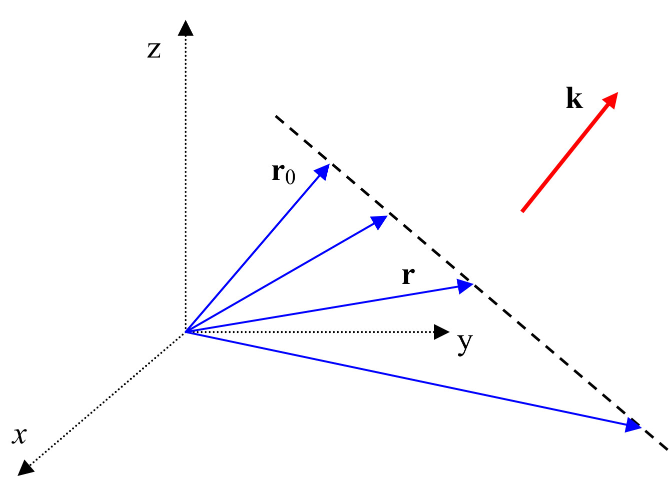
$$\omega = c|\mathbf{k}|. \tag{9}$$

It is also the case that the wavelength λ is related to $\mathbf{k}$ via $|\mathbf{k}| = 2\pi/\lambda$.

Analogous to the discussion about the direction of the 1D solutions, the wave in Eq. (8a) propagates in the $+\mathbf{k}$ direction while the wave in Eq. (8b) propagates in the $-\mathbf{k}$ direction. This is why one usually sees the form in Eq. (8a): the wave simply propagates in the direction that $\mathbf{k}$ points in this case.

These propagating solutions in Eq. (8) are known as **plane waves**. Why is that, you may ask? It is because at any given time the planes perpendicular to the propagation direction have the same value of the displacement of q.

Let's see that this is so. Consider the following picture.



Keep in mind that the wave vector $\mathbf{k}$ is a fixed quantity (for a given plane wave); its direction is indicated in the figure. The dotted line in the picture represents a plane that is perpendicular to $\mathbf{k}$ and passes through the point in space defined by the vector $\mathbf{r}$. Now consider the dot product

$$\mathbf{k}\cdot\mathbf{r} = |\mathbf{k}|\cdot|\mathbf{r}|\cos(\theta) \tag{10}$$

This is simply equal to $|\mathbf{k}|\cdot|\mathbf{r}_0|$, where $\mathbf{r}_0$ is the position vector in the plane that is parallel to $\mathbf{k}$. Furthermore, for *any* position vector in the plane the dot product with $\mathbf{k}$ has this same value. That is, for any vector $\mathbf{r}$ in the plane $\mathbf{k}\cdot\mathbf{r}$ is constant. Thus, the plane-wave function $Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ has the same value for all points $\mathbf{r}$ in the plane.

A simple example of a plane wave is one that is propagating in the z direction. In that case the q^+ plane wave is $q^+_{0,0,k_z}(z,t) = Ae^{i(k_z z-\omega t)}$. Notice that this wave does not depend upon x or y. That is, for a given value of z, the wave has the same displacement for all values of x and y. That is, it has the same displacement for any point on a plane with the same value of z.

II. Some 3D Linear Differential Operators

A. The Laplacian

The combination of spatial derivatives on the rhs of Eq. (3),

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \tag{11}$$

is the Cartesian-coordinate version of the linear differential operator know as the **Laplacian**, generically designated as either Δ or ∇^2 (**del squared**). The del-squared representation is often used because the Laplacian can be though of as two successive (although different) applications of the differential expression that is simply known as **del** , which is represented by the symbol ∇.[2]

In Cartesian coordinates

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial x}\hat{\mathbf{z}}. \tag{12}$$

The Laplacian ∇^2 can thus be written in Cartesian coordinates as

[2] As we shall see below, ∇ can be used in the representation of several operators. It is thus probably best not to think of ∇ itself as an operator.

$$\nabla\cdot\nabla=\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial x}\hat{\mathbf{z}}\right)\cdot\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial x}\hat{\mathbf{z}}\right). \tag{13}$$

We now consider the application of ∇^2 to a function $f(x,y,z)$, but we do it "one del at a time." That is, writing $\nabla^2 f(x,y,z)$ as $\nabla\cdot[\nabla f(x,y,z)]$, we first consider the piece $\nabla f(x,y,z)$. Afterwards we look at $\nabla\cdot[\nabla f(x,y,z)]$, which we usually simply write as $\nabla\cdot\nabla f(x,y,z)$.

However, before we do this, let's make sure that we understand the concept of a field. A **field** is simply a mathematical quantity that has a value assigned to each point in space. The function $f(x,y,z)$ is known as a **scalar field**, because $f(x,y,z)$ assigns a scalar to each point in space. A **vector field** is a function that assigns a vector to each point in space. An electric field is an example of a vector field.

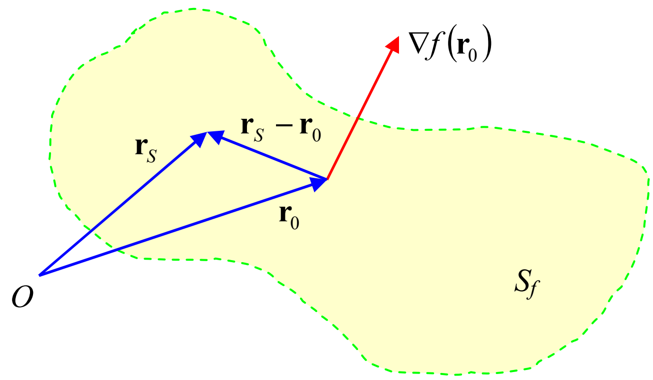
B. The Gradient

The quantity ∇f is know as the **gradient** of f. Let's take a closer look at ∇f. Applying Eq. (12) (the Cartesian-coordinate version of ∇) to $f(x,y,z)$ produces

$$\nabla f(x,y,z)=\frac{\partial f}{\partial x}\hat{\mathbf{x}}+\frac{\partial f}{\partial y}\hat{\mathbf{y}}+\frac{\partial f}{\partial z}\hat{\mathbf{z}}. \tag{14}$$

Although not explicitly shown, each term on the rhs is a function of x, y, and z. Thus ∇f is a vector field because it assigns a vector to each point in space. Simply put, the gradient of a scalar field is a vector field.

An important property of $\nabla f(\mathbf{r})$ is that $\nabla f(\mathbf{r})$ is perpendicular to the surface of constant f that contains $\mathbf{r}$ (where $\mathbf{r}$ is any position vector). Let's use a bit of vector calculus to show this. Consider the picture below. The surface S_f is the surface of constant f that contains (the end of) $\mathbf{r}_0$. The vector $\mathbf{r}_S$ is also assumed to lie on this surface. So we wish to show that $\nabla f(\mathbf{r}_0)$ is perpendicular to S_f at $\mathbf{r}_0$.



We now assume that $\mathbf{r}_S$ is close enough to $\mathbf{r}_0$ that we can write the function $f(\mathbf{r}_S) = f(x_S, y_S, z_S)$ as a Taylor series expanded about the point $\mathbf{r}_0 = (x_0, y_0, z_0)$,

$$f(x_S, y_S, z_S) = f(x_0, y_0, z_0) + \left.\frac{\partial f}{\partial x}\right|_{\mathbf{r}_0}(x_S - x_0) + \left.\frac{\partial f}{\partial y}\right|_{\mathbf{r}_0}(y_S - y_0) + \left.\frac{\partial f}{\partial z}\right|_{\mathbf{r}_0}(z_S - z_0) + \ldots \tag{15}$$

Now this equation can be expressed in coordinate-independent form as

$$f(\mathbf{r}_S) = f(\mathbf{r}_0) + \nabla f(\mathbf{r}_0) \cdot (\mathbf{r}_S - \mathbf{r}_0) + \ldots \tag{16}$$

If we now assume that $\mathbf{r}_S$ is close enough to $\mathbf{r}_0$ so that the curvature of the surface is negligible, then the higher-order terms can be neglected. Then, because $f(\mathbf{r}_S) = f(\mathbf{r}_0)$ (both vectors are on the surface S_f), we have

$$\nabla f(\mathbf{r}_0) \cdot (\mathbf{r}_S - \mathbf{r}_0) = 0\,. \tag{17}$$

And because $(\mathbf{r}_S - \mathbf{r}_0)$ lies in the surface[3] we have the result that $\nabla f(\mathbf{r}_0)$ is perpendicular to the surface S_f at the point $\mathbf{r}_0$. *QED*.

A concept closely associated with the gradient is the **directional derivative**. If we have some unit position vector $\hat{\mathbf{r}}_d$, then the directional derivative of $f(\mathbf{r})$ in the direction of $\hat{\mathbf{r}}_d$ is defined as

$$\nabla f(\mathbf{r}) \cdot \hat{\mathbf{r}}_d \tag{18}$$

[3] The surface is essentially planar in the vicinity of $\mathbf{r}_S$ and $\mathbf{r}_0$ because of the proximity of $\mathbf{r}_S$ to $\mathbf{r}_0$.

Physically, the directional derivative tells you how fast the function $f(\mathbf{r})$ changes in the direction of $\hat{\mathbf{r}}_d$. What kind of field is this quantity? Notice that you can also think of the directional derivative as the scalar component of $\nabla f(\mathbf{r})$ in the $\hat{\mathbf{r}}_d$ direction.

A straightforward application of the gradient is found in classical mechanics. If $U(x,y,z)$ is a potential-energy function associated with a particle, then the force on the particle associated with that potential energy is given by $\mathbf{F}(x,y,z) = -\nabla U(x,y,z)$, which is a vector field (a force field!).

As an example, let's consider the gravitational force on a particle near the Earth's surface. If we define our coordinate system such that z points upwards and x and y lie in a horizontal plane, then the gravitational potential energy of a particle with mass m is given by $U(x,y,z) = mg(z - z_0)$, where z_0 is an arbitrary constant and $g \approx 9.8\ \text{m/s}^2$. The force on the particle is then given by $-\nabla U(x,y,z) = -mg\hat{\mathbf{z}}$. What are surfaces of constant $U(x,y,z)$ in this case? These are simply horizontal planes (each one at a constant value of z). Notice that these planes are indeed perpendicular to the force field $-mg\hat{\mathbf{z}}$, which points downward at all points in space.

C. Divergence

Now that we have some feel for the meaning of $\nabla f(\mathbf{r})$, let's now apply the second del to $\nabla f(\mathbf{r})$, which gives us $\nabla \cdot \nabla f(\mathbf{r}) = \nabla^2 f(\mathbf{r})$. But before we do this, maybe we should first say a few words about $\nabla\cdot$ operating on any vector field $\mathbf{V}(\mathbf{r})$. Writing $\mathbf{V}(\mathbf{r})$ as $V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}$ and using the Cartesian-coordinate form of $\nabla\cdot$, we have

$$\begin{aligned} \nabla \cdot \mathbf{V}(\mathbf{r}) &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial x}\hat{\mathbf{z}}\right)\cdot\left(V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}\right) \\ &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial x} \end{aligned} . \tag{19}$$

The quantity $\nabla \cdot \mathbf{V}(\mathbf{r})$ is called the **divergence** of the vector field $\mathbf{V}(\mathbf{r})$. So what kind of object is $\nabla \cdot \mathbf{V}(\mathbf{r})$? Because it assigns a scalar to each point in space it is a scalar field.[4] Thus we see that the gradient of a scalar field is a vector field, while the divergence of a vector field is a scalar field.

If we now let $\mathbf{V}(\mathbf{r})$ equal $\nabla f(\mathbf{r})$, we then get [using Eq. (14)]

[4] We will discuss the divergence in more detail in a later lecture. Stay tuned!

$$\begin{aligned}\nabla \cdot \nabla f(\mathbf{r}) &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial x}\hat{\mathbf{z}} \right) \cdot \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial x}\hat{\mathbf{z}} \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned} . \tag{20}$$

So we see that the Laplacian of f, $\nabla^2 f$, is the divergence of the gradient of f. Thus, $\nabla^2 f$ is a scalar field.

III. Some Final Remarks

Using the generic form of the Laplacian, Eq. (3) can be written in coordinate-independent form as

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = \nabla^2 q . \tag{21}$$

In the next lecture we will look at some more solutions to Eq. (21) using Eq. (3), the Cartesian-coordinate representation, but after that we will look at solutions to Eq. (21) using some different coordinate systems – cylindrical and spherical-polar. The representation of Equation (21) in each coordinate system will look vastly different. Nonetheless, in each case we will be solving a version of Eq. (21), the 3D wave equation.

Exercises

***18.1 Plane Waves**. Consider the solution $q_{\mathbf{k}}^{+}(\mathbf{r},t) = A\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ to the 3D wave equation. Assume that $\omega > 0$.

(**a**) Calculate $\nabla q_{\mathbf{k}}^{+}(\mathbf{r},t)$. In what direction does $\nabla q_{\mathbf{k}}^{+}(\mathbf{r},t)$ point? Thus describe constant surfaces of $q_{\mathbf{k}}^{+}(\mathbf{r},t)$ (for some fixed value of t).

(**b**) In what direction does $q_{\mathbf{k}}^{+}(\mathbf{r},t) = A\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ move? What is the wavelength λ of this wave? What is the dispersion relation $\omega(\mathbf{k})$ for this wave?

(**c**) Show that the sum of $q_{\mathbf{k}}^{+}(\mathbf{r},t) = A\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ and $q_{\mathbf{k}}^{-}(\mathbf{r},t) = B\, e^{i(\mathbf{k}\cdot\mathbf{r} + \omega t)}$ is a standing wave if $B = A$. (Make sure that you write down the specific form of this standing wave.) What property of the wave equation allows you to combine solutions to produce another solution?

*__18.2 Divergence and Gradient__. Consider the function $f(x,y,z)=\frac{1}{x^2}+\frac{2}{y^2}+\frac{3}{z^2}$.

(**a**) Is it appropriate to calculate the divergence or gradient of this function? Calculate whichever is appropriate.
(**b**) You should now have a new function that you calculated in part (**a**). What kind of function is it? Calculate either the gradient or divergence of this new function, whichever is appropriate.

*__18.3 Divergence__. Using the Cartesian-coordinate form of the divergence, $\nabla\cdot=\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial x}\hat{\mathbf{z}}\right)\cdot$, compute the following.

(**a**) $\mathbf{E}(\mathbf{r})=\frac{\mathbf{r}}{r^3}$, where $r=\sqrt{x^2+y^2+z^2}>0$. (Coulomb electric field)

(**b**) $\mathbf{B}(\mathbf{r})=-\frac{y}{x^2+y^2}\hat{\mathbf{x}}+\frac{x}{x^2+y^2}\hat{\mathbf{y}}$, $x^2+y^2>0$. (magnetic field outside a long wire)

*__18.4__ Let f and g be two scalar fields. Using the Cartesian-coordinate form of del, show that $\nabla\cdot(f\nabla g)=\nabla f\cdot\nabla g+f\nabla^2 g$. What kind of field is $\nabla\cdot(f\nabla g)$?

*__18.5__ Using (a) Cartesian coordinates, and then (b) spherical-polar coordinates, calculate the divergence of $\mathbf{E}(\mathbf{r})=\mathbf{r}$ (electric field inside a uniform ball of charge). For a field $V(\mathbf{r})$that only depends upon the coordinate r, the divergence in spherical-polar coordinates is given by $\nabla\cdot\mathbf{F}(\mathbf{r})=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)$.