Complex Fourier Series

Overview and Motivation: We continue with our discussion of Fourier series, which is all about representing a function as a linear combination of harmonic functions. The new wrinkle is that we now use complex forms of the harmonic functions.

Key Mathematics: More Fourier Series! And a cute trick that often comes in handy when calculating integrals.

I. The Complex Fourier Series

Last time we introduced Fourier Series and discussed writing a function $f(x)$ defined on the interval $-L \le x \le L$ as

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty}\left[\alpha_n \cos\left(\tfrac{n\pi x}{L}\right) + \beta_n \sin\left(\tfrac{n\pi x}{L}\right)\right], \tag{1}$$

where the Fourier coefficients are given as

$$\alpha_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx\,, \tag{2a}$$

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi x}{L}\right) dx\,, \tag{2b}$$

$$\beta_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi x}{L}\right) dx\,. \tag{2c}$$

While there is nothing wrong with this description of Fourier Series, it is often advantageous to use the complex representations of the sine and cosine functions,

$$\cos\left(\tfrac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right), \tag{3a}$$

$$\sin\left(\tfrac{n\pi x}{L}\right) = \frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right). \tag{3a}$$

If we insert these expressions into Eq. (1) we obtain

$$f(x) = \alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\alpha_n\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) - i\beta_n\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)\right], \tag{4}$$

which can be rearranged as

$$f(x) = \alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\alpha_n - i\beta_n\right)e^{in\pi x/L} + \left(\alpha_n + i\beta_n\right)e^{-in\pi x/L}\right]. \tag{5}$$

This doesn't look any simpler, but notice what happens if we define a new set of coefficients (which are simply linear combinations of the of the current coefficients a_n and β_n),

$$c_0 = \alpha_0 \tag{6a}$$

$$c_n = \tfrac{1}{2}\left(\alpha_n - i\beta_n\right) \tag{6b}$$

$$c_{-n} = \tfrac{1}{2}\left(\alpha_n + i\beta_n\right) \tag{6c}$$

Then we can write Eq. (5) as

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{in\pi x/L} + c_{-n}e^{-in\pi x/L}\right], \tag{7}$$

or even more simply as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} . \tag{8}$$

Using Eq. (2) it is not hard to show that the coefficients c_n in Eq. (6) are given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx . \tag{9}$$

Equations (8) and (9) are known as the **complex Fourier series** representation of the function $f(x)$. Notice that with the complex representation there is only one expression needed for all of the Fourier coefficients.

There is another way to obtain Eq. (9), which is to use the same trick that we have used several times before to find coefficients of the harmonic functions: multiply Eq. (8) by the proper function and integrate! Let's say we want to find the mth coefficient c_m. We then multiply Eq. (8) by $e^{-im\pi x/L}$ (notice the minus sign in the exponent!) and integrate on x from $-L$ to L, which produces

$$\int_{-L}^{L} f(x)e^{-im\pi x/L}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{i(n-m)\pi x/L}\,dx\,. \tag{10}$$

Now, as before, only one integral on the rhs is nonzero. That is the integral with $n = m$, and its value is $2L$. Eq. (10) thus simplifies to

$$\int_{-L}^{L} f(x)e^{-im\pi x/L}\,dx = c_m 2L\,, \tag{11}$$

which is equivalent to Eq. (9). That is pretty much it for the setup of the complex Fourier series.

II. An Example Revisited

Let's look at an example that we looked at last time, the triangle function

$$f_1(x) = \begin{cases} \frac{A}{L}x + A & -L \le x < 0 \\ -\frac{A}{L}x + A & 0 \le x \le L \end{cases}, \tag{12}$$

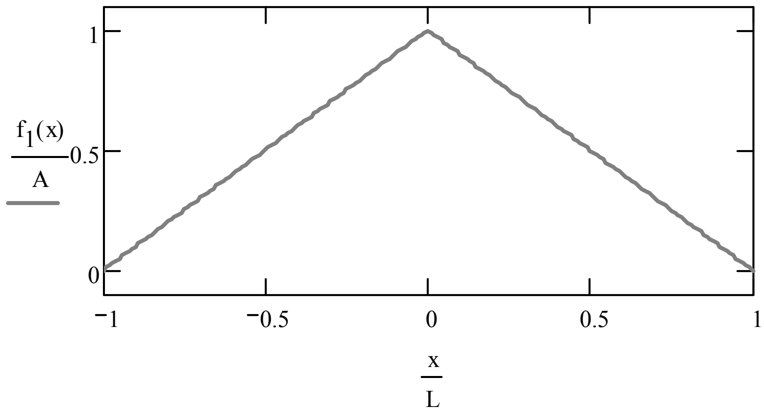
which is plotted on the top of the next page.

Let's use Mathcad to evaluate the c_n's. Inserting Eq. (12) into Eq. (9)

$$c_n = \frac{1}{2L}\left\{\int_{-L}^{0}\left(\tfrac{A}{L}x + A\right)e^{-in\pi x/L}\,dx + \int_{0}^{L}\left(-\tfrac{A}{L}x + A\right)e^{-in\pi x/L}\,dx\right\} \tag{13}$$

and asking Mathcad to evaluate this expression results in

$$c_n = A\frac{1-\cos(n\pi)}{n^2\pi^2}\,. \tag{14}$$



Not this is OK, as long as we do not use it for more than it is worth: for any nonzero value of n Eq. (14) is perfectly fine. But what about the case of $n=0$? Then this expression is undefined. What this means is that we must explicitly set $n=0$ in Eq. (13) and reevaluate it. But for $n=0$, we see from Eq. (9) that c_0 is just the average value of the function, which is $A/2$. Putting this all together we can represent the function $f_1(x)$ as

$$f_1(x) = \frac{A}{2} + A\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{1-\cos(n\pi)}{n^2\pi^2} e^{in\pi x/L} \tag{15}$$

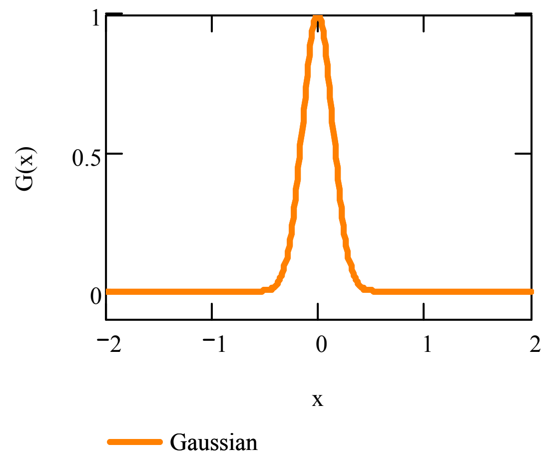
Now this is a valid representation of the function $f_1(x)$, but you may be wondering about something. We know that the function $f_1(x)$ is real, but the rhs of Eq. (15) appears to have an imaginary part, because $e^{in\pi x/L} = \cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)$. So what is the deal? Well, it is not too difficult to see that the imaginary part of each positive-n term is exactly cancelled by the imaginary part of the corresponding negative-n term. So, Eq. (15) is indeed real.

III. The Gaussian Function

Let's take another look at the Gaussian function and think a bit about representing it as a Fourier Series. You should recall that the Gaussian function is defined as

$$G_\sigma(x) = e^{-x^2/\sigma^2}, \tag{16}$$

where σ is known as the width parameter. Let's assume in this example that $L >> \sigma$. Then we have something like the following picture, where we have set $\sigma = 0.2$ and $L = 2$.



Let's calculate the coefficients c_n. Using Eq. (9) we have

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-x^2/\sigma^2}\, e^{-in\pi x/L}\, dx\,. \tag{17}$$

Now this integral can be expressed in terms of the error function, which is the integral of the Gaussian function, but the expression is pretty messy. However, there is an approximate solution to the integral in Eq. (17) that is quite simple and very accurate, as we now show. For the conditions that we assumed, namely $L >> \sigma$, the Gaussian function is nearly zero for $|x| \geq L$. Because of this we can extend the limits of integration in Eq. (17) to $\mp\infty$, and with very little loss of accuracy we can write

$$c_n = \frac{1}{2L}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\, e^{-in\pi x/L}\, dx\,, \tag{18}$$

We can also take advantage of the properties of integrals of odd and even functions if we write $e^{-in\pi x/L} = \cos(n\pi x/L) - i\sin(n\pi x/L)$, which turns Eq. (18) into

$$c_n = \frac{1}{2L}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\left[\cos(n\pi x/L) - i\sin(n\pi x/L)\right] dx\,. \tag{19}$$

Now the Gaussian function is even, so the integral of $e^{-x^2/\sigma^2}\left[-i\sin(n\pi x/L)\right]$ is zero, and so we are left with

$$c_n = \frac{1}{2L}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\,[\cos(n\pi x/L)]\,dx\,. \tag{19}$$

It so happens that this integral has a nice analytic solution. Using the (fairly well-known) result (which you can find in any table of integrals)

$$\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\cos(\beta x)dx = \sqrt{\pi}\sigma\, e^{-\beta^2\sigma^2/4}\,, \tag{20}$$

we can identify $n\pi/L$ in Eq. (19) as β in Eq. (20), so we have for the coefficients

$$c_n = \frac{\sqrt{\pi}\sigma}{2L} e^{-n^2/(2L/\pi\sigma)^2}\,. \tag{21}$$

Now this is pretty cool: as a function of n, c_n is also a Gaussian, and its width parameter is $2L/(\pi\sigma)$. Notice that this width parameter is inversely proportional to the width parameter σ of the original Gaussian function $G_\sigma(x) = e^{-x^2/\sigma^2}$.[1]

We can now use Eq. (21) in Eq. (8) and represent a Gaussian function (on the interval $-L \le x \le L$) as

$$G_\sigma(x) = \frac{\sqrt{\pi}\sigma}{2L}\sum_{n=-\infty}^{\infty} e^{-n^2/(2L/\pi\sigma)^2} e^{in\pi x/L}\,. \tag{22}$$

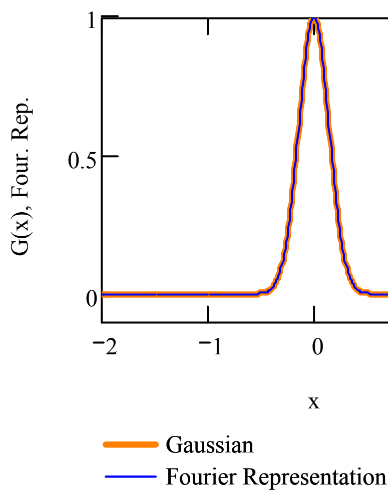
Let's look at the original Gaussian function $G_\sigma(x)$ and its (truncated) Fourier representation,

$$G_\sigma(x)_M = \frac{\sqrt{\pi}\sigma}{2L}\sum_{n=-M}^{M} e^{-n^2/(2L/\pi\sigma)^2} e^{in\pi x/L}\,. \tag{23}$$

So how many terms do we need; that is, how large does M need to be in Eq. (23)? We can get some idea by considering what happens to the coefficients c_n [see Eq. (21)] as $|n|$ gets larger. For a Gaussian function, if the argument is several times larger than the width parameter, then the Gaussian function is very close to zero. Thus, we need to choose M such that it is a few times larger than the width parameter $2L/(\pi\sigma)$. This is illustrated in the next figure, where we have used $M = 20$, which is

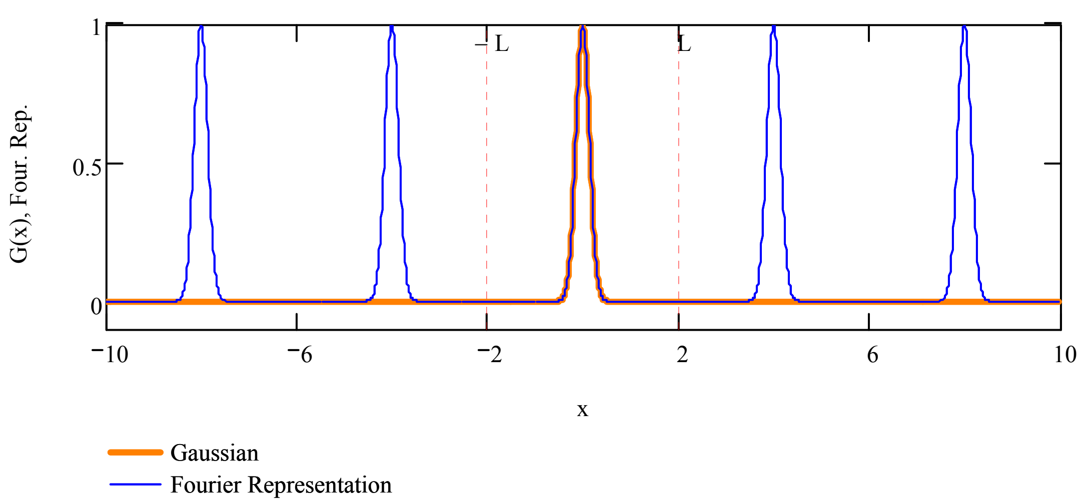
[1] We will see later that this observation is essentially the uncertainty principle of quantum mechanics!

approximately $3\times 2L/(\pi\sigma)$. On the scale of this graph, the truncated Fourier series certainly does a good job of representing the Gaussian function. (As above, we have again set $\sigma = 0.2$ and $L = 2$).



However, there is an inherent limitation to using Fourier series to represent a nonperiodic function such as a Gaussian. That limitation is illustrated in the next figure, which plots the Gaussian and its Fourier series over an interval larger than $-L \le x \le L$. Within the interval the match is very good (as we saw in the last graph), but outside the interval the match is pretty lousy. Why? Well, that is because the harmonic functions that make up the Fourier series all repeat on any interval with length $2L$. Thus, the Fourier representation of the Gaussian function has periodicity $2L$.

Well, you might say that there is no problem here. I'll just pick a value of L that is



larger than any value of x where I might want to evaluate the original function. That might work in practice, but we might also ask the question: is there a Fourier-series representation that will work for all x? The answer is yes, and we will discuss that in a few lectures after this one.

Right now I just want to point out that getting to such a representation is not at all trivial. Consider the following. We have represented the Gaussian as a linear combination of harmonic functions $e^{in\pi x/L}$. If we want to use a Fourier representation for all x, then somehow we must take the limit where $L \to \infty$. What does that mean for the harmonic functions? It looks like all of the harmonic functions will simply become equal to 1 (which seems pretty bad!). There is a resolution to this dilemma, but this illustrates that taking the $L \to \infty$ limit of the Fourier-series representation is somewhat nontrivial.

Exercises

*__12.1__ Calculate the integral on the rhs of Eq. (10) and show that it is nonzero only if $n = m$.

*__12.2__ Consider the result for the coefficient for the triangle function, $c_n = A\dfrac{1-\cos(n\pi)}{n^2\pi^2}$, which is undefined for $n = 0$. Use l'Hôspital's rule to show that as $n \to 0$, $c_n \to A/2$, the result for c_0.

*__12.3__ Using Eq. (2) in Eq. (6b), show that c_n is given by Eq. (9).

**__12.4 Fourier series example__. Consider the function $f(x) = \begin{cases} e^{x} & x \le 0 \\ e^{-x} & x > 0 \end{cases}$.

(a) Plot this function. Explain why this function is even?
(b) Find a *real* analytic expression for the Fourier coefficients c_n for this function. (Hint: You can use the fact that $f(x)$ is even to simplify your determination of the coefficients.)
(c) Let $L = 5$. Plot the function and its truncated Fourier representation for several values of M. What is the minimum reasonable value for M necessary to represent $f(x)$ on this interval?