Introduction to Fourier Series

Overview and Motivation: Fourier series is based on the idea that many functions of interest can be represented as a linear combination of harmonic functions. Is this cool, or what?

Key Mathematics: Fourier Series! And some facts about integrals of odd and even functions.

I. An Observation

We have already been treading in Fourier-series territory. You should recall in the last lecture that we wrote the general solution $q(x,t)$ to the wave equation as a linear combination of normal-mode solutions. Well, these normal mode solutions are harmonic (in both space and time). To write another function as a linear combination of harmonic functions is the basic idea of Fourier series.

As a fairly simple example from last time, let's consider Eq. (8) from those lecture notes, which can be written as

$$a(x) = \sum_{n=1}^{\infty} \mathrm{Re}(a_n) \sin\left(\tfrac{n\pi}{L} x\right) . \tag{1}$$

This is a profound equation. It says that we can write the function $a(x)$, which is fairly arbitrary, as a linear combination of the harmonic functions $\sin\left(\tfrac{n\pi}{L} x\right)$. The price we must pay is that we need an infinite number of these functions to describe $a(x)$. However, as we discussed in the last lecture notes, we often need only a few of these functions to accurately describe the function $a(x)$.

You should also recall that last time we found an equation for the coefficient $\mathrm{Re}(a_n)$ of each harmonic function. Without such an equation Eq. (1) might be theoretically interesting, but it would not be of much use. That equation is

$$\mathrm{Re}(a_n) = \frac{2}{L}\int_0^L \sin\left(\tfrac{n\pi}{L} x\right) a(x)\, dx . \tag{2}$$

As we shall see below, equations such as Eqs. (1) and (2) are the essence of Fourier Series theory.

Our formal discussion of Fourier series will be limited to one independent variable, which we call x. The variable x does not necessarily represent a spatial position, however. There are many cases when one is interested in using Fourier series to represent what is happening in time.

II. Fourier Series Equations

The theory of Fourier series starts by considering a function, which we will call $f(x)$, on the symmetric interval $-L \le x \le L$. If $f(x)$ is a "good" function[1] then we can represent $f(x)$ as a linear combination of harmonic functions,

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \left[\alpha_n \cos\left(\tfrac{n\pi}{L} x\right) + \beta_n \sin\left(\tfrac{n\pi}{L} x\right) \right]. \tag{3}$$

The amplitudes α_n and β_n are known as the Fourier coefficients of the function $f(x)$. There are a several things to point out here. The first is that the harmonic functions in the series have a period (or wavelength) of $2L/n$. Thus each harmonic function has the periodicity $2L$ of the interval. In fact, the sum in Eq. (3) includes all linearly independent harmonic functions with periodicity $2L$. Second, the average value of an harmonic function over an interval of periodicity is zero. Thus, the coefficient α_0 is needed to represent functions whose average value is not zero. Indeed, as we shall see, α_0 is the average value of the function $f(x)$.

As mentioned above, the representation of a function by a linear combination of harmonic functions isn't that useful unless we know how to calculate the coefficients α_0, α_n, and β_n. Fortunately, expressions for the coefficients are fairly simple and are given by

$$\alpha_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \tag{4a}$$

$$\alpha_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\tfrac{n\pi}{L} x\right) dx, \tag{4b}$$

$$\beta_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\tfrac{n\pi}{L} x\right) dx. \tag{4c}$$

[1] In typical physicist fashion we will dodge the question of what exactly makes a function "good". If you are interested, there are plenty of text books that discuss this point, including Dr. Torre's text *FWP*.

Now you may be wondering where these equations came from, but you have seen the derivation of formulae equivalent to Eq. (4) several times before. The last time was in the last lecture notes when we obtained Eq. (2) from Eq. (1). *The key is to multiply Eq. (3) by one of the harmonic functions and integrate over the proper interval.*

As an example, let's derive Eq. (4b). Starting with Eq. (3), we multiply it by $\cos\left(\frac{m\pi x}{L}\right)$ (notice the m) and integrate from $-L$ to L, which gives us

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi}{L}x\right)dx = \alpha_0\int_{-L}^{L}\cos\left(\tfrac{m\pi}{L}x\right)dx + \sum_{n=1}^{\infty}\left[\alpha_n\int_{-L}^{L}\cos\left(\tfrac{n\pi}{L}x\right)\cos\left(\tfrac{m\pi}{L}x\right)dx + \beta_n\int_{-L}^{L}\sin\left(\tfrac{n\pi}{L}x\right)\cos\left(\tfrac{m\pi}{L}x\right)dx\right] \tag{5}$$

For $m \geq 1$ there is only one nonzero integral on the rhs of this equation,

$$\int_{-L}^{L}\cos\left(\tfrac{m\pi}{L}x\right)\cos\left(\tfrac{m\pi}{L}x\right)dx = L\,. \tag{6}$$

Equation (5) thus greatly simplifies to

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi}{L}x\right)dx = \alpha_m L\,, \tag{7}$$

which can be solved for α_m, resulting in Eq. (4b) (after replacing m by n).

III. Some Examples

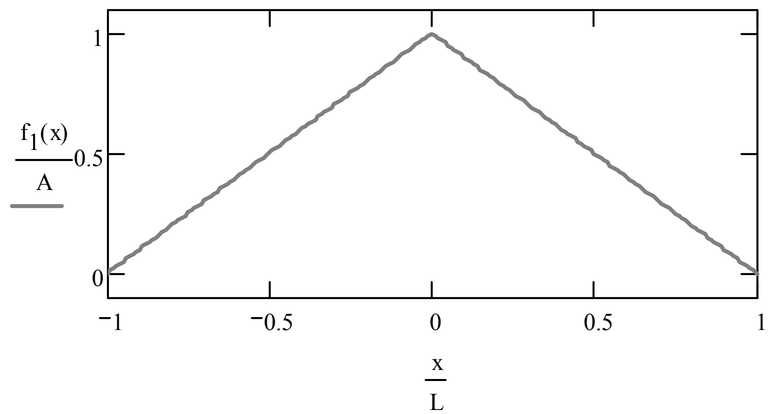
A. Triangle Function

Let's first look at the function,

$$f_1(x) = \begin{cases} \frac{A}{L}x + A & -L \leq x < 0 \\ -\frac{A}{L}x + A & 0 \leq x \leq L \end{cases}, \tag{8}$$

which is plotted in the figure on the top of the next page.

To use the Fourier-series representation of this function we must first calculate the Fourier coefficients using Eq. (4). Before we go ahead and try to calculate the integrals, let's notice a few things that will make the calculations simpler. First, Eq.



(4a) tells us that α_0 is simply the average of the function $f(x)$ on the interval $-L$ to L. From the graph we see that this is $A/2$, so without doing any math we have

$$\alpha_0 = \frac{A}{2}. \tag{9}$$

Second, notice that $f(x)$ is an even function. [An even function has the property $f_{even}(-x) = f_{even}(x)$.] Now Eq. (4c) is the integral of the product of this even function with the odd function $\sin\left(\frac{n\pi}{L}x\right)$. [An odd function is defined via $f_{odd}(-x) = -f_{odd}(x)$.] Now the product of an odd function and an even function is an odd function, and the integral of an odd function over a symmetric interval about zero (such as $-L$ to L) is zero. Thus, again without explicitly calculating the integral in Eq. (4c) we have for this example

$$\beta_n = 0 \tag{10}$$

We are left with determining the coefficients α_n. Even here things are simpler than at first glance: we can use a simplifying fact about integrals of even functions over a symmetric interval about $x = 0$. The simplification is that the integral of an even function over a symmetric interval is equal to twice the integral of the function over the positive (or negative) portion of the interval. Now $\cos\left(\frac{m\pi}{L}x\right)$ is an even function, and the product of two even functions is an even function. With the simplifying fact and Eqs. (4b) and (8) we have

$$\alpha_n = \frac{2}{L}\int_0^L \left(-\tfrac{A}{L}x + A\right)\cos\left(\tfrac{n\pi}{L}x\right)dx. \tag{11}$$

Using Mathcad, for example, the integral is easily evaluated, resulting in

$$\alpha_n = \frac{2A}{n^2\pi^2}\left[1-\cos(n\pi)\right]. \tag{12}$$

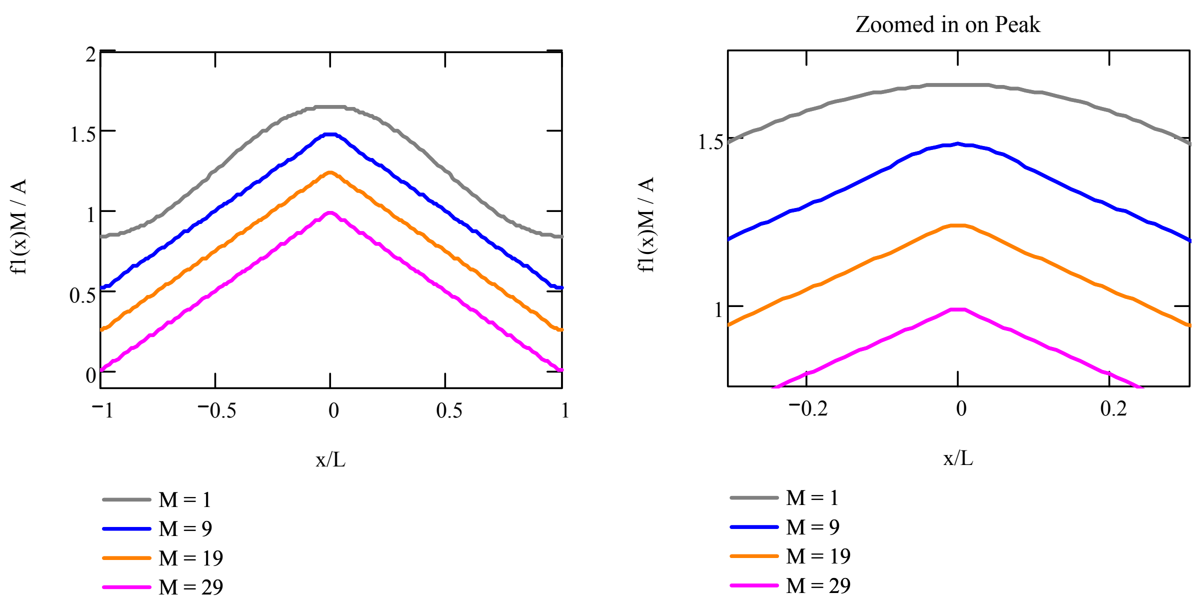
Using Eqs. (9), (10), and (12) in Eq. (3) produces the Fourier representation of $f_1(x)$,

$$f_1(x) = \frac{A}{2} + 2A\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n^2\pi^2}\cos\left(\tfrac{n\pi}{L}x\right)\right]. \tag{13}$$

As discussed in the last lecture, in practice we use a truncated version of an infinite-series representation such as that in Eq. (13). Following that lecture, we write the truncated version of Eq. (13) as

$$f_1(x)_M = \frac{A}{2} + 2A\sum_{n=1}^{M}\left[\frac{1-\cos(n\pi)}{n^2\pi^2}\cos\left(\tfrac{n\pi}{L}x\right)\right]. \tag{14}$$

So where should we cut off the series? To get some idea, let's graphically look at Eq. (14) for several values of M. As shown in the following figure, the series with $M = 9$, 19, and 29 all do a reasonable job of representing the original function, with the major difference being the sharpness of the peak at $x = 0$, which is clearly visible in the rhs graph.[2] That the function near this point is hard to represent with an harmonic series isn't surprising. Because $f_1(x)$ has a kink at $x = 0$, its first derivative is undefined there.



[2] Although hard to see in the lhs graph, there is also some rounding of the function at the ends of the interval.

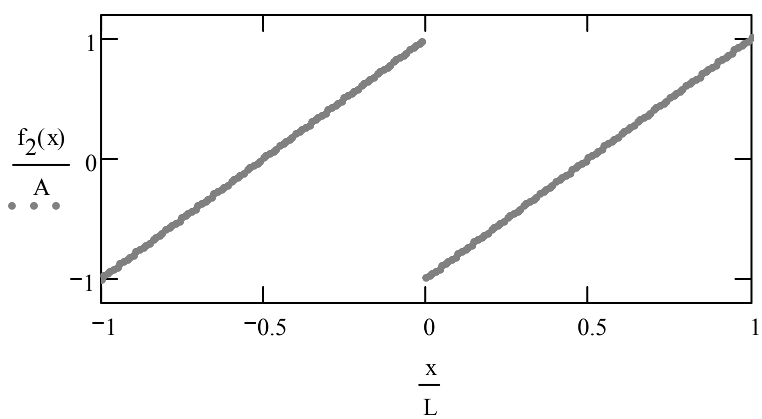
Conversely, the harmonic functions that make up the Fourier series are differentiable at that point; thus for any finite value of M, $f_1(x)_M$ is also be differentiable at $x = 0$.

B. Sawtooth Function

Let's finish up this introduction to Fourier series with another example. This time we look at the function

$$f_2(x) = \begin{cases} \frac{2A}{L}x + A & -L \le x < 0 \\ \frac{2A}{L}x - A & 0 \le x \le L \end{cases}, \tag{15}$$

which is plotted in the next figure. Notice that the function is discontinuous at $x = 0$. Because the harmonic functions are all continuous, you might expect some difficulty in representing this function with a Fourier series. Indeed, there is a major problem, as we shall shortly see.



Again we use Eq. (4) to calculate the Fourier coefficients. As before, the function $f_2(x)$ has enough symmetry to make some of the calculations trivial. We first note from the graph that the average value of $f_2(x)$ is zero, so $\alpha_0 = 0$. Notice also that $f_2(x)$ is odd, which means $f_2(x)\cos\left(\frac{n\pi}{L}x\right)$ is odd, and so this time $\alpha_n = 0$. Similarly, $f_2(x)\sin\left(\frac{n\pi}{L}x\right)$ is even, so we can the equation for β_n simplifies to

$$\beta_n = \frac{2}{L}\int_0^L \left(\frac{2A}{L}x - A\right)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{16}$$

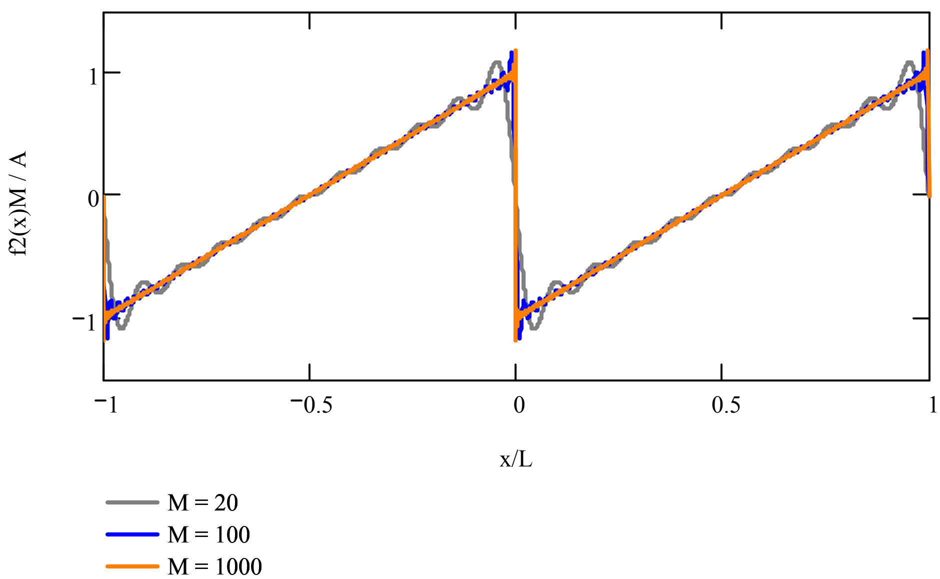
Again, Mathcad can do the integral, and it gives us

$$\beta_n = -\frac{2A}{n\pi}\left[1+\cos(n\pi)\right], \tag{17}$$

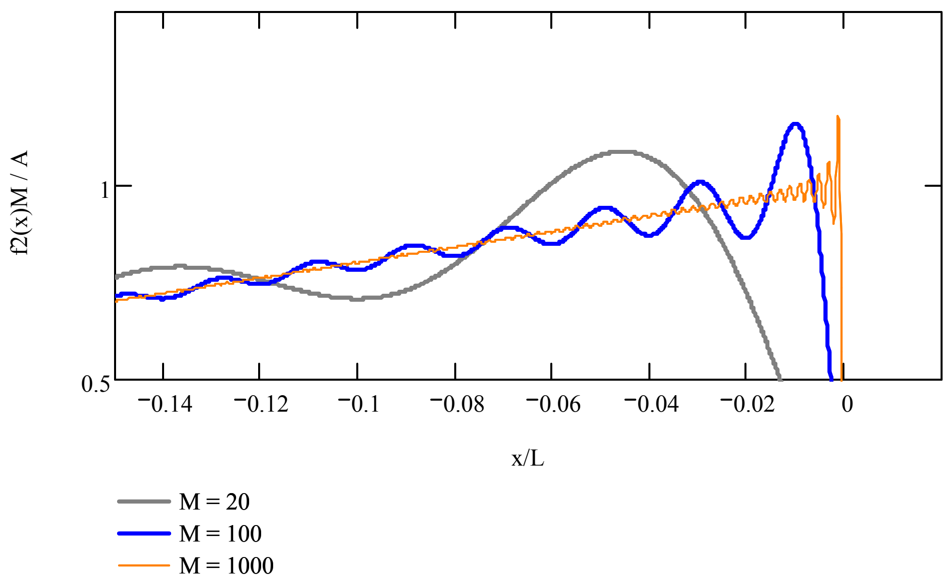
So the truncated Fourier representation of the function $f_2(x)$ can be written as

$$f_2(x)_M = -2A\sum_{n=1}^{M}\left[\frac{1+\cos(n\pi)}{n\pi}\sin\left(\tfrac{n\pi}{L}x\right)\right]. \tag{18}$$

This function is plotted in the next figure for several values of M. Notice that using even a large number of terms does not do justice to the original function. The problem is due to the discontinuity, as was alluded to above.



In fact, something quite pathological happens near the discontinuity, as illustrated in the next figure, where we zoom in on a section of the above graph. As the figure clearly illustrates, there is an overshoot of the truncated Fourier series, and the size of the overshoot does not decrease as the number of terms increases. This overshoot, which is known as the **Gibbs phenomenon**, happens whenever we try to represent a discontinuous function with a (truncated) Fourier series. Notice that the Gibbs phenomenon also occurs for this function at the two ends of the interval. This is because $f_2(L) \neq f_2(-L)$, whereas, the harmonic functions in the Fourier series all have the same value at $-L$ and L.



Summarizing, we have seen how to represent a function on a symmetric interval as a linear combination of harmonic functions that have the periodicity of that interval. If the function is continuous, the representation works well. If the function has a discontinuity, then the representation it not without its difficulties.

Exercises

***11.1** Obtain Eq. (4a), the expression for α_0, from Eq. (3).

***11.2 An integral involving harmonic functions**. In deriving Eq. (6) from Eq. (5) we used the fact that $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = 0$ for all integers n and m. Using the trig identities for $\sin(x+y)$ and $\sin(x-y)$, do this integral and show that this equation is indeed true.

***11.3 Odd and even functions**. Using the basic definitions of even and odd function, $f_{even}(-x) = f_{even}(x)$ and $f_{odd}(-x) = -f_{odd}(x)$, show that the following statements are true.
(**a**) The product of two even functions is even.
(**b**) The product of two odd functions is even.
(**c**) The product of an odd function and an even function is odd.

*11.4 **Integrals of odd and even functions**.

(**a**) If $f(x)$ is an odd function, show that $\int_{-L}^{L} f(x)\,dx = 0$.

(**b**) If $f(x)$ is an even function, show that $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

*11.5 Starting with Eq. (11) and using integration by parts (where appropriate) derive Eq. 12).

11.6 **A Fourier series example. Consider the function

$$f_3(x) = \begin{cases} 0 & -L \le x < -L/2 \\ A & -L/2 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}, \tag{15}$$

(**a**) Carefully graph this function.
(**b**) Find the Fourier coefficients of this function.
(**c**) Plot $f_3(x)$ and the truncated Fourier expansions of $f_3(x)$ for $M = 1$, 5, and 10.
(**d**) Identify all places where the Gibbs phenomenon occurs.

*11.7 Identify whether the following functions are odd, even, or neither. x^3, e^{-x2}, $erf(x)$, $\cosh(x)$, $\sinh(x)$.