Introduction to Fourier Transforms

Overview and Motivation: Fourier transform theory is the extension of Fourier series theory to functions that are defined for all values of x. Thus, we will be able to represent a function defined for $-\infty \le x \le \infty$ as a linear combination of harmonic functions.

Key Mathematics: Fourier transforms and more vector-space theory.

I. Fourier Series vs the Fourier Transform

By now you should be intimately familiar with the Fourier series representation of a function $f(x)$ on the interval $-L \le x \le L$. A representation that uses the normalized harmonic functions $\frac{1}{\sqrt{2L}} e^{in\pi x/L}$ (introduced in Lecture 14) is

$$f(x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} , \tag{1a}$$

$$c_n = \frac{1}{\sqrt{2L}} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx . \tag{1b}$$

As we know the Fourier series representation is useful for any function that we only need to define within the bounds $-L \le x \le L$. Outside that interval, the representation is periodic with period $2L$ because the rhs of Eq. (1a) has a period of $2L$.

There are many times, however, when we wish to represent a (nonperiodic) function on the entire real line as a linear combination of harmonic functions. To do this we can take the $L \to \infty$ limit of Eq. (1). This limit (which we will not go through, but is well defined) yields the following pair of relationships

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(k) e^{ikx} \, dk , \tag{2a}$$

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx , \tag{2b}$$

There are several things to notice about Eq. (2). First, we have traded in the discrete index n in Eq. (1) for the continuous variable $k = (n\pi/L)(= 2\pi/\lambda)$, which is already familiar as the wave vector. Second, if we compare Eqs. (1) and (2), we might conclude that $\frac{1}{\sqrt{2\pi}}e^{ikx}$ are now our normalized harmonic functions. That is correct, as we discuss in further detail below. With that, we can then interpret the function $h(k)$ as the coefficient (or component) of the harmonic function $\frac{1}{\sqrt{2\pi}}e^{ikx}$. This function $h(k)$ has a special name: it is known as the **Fourier transform** of the function $f(x)$. The Fourier transform $h(k)$ is thus analogous to the Fourier coefficients c_n that appear in the Fourier series. The other feature of Eq. (2) that you undoubtedly noticed is that $f(x)$ is expressed as a continuous sum (integral) over basis functions rather than a discrete sum over basis functions. This is a necessary consequence of the $L \to \infty$ limit.

Because of the resultant symmetry in the two relationships in Eq. (2), the function $f(x)$ is also known as the **inverse Fourier transform** of $h(k)$. In fact, because $f(x)$ and $h(k)$ are obtainable from each other, they each contain the same information, just in a different form.[1]

We remark that there are technical criteria that the function $f(x)$ must meet for Eq. (2) to be valid. A sufficient condition is that $f(x)$ be square integrable,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty . \tag{3}$$

If Eq. (3) is true, then $h(k)$ is also square integrable and it can be shown that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |h(k)|^2 dk \tag{4}$$

The proof of Eq. (4) is left as an exercise.

II. The Fourier Transform and Vector Space Theory

As we also discussed in Lecture 14, the Fourier series [Eq. (1)] can be thought of as a pair of vector-space relationships

[1] Of course, the same is true about the function $f(x)$ and the coefficients c_n in the Fourier series. Knowing the c_n's is equivalent to knowing the function $f(x)$ itself.

$$\mathbf{f} = \sum_{n=-\infty}^{\infty} c_n \hat{\mathbf{u}}_n \,, \tag{5a}$$

$$c_n = \left(\hat{\mathbf{u}}_n, \mathbf{f}\right) \tag{5b}$$

where the vector $\mathbf{f}$ is the function $f(x)$, $\hat{\mathbf{u}}_n = \frac{1}{\sqrt{2L}} e^{in\pi x/L}$ is an orthonormal basis vector, c_n is the corresponding component of $\mathbf{f}$, and the inner product is defined as

$$(\mathbf{g}, \mathbf{f}) = \int_{-L}^{L} g^*(x) f(x) dx \,. \tag{6}$$

Further, because the basis functions are orthonormal, we have the relationship for their inner product

$$\left(\hat{\mathbf{u}}_m, \hat{\mathbf{u}}_n\right) = \delta_{mn} \,, \tag{7}$$

where δ_{mn}, known as the **Kronecker delta**, equals 1 if $m = n$ and equals 0 otherwise. Eq. (7) is the standard way of expressing the orthonormality of the basis vectors.

We now want to put Fourier-transform theory on the same vector-space footing as Fourier series. This is actually fairly straightforward, except that there is a bit of subtlety needed in defining the inner product, as we shall see. First, if we identify the basis vectors as the harmonic functions[2]

$$\hat{u}(k, x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \,, \tag{8}$$

then Eq. (2) can be written as

$$f(x) = \int_{-\infty}^{\infty} h(k) \hat{u}(k, x) dk \,, \tag{9a}$$

$$h(k) = \int_{-\infty}^{\infty} \hat{u}^*(k, x) f(x) \, dx \tag{9b}$$

[2] To keep the notation as simple as possible, we drop the formal vector notation and just use the functional form of the vectors for this space.

A comparison of Eqs. (5) and (9) then suggests that we define the inner product on this vector space as

$$(g(x), f(x)) = \int_{-\infty}^{\infty} g^*(x) f(x) dx \tag{10}$$

Let's see what this gives us if we calculate $(\hat{u}(k',x), \hat{u}(k,x))$ for this vector space. If life is good then we expect to get $(\hat{u}(k',x), \hat{u}(k,x)) = \delta_{k'k}$, similar to Eq. (7). Let's see what happens. Using Eq. (10) we have

$$(\hat{u}(k',x), \hat{u}(k,x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx \tag{11}$$

We consider two cases separately, $k = k'$ and $k \neq k'$.
(*i*) If $k = k'$, then Eq. (11) integrates to

$$(\hat{u}(k,x), \hat{u}(k,x)) = \frac{1}{2\pi} x\Big|_{-\infty}^{\infty}, \tag{12a}$$

which is undefined. Hum… Not too good. Let's look at the other case.
(*ii*) If $k \neq k'$, then Eq. (11) integrates to

$$(\hat{u}(k,x), \hat{u}(k,x)) = \frac{1}{2\pi} \frac{1}{i(k-k')} e^{i(k-k')x}\Bigg|_{-\infty}^{\infty} \tag{12b}$$

Unfortunately, this is not defined either! So it looks like either the basis functions or the inner product is unsuitable.

As it turns out, we can fix this dilemma by defining the inner product slightly differently as

$$(g(x), f(x)) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-x^2/n} g^*(x) f(x) dx . \tag{13}$$

Notice what the function $e^{-x^2/n}$ does for us. For any finite n this Gaussian function cuts off the integrand fast enough to make the integral converge. Furthermore in the limit $n \to \infty$ the function itself simply approaches 1.[3]

Let's see what happens with this definition of the inner product. We now have

$$(\hat{u}(k',x),\hat{u}(k,x)) = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n}\ e^{i(k-k')x}dx\,. \tag{14}$$

To take advantage of the symmetry of the Gaussian we rewrite this as

$$(\hat{u}(k',x),\hat{u}(k,x)) = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n}\ \{\cos[(k-k')x] + i\sin[(k-k')x]\}dx\,. \tag{15}$$

Because $e^{-x^2/n}$ is even, the integral involving the sine function is zero, so this simplifies to

$$(\hat{u}(k',x),\hat{u}(k,x)) = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n}\ \cos[(k-k')x]dx\,. \tag{15}$$

We have seen this integral before (see Lecture 12). Calculating the integral, Eq. (15) becomes

$$(\hat{u}(k',x),\hat{u}(k,x)) = \frac{1}{2}\lim_{n\to\infty}\sqrt{\tfrac{n}{\pi}}\ e^{-n(k-k'/2)^2} \tag{16}$$

Now you may remember from the last lecture that the limit of a similar sequence of Gaussian functions is the Dirac delta function. If you closely compare Eq. (16) with Eq. (8) from the Lecture 15 notes, you will see that Eq. (16) can be expressed as

$$(\hat{u}(k',x),\hat{u}(k,x)) = \frac{1}{2}\,\delta\!\left(\frac{k-k'}{2}\right) \tag{17}$$

And using the relationship $\delta(x/a) = |a|\,\delta(x)$, this simplifies to

[3] In fact, $\lim_{n\to\infty} e^{-x^2/n}$ is one definition of the unit distribution.

$$\left(\hat{u}(k',x),\hat{u}(k,x)\right)=\delta(k-k') \tag{18}$$

This, then, is the **orthogonality relationship** for the basis functions $\hat{u}(k,x)=\frac{1}{\sqrt{2\pi}}e^{ikx}$. Notice that it is similar to Eq. (7) for the Fourier series basis functions, but instead of the Krocecker delta, we have the Dirac delta function. That the orthogonality relationship is a distribution rather than a simple function is a result of the variable k being continuous rather than discrete.

You will often see written

$$\delta(k-k')=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k-k')x}\,dx\,, \tag{19}$$

but this is really shorthand for the limiting procedure that we did above. That is, it is really shorthand for

$$\delta(k-k')=\frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{-x^2/n}\,e^{i(k-k')x}\,dx\,. \tag{20}$$

Note that Eq. (20) is a definition of the delta function as the limit of a sequence of functions (which is exactly equivalent to our original definition using the sequence of Gaussian functions).

Now that we have the inner product suitably defined, let go back to the Fourier transform equation and see that $h(k)$ is indeed equal to the inner product $\left(\hat{u}(k,x),f(x)\right)$. So using Eq. (13) we calculate the inner product of $\hat{u}(k,x)$ with Eq. (2a)[4]

$$\left(\hat{u}(k,x),f(x)\right)=\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{-x^2/n}\,\tfrac{1}{\sqrt{2\pi}}e^{-ikx}\left[\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(k')e^{ik'x}\,dk'\right]dx \tag{21}$$

If we switch the order of integration in this equation we get something that should look familiar,

[4] Notice that we have renamed the integration variable on the rhs of Eq. (2a) because we have another variable k in this equation.

$$\left(\hat{u}(k,x), f(x)\right) = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-x^2/n}\, e^{i(k'-k)x}\, dx\right] h(k')\,dk' \tag{22}$$

But from Eq. (20) we see that Eq. (22) is simply

$$\left(\hat{u}(k,x), f(x)\right) = \int_{-\infty}^{\infty}\delta(k'-k)\,h(k')\,dk', \tag{23}$$

which gives us the result that we want,

$$\left(\hat{u}(k,x), f(x)\right) = h(k). \tag{24}$$

Thus, as with the c_n's in the Fourier series representation of a function, the Fourier transform $h(k)$ can be though of as the inner product of the normalized basis function with the original function $f(x)$.

Exercises

***16.1** Show that Eq. (4), $\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|h(k)|^2\,dk$, is true.

***16.2** Calculate the Fourier transform of the function $f(x) = e^{-|x|}$. Plot the resulting function vs k.

****16.3** As the notes discuss, the original attempt at defining the inner product as $\left(g(x), f(x)\right) = \int_{-\infty}^{\infty} g^*(x) f(x)\,dx$ needs to be slightly modified. We chose one particular way that this can be done. Another choice that we could have made is $\left(g(x), f(x)\right) = \lim_{n\to\infty}\int_{-n}^{n} g^*(x) f(x)\,dx$. Show that this definition of the inner product also gives the result $\left(\hat{u}(k',x), \hat{u}(k,x)\right) = \delta(k-k')$ for the inner product of two basis functions. {Hint: you will need to use the second definition of the delta function from Lecture 15 [Eq. (9) on p. 4]}.