Fourier Transforms and the Wave Equation

Overview and Motivation: We first discuss a few features of the Fourier transform (FT), and then we solve the initial-value problem for the wave equation using the Fourier transform.

Key Mathematics: More Fourier transform theory, especially as applied to solving the wave equation.

I. FT Change of Notation

In the last lecture we introduced the FT of a function $f(x)$ through the two equations

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk, \tag{1a}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx. \tag{1b}$$

Note that we have changed notation compared to the last lecture. Hereafter we designate the FT of any function by the same symbol, but with an overhead caret included. That is, the FT of $f(x)$ we now write as $\hat{f}(k)$. As we shall see, this is useful when dealing with equations that include FTs of several functions. [1]

II. Some Properties of the Fourier Transform

We now discuss several useful properties of the Fourier transform.

A. Translation

The first property has to do with translation of the function $f(x)$. Let's say we are interested in $f(x - x_0)$, which corresponds to translation of $f(x)$ by x_0. Then, using Eq. (1a) we can write

$$f(x - x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ik(x - x_0)} \, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(k) e^{-ikx_0} \right] e^{ikx} \, dk \tag{2}$$

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[1] This notation is fairly common practice. At some point you may even see the FT of $f(x)$ written as $f(k)$. At least we won't be doing that here!

Thus, we see that the FT of $f(x - x_0)$ is $\hat{f}(k)e^{-ikx_0}$. In other words, translation of $f(x)$ by x_0 corresponds to multiplying the FT $\hat{f}(k)$ by e^{-ikx_0}.

B. Differentiation

The second property has to do with the FT of $f'(x)$, the derivative of $f(x)$. Again, using Eq. (1a) we have

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ik \, \hat{f}(k) \right] e^{ikx} \, dk \, .$$
(3)

So we see that FT of $f'(x)$ is $ik \, \hat{f}(k)$. That is, differentiation of $f(x)$ corresponds to multiplying $\hat{f}(k)$ by ik.

C. Integration

Let's consider the definite integral of $f(x)$,

$$\int_{x_1}^{x_2} dx \, f(x) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} dx \left[\int_{-\infty}^{\infty} dk \, \hat{f}(k) e^{ikx} \right].$$
(4)

Switching the order of integration on the rhs produces

$$\int_{x_1}^{x_2} dx \, f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \hat{f}(k) \left[\int_{x_1}^{x_2} dx \, e^{ikx} \right]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{\hat{f}(k)}{ik} \left(e^{ikx_2} - e^{ikx_1} \right)$$
(5)

So if we define $If(x)$ to be the indefinite integral of $f(x)$, we can rewrite Eq. (5) as

$$If(x_2) - If(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{\hat{f}(k)}{ik} \left(e^{ikx_2} - e^{ikx_1} \right)$$
(6)

So integration of $f(x)$ essentially corresponds to dividing the Fourier transform $\hat{f}(k)$ by ik.[2]

D. Convolution
The last property concerning the a function and its FT has to do with convolution. Because you may not be familiar with convolution, let's first define it. Simply put, the convolution of two functions $f(x)$ and $g(x)$, which we denote $(f * g)(x)$, is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - x') g(x') dx' \tag{7}$$

Perhaps the most common place that convolution arises is in spectroscopy, where $g(x)$ is some intrinsic spectrum that is being measured, and $f(x)$ is the resolution function of the spectrometer that is being used to measure the spectrum.[3] The convolution $(f * g)(x)$ is the spectrum that is then measured.

Note that $(f * g)(x)$ is indeed a function of x, and so we can calculate its FT, which we denote $(f \hat{*} g)(k)$. Using Eq. (1b) we can write

$$(f \hat{*} g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left[\int_{-\infty}^{\infty} dx' \, f(x - x') g(x') \right] e^{-ikx}, \tag{8}$$

which can be rearranged as

$$(f \hat{*} g)(k) = \int_{-\infty}^{\infty} dx' \, g(x') \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, f(x - x') e^{-ikx} \right]. \tag{9}$$

Now the quantity in brackets is the FT of $f(x)$ translated by x'. From Sec. II.B above we know that this is $\hat{f}(k) e^{-ikx'}$, and so Eq. (9) can be expressed as

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[2] You might think that Eq. (6) could be simplified to $If(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \dfrac{\hat{f}(k)}{ik} e^{ikx}$, but this cannot be done because indefinite integration produces an undetermined integration constant. The constant does not appear in Eq. (6) because it is an equation for the difference of $If(x_2)$ and $If(x_1)$.

[3] The resolution function is often quite close to a Gaussian of a particular, fixed width.

$$(f \hat{*} g)(k) = \hat{f}(k) \int_{-\infty}^{\infty} dx'\, g(x') e^{-ikx'}. \tag{10}$$

Recognizing the integral as $\sqrt{2\pi}\,\hat{g}(k)$ we finally have

$$(f \hat{*} g)(k) = \sqrt{2\pi}\, \hat{f}(k)\hat{g}(k). \tag{11}$$

So we see that the FT of the convolution is the product of the FT's of the individual functions (along with a factor of $\sqrt{2\pi}$). One way you may hear this result expressed is that convolution in real space (x) corresponds to multiplication in k space. Equation (11) is known as the **convolution theorem**.

III. Solution to the Wave Equation Initial Value Problem
Way back in Lecture 8 we discussed the initial value problem for the wave equation

$$\frac{\partial^2 q(x,t)}{\partial t^2} = c^2 \frac{\partial^2 q(x,t)}{\partial x^2} \tag{12}$$

on the interval $-\infty < x < \infty$. For the initial conditions

$$q(x,0) = a(x), \tag{13a}$$

$$\frac{\partial q}{\partial t}(x,0) = b(x), \tag{13b}$$

we found that the solution to Eq. (12) can be written as

$$q(x,t) = \frac{1}{2}\left[a(x+ct) + a(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} b(x')dx' \right]. \tag{14}$$

With the help of the Fourier transform we are now going to rederive this solution, and along the way we will learn something very interesting about the FT of $q(x,t)$.

We start by defining the (spatial) FT of $q(x,t)$ as

$$\hat{q}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x,t) e^{-ikx}\, dx, \tag{15a}$$

so that we also have

$$q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{q}(k,t) e^{ikx}\, dk \,. \tag{15b}$$

We also define the FT of Eq. (13), the initial conditions,

$$\hat{q}(k,0) = \hat{a}(k), \tag{16a}$$

$$\frac{\partial \hat{q}}{\partial t}(k,0) = \hat{b}(k). \tag{16b}$$

Now each side of the Eq. (12) is a function of x and t, so we can calculate the FT of both sides of Eq (12),

$$\int_{-\infty}^{\infty} \frac{\partial^2 q(x,t)}{\partial t^2} e^{-ikx} dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 q(x,t)}{\partial x^2} e^{-ikx} dx \tag{17}$$

On the lhs of this equation we can pull the time derivative outside the integral. The lhs is then just the second time derivative of $\hat{q}(k,t)$. The rhs can be simplified by remembering that the FT of the (x) derivative of a function is ik times the FT of the original function. Thus the FT of $\partial^2 q(x,t)/\partial x^2$ is just $-k^2$ times $\hat{q}(k,t)$, the FT of $q(x,t)$. Thus we can rewrite Eq. (17) as

$$\frac{\partial^2 \hat{q}(k,t)}{\partial t^2} = -k^2 c^2 \hat{q}(k,t) \tag{18}$$

This equation should look very familiar to you. What equation is it? None other than the harmonic oscillator equation! What does this tell us about $\hat{q}(k,t)$? It tells us that $\hat{q}(k,t)$ (for a fixed value of k) oscillates harmonically at the frequency $\omega = kc$. Thus we can interpret the function $\hat{q}(k,t)$ as the set of normal modes coordinates for this problem. This further means that the FT has decoupled the equations of motion for this system [as represented by Eq. (12), the wave equation.] Notice also that the dispersion relation $\omega = kc$ has also fallen into our lap by considering the FT of Eq. (12).

As we should know by this point, the solution to Eq. (18) can be written as

$$\hat{q}(k,t) = A(k) e^{ikct} + B(k) e^{-ikct} \,, \tag{19}$$

where $A(k)$ and $B(k)$ are functions of k. And as you should suspect, these two functions are determined by the initial conditions, as follows. First, setting $t = 0$ in Eq. (19) and using Eq. (16a) produces

$$\hat{a}(k) = A(k) + B(k),$$ (20a)

and calculating the time derivative of Eq. (19), setting $t = 0$, and using Eq. (16b) gives us

$$\frac{\hat{b}(k)}{ikc} = A(k) - B(k)$$ (20b)

We can solve Eqs. (20a) and (20b) for $A(k)$ and $B(k)$ by taking their sum and difference, which yields

$$A(k) = \frac{1}{2}\left[\hat{a}(k) + \frac{\hat{b}(k)}{ikc}\right],$$ (21a)

$$B(k) = \frac{1}{2}\left[\hat{a}(k) - \frac{\hat{b}(k)}{ikc}\right],$$ (21b)

which gives us the solution for $\hat{q}(k,t)$ in terms of the initial conditions

$$\hat{q}(k,t) = \frac{1}{2}\left[\hat{a}(k) + \frac{\hat{b}(k)}{ikc}\right]e^{ikct} + \frac{1}{2}\left[\hat{a}(k) - \frac{\hat{b}(k)}{ikc}\right]e^{-ikct}$$ (22)

We are essentially done. We have now expressed the FT of $q(x,t)$ in terms of the FT's of the initial conditions for the problem. The solution $q(x,t)$ is just the inverse FT of Eq. (22) [see Eq. (15b)],

$$q(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-\infty}^{\infty}\left\{\left[\hat{a}(k) + \frac{\hat{b}(k)}{ikc}\right]e^{ik(x+ct)} + \left[\hat{a}(k) - \frac{\hat{b}(k)}{ikc}\right]e^{ik(x-ct)}\right\}dk.$$ (23)

This is the initial-value-problem solution.[4] We can make it look exactly like Eq. (14) with a little bit more manipulation. To see this let's first rewrite Eq. (23) as

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[4] Notice that $q(x,t)$ as expressed in Eq. (23) is the sum of two functions, $f(x+ct)$ and $g(x-ct)$!

$$q(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \left[\hat{a}(k) e^{ik(x+ct)} + \hat{a}(k) e^{ik(x-ct)} \right] + \frac{1}{c} \left[\frac{\hat{b}(k)}{ik} \left(e^{ik(x+ct)} - e^{ik(x-ct)} \right) \right] \right\} dk . \qquad (24)$$

The first two terms we recognize as $(1/2)[a(x+ct) + a(x-ct)]$, while we can use Eq. (5) to recognize the second half of the rhs of Eq. (24) as $(1/2c)\int_{x-ct}^{x+ct} b(x') dx'$. Thus Eq. (24) can be re-expressed as

$$q(x,t) = \frac{1}{2} \left[a(x+ct) + a(x-ct) + \frac{1}{c} \int_{x-ct}^{x+ct} b(x') dx' \right], \qquad (25)$$

which is identical to Eq. (14).

Exercises

*$**17.1**$ **FT Properties**. If the FT of $f(x)$ is $h(k)$,
(**a**) show that the FT of $\left[e^{ik_0 x} f(x) \right]$ is $h(k - k_0)$;
(**b**) show that the FT of $\left[x f(x) \right]$ is $ih'(k)$;
(**c**) show that the FT of $\left[x^2 f(x) \right]$ is $-h''(k)$.

*$**17.2**$ Show that $(f * g)(x) = (g * f)(x)$ by
(**a**) directly by manipulating Eq. (7), the definition of convolution;
(**b**) by using Eq. (11), the result for the FT of $(f * g)(x)$.

*$**17.3**$ **Convolution and the Gaussian**. The function that has the same form as its Fourier transform is the Gaussian. Specifically if $f(x) = e^{-x^2/\sigma^2}$, its FT is given by $h(k) = \frac{\sigma}{\sqrt{2}} e^{-k^2(\sigma^2/4)}$. Using this fact, show that the convolution $(f_1 * f_2)(x)$ of two Gaussian functions $f_1(x) = e^{-x^2/\sigma_1^2}$ and $f_2(x) = e^{-x^2/\sigma_2^2}$ is proportional to the Gaussian function $e^{-x^2/(\sigma_1^2 + \sigma_2^2)}$. [Hint: you need not calculate any integrals to do this problem.]

17.4 The Rectangular Pulse.

Here you will explore the convolution theorem as it applies to a rectangular pulse and its convolution, a triangular pulse. We start with the function $f(x)$, a rectangular pulse of height H and width $2L$ centered at $x = 0$. Elsewhere the function is zero.

(**a**) Graph $f(x)$

(**b**) Calculate Find $\hat{f}(k)$, the FT of $f(x)$. (This is a real function; express it as such.)

(**c**) Graph $\hat{f}(k)$.

(**d**) The function $(f * f)(x)$, the convolution of $f(x)$ with itself, is a triangle function of height $2LH^2$ and base $4L$ centered at zero. It is zero elsewhere. Graph this function.

(**e**) Write down the mathematical expression for $(f * f)(x)$ [that you graphed in (d)]. Then *directly* calculate $\widehat{(f * f)}(k)$ using your functional form for $(f * f)(x)$. (Do not set H and L to specific values. Again, this is a real function; express it as such.)

(**f**) Graph your calculated transform $\widehat{(f * f)}(k)$.

(**g**) Lastly, use $\hat{f}(k)$ and the convolution theorem to find $\widehat{(f * f)}(k)$. Show that this is equal to the result in part (e).

17.5 FT Solution to the 1D Wave Equation. Eq. (23),

$$q(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \left[\hat{a}(k) + \frac{\hat{b}(k)}{ikc} \right] e^{ik(x+ct)} + \left[\hat{a}(k) - \frac{\hat{b}(k)}{ikc} \right] e^{ik(x-ct)} \right\} dk , \qquad (23)$$

is the formal solution to the initial-value problem.

(**a**) What kind of waves are described by the functions $\exp[ik(x+ct)]$ and $\exp[ik(x-ct)]$? Be as specific as possible!

(**b**) From a vector-space point of view, the functions $\exp[ik(x+ct)]$ and $\exp[ik(x-ct)]$ can be considered basis functions for the vector space that consists of solutions to the wave equation. Given this, what do the terms $\frac{1}{2}\left[\hat{a}(k) - \frac{i\hat{b}(k)}{kc} \right]$ and $\frac{1}{2}\left[\hat{a}(k) + \frac{i\hat{b}(k)}{kc} \right]$ represent?

(**c**) Given your answer in (b), how would you describe the solution $q(x,t)$ as written above? [Hint: the term *linear combination* should appear in your answer.]

(**d**) How are $\hat{a}(k)$ and $\hat{b}(k)$ related to the initial conditions $q(x,0)$ and $\frac{\partial q}{\partial t}(x,0)$? That is, write down expressions for $\hat{a}(k)$ and $\hat{b}(k)$ in terms of $q(x,0)$ and $\frac{\partial q}{\partial t}(x,0)$.

(e) Assume that the initial conditions $q(x,0)$ and $\frac{\partial q}{\partial t}(x,0)$ are real. Using your answer to (d), show that $\hat{a}(k) = \hat{a}^*(-k)$ and $\hat{b}(k) = \hat{b}^*(-k)$.

(f) Using the results from (e) you can now show that $q(x,t)$ is real if the initial conditions $q(x,0)$ and $\dot{q}(x,0)$ are both real, as follows. First, in Eq. (23) replace $\hat{a}(k)$ and $\hat{b}(k)$ by $\hat{a}^*(-k)$ and $\hat{b}^*(-k)$, respectively. Then make the change of variable $k \rightarrow -k$, $dk \rightarrow -dk$ in the integral (taking care with the limits of integration). Then compare Eq. (23) with your new expression for $q(x,t)$ and notice how they are related. From this comparison you should be able to conclude that $q(x,t)$ is real.