General Solution using Normal Modes

Overview and Motivation: Last time we solved the initial-value problem (IVP) for the 1D wave equation on a finite domain with "closed-closed" bc's using the general form of the solution $q(x,t) = f(x+ct) + g(x-ct)$. Today we solve the same problem using the normal mode solutions for this system.

Key Mathematics: We utilize some integrations involving harmonic functions.

I. The Problem Defined

We are looking for the general solution to the wave equation

$$\frac{\partial^2 q(x,t)}{\partial t^2} = c^2 \frac{\partial^2 q(x,t)}{\partial x^2} \tag{1}$$

on the finite domain $0 \le x \le L$ subject to the initial conditions

$$q(x,0) = a(x) \text{ and } \frac{\partial q}{\partial t}(x,0) = b(x) \qquad \text{(2a), (2b)}$$

and the boundary conditions

$$q(0,t) = 0 \text{ and } q(L,t) = 0 . \qquad \text{(3a), (3b)}$$

This time we write the solution $q(x,t)$ as linear combination[1] of the normal-mode solutions $q_n(x,t)$

$$q(x,t) = \sum_{n=1}^{\infty} q_{(n)}(x,t), \tag{4}$$

where the normal modes can be expressed as

$$q_{(n)}(x,t) = \sin(k_n x)\left(A_n e^{i\omega_n t} + B_n e^{-i\omega_n t}\right). \tag{5}$$

Here $k_n = n\pi/L$ is the wave vector and $\omega_n = ck_n$ is the angular frequency. As before, let's make the normal-mode solutions explicitly real by setting $B_n = A_n^*$. Then Eq. (5) can be written as

[1] As written, Eq. (4) looks like a simple sum, not a linear combination, but as Eq. (5) shows, we have kept undetermined amplitudes as part of our normal modes, so Eq. (4) may be justifiably thought of as a linear combination of normal modes.

$$q_{(n)}(x,t) = \sin(k_n x)[\mathrm{Re}(a_n)\cos(\omega_n t) - \mathrm{Im}(a_n)\sin(\omega_n t)], \tag{6}$$

where we have defined a new amplitude $a_n = A_n/2$. Using Eq. (6) we may thus express the general solution [via Eq. (4)] as

$$q(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)[\mathrm{Re}(a_n)\cos(\omega_n t) - \mathrm{Im}(a_n)\sin(\omega_n t)], \tag{7}$$

As we shall shortly see, the amplitudes a_n are determined by the initial conditions. Recall, however, that normal modes already satisfy the bc's, and so the general solution as expressed by Eq. (7) automatically satisfies those bc's. We thus need not consider the bc's any further.

II. The Initial Value Problem (Yet Again!)

Let's now apply the initial conditions and see what we get. From Eq. (7) we obtain for the initial displacement

$$a(x) = \sum_{n=1}^{\infty} \sin\left(\tfrac{n\pi}{L} x\right)\mathrm{Re}(a_n), \tag{8}$$

and for the initial velocity, we obtain after differentiating Eq. (7)

$$b(x) = \sum_{n=1}^{\infty} -\omega_n \sin\left(\tfrac{n\pi}{L} x\right)\mathrm{Im}(a_n). \tag{9}$$

In Eqs. (8) and (9) we have used $k_n = n\pi/L$. So what do we have here? Well, perhaps not surprisingly, we have two equations for the amplitudes a_n in terms of the initial conditions. However, unlike the (finite) N-oscillator case, the rhs's of Eqs. (8) and (9) have an infinite number of amplitudes because the wave equation has an infinite number of normal modes!

Aside: The N -oscillator problem

This looks pretty grim, but perhaps a look back at the N-oscillator case will give us some insight into the current problem. For the N oscillator problem the equation equivalent to Eq. (8) is the extension of Eq. (19) from the Lecture (6) notes to N oscillators, which we can write as

$$\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ \vdots \\ q_N(0) \end{pmatrix} = \sum_{n=1}^{N} \begin{pmatrix} \sin\left(\frac{n\pi}{N+1}1\right) \\ \sin\left(\frac{n\pi}{N+1}2\right) \\ \sin\left(\frac{n\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{n\pi}{N+1}N\right) \end{pmatrix} \mathrm{Re}(a_n), \tag{10}$$

where we have again made the assignment $a_n = A_n/2$. Now recall what we did there. To find any particular amplitude $\mathrm{Re}(a_m)$ (which we label by m) we take the m th eigenvector expressed as a row vector and multiply Eq. (10) by that vector. We can write this multiplication as

$$\begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) & \sin\left(\frac{m\pi}{N+1}2\right) & \sin\left(\frac{m\pi}{N+1}3\right) & \ldots & \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix} \left[\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ \vdots \\ q_N(0) \end{pmatrix} = \sum_{n=1}^{N} \begin{pmatrix} \sin\left(\frac{n\pi}{N+1}1\right) \\ \sin\left(\frac{n\pi}{N+1}2\right) \\ \sin\left(\frac{n\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{n\pi}{N+1}N\right) \end{pmatrix} \mathrm{Re}(a_n) \right], \tag{11}$$

Now recall what happens in that case: when the rhs is multiplied by the m th eigenvector, the only term in the sum that survives in the sum is the on with the same eigenvector. That is, only the $n = m$ term survives, which transforms Eq. (11) into

$$\begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) & \sin\left(\frac{m\pi}{N+1}2\right) & \sin\left(\frac{m\pi}{N+1}3\right) & \ldots & \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix} \left[\begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ \vdots \\ q_N(0) \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) \\ \sin\left(\frac{m\pi}{N+1}2\right) \\ \sin\left(\frac{m\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix} \mathrm{Re}(a_m) \right], \tag{12}$$

Notice that this equation only has one coefficient $\mathrm{Re}(a_m)$on the rhs and so it can now be solved for that coefficient,

$$\mathrm{Re}(a_m) = \frac{\begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) & \sin\left(\frac{m\pi}{N+1}2\right) & \sin\left(\frac{m\pi}{N+1}3\right) & \ldots & \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix} \begin{pmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ \vdots \\ q_N(0) \end{pmatrix}}{\begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) & \sin\left(\frac{m\pi}{N+1}2\right) & \sin\left(\frac{m\pi}{N+1}3\right) & \ldots & \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix} \begin{pmatrix} \sin\left(\frac{m\pi}{N+1}1\right) \\ \sin\left(\frac{m\pi}{N+1}2\right) \\ \sin\left(\frac{m\pi}{N+1}3\right) \\ \vdots \\ \sin\left(\frac{m\pi}{N+1}N\right) \end{pmatrix}}. \tag{13}$$

The key point here is that multiplying Eq. (10) by the mth eigenvector allows us to find $\mathrm{Re}(a_m)$ in terms of the initial condition on the displacement of the system. Notice that Eq. (10) is an equation for the initial displacement of the system in terms of the set of coefficients $\mathrm{Re}(a_n)$, while Eq. (13) is an equation for any coefficient $\mathrm{Re}(a_m)$ (labeled by m) in terms of the initial displacement of the system. We can thus think of Eq. (13) as the **inversion** of Eq. (10).

Now the notation used in Eqs. (10) – (13) is rather cumbersome. Fortunately there is a more succinct way to express these equations. Looking at Eq. (10) we first note that the jth element of that equation can be written as

$$q_j(0) = \sum_{n-1}^{N} \sin\left(\tfrac{n\pi}{N+1} j\right) \mathrm{Re}(a_n). \tag{14}$$

We now notice that if we multiply this equation by $\sin\left(\frac{m\pi}{N+1} j\right)$ and them sum the equation on j,

$$\sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) q_j(0) = \sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) \sum_{n=1}^{N} \sin\left(\tfrac{n\pi}{N+1} j\right) \mathrm{Re}(a_n), \tag{15}$$

then this is indeed multiplication of Eq. (10) by the transpose of the mth eigenvector! That is, Eq. (15) is the same equation as Eq. (11), only in a much more succinct form. We now switch the order of the sums on the rhs

$$\sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) q_j(0) = \sum_{n=1}^{N} \mathrm{Re}(a_n) \sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) \sin\left(\tfrac{n\pi}{N+1} j\right) \tag{16}$$

and notice that the sum on j on the rhs is simply the product of the (transpose of) m th eigenvector with the n th eigenvector, which is nonzero only if they are the same eigenvector. That is, the sum on j on the rhs is nonzero only if $n = m$. Thus Eq. (16) simplifies to

$$\sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) q_j(0) = \mathrm{Re}(a_m) \sum_{j=1}^{N} \sin^2\left(\tfrac{m\pi}{N+1} j\right) . \tag{17}$$

This is equivalent to Eq. (12), only in much more elegant notation. As with Eq. (12) we now only have one amplitude, $\mathrm{Re}(a_m)$, on the rhs. We can thus solve Eq. (12) for this amplitude, obtaining

$$\mathrm{Re}(a_m) = \frac{\sum_{j=1}^{N} \sin\left(\tfrac{m\pi}{N+1} j\right) q_j(0)}{\sum_{j=1}^{N} \sin^2\left(\tfrac{m\pi}{N+1} j\right)} , \tag{18}$$

[which is equivalent to Eq. (13)].

Back to the wave-equation IVP

So how does this apply to the current problem? Well, the main difference between the two problems is that x is a continuous variable while j is discrete. But if we remember a bit of calculus, we should recall that an integral is really just a sum over a continuous variable. We thus might expect the sums over j in Eq. (15) to be replaced by integrals over x. Let's try it and wee what happens. Multiplying Eq. (8) by $\sin\left(\frac{m\pi}{L} x\right)$ and integrating on x from 0 to L gives us

$$\int_0^L \sin\left(\tfrac{m\pi}{L} x\right) a(x)\, dx = \int_0^L \sin\left(\tfrac{m\pi}{L} x\right) \sum_{n=1}^{\infty} \sin\left(\tfrac{n\pi}{L} x\right) \mathrm{Re}(a_n)\, dx , \tag{19}$$

and switching the order of integration and summation on the rhs produces

$$\int_0^L \sin\left(\tfrac{m\pi}{L} x\right) a(x)\, dx = \sum_{n=1}^{\infty} \mathrm{Re}(a_n) \int_0^L \sin\left(\tfrac{m\pi}{L} x\right) \sin\left(\tfrac{n\pi}{L} x\right) dx , \tag{20}$$

the continuous-variable analog to Eq. (16). Further, analogous to what happened with the sum on j on the rhs of Eq (16), the integral on the rhs of Eq. (20) is nonzero only if $n = m$. Thus Eq. (20) simplifies to

$$\int_0^L \sin\left(\tfrac{m\pi}{L} x\right) a(x)\, dx = \mathrm{Re}(a_m) \int_0^L \sin^2\left(\tfrac{m\pi}{L} x\right) dx\,, \tag{21}$$

and so, as above, we can solve for $\mathrm{Re}(a_m)$

$$\mathrm{Re}(a_m) = \frac{\int_0^L \sin\left(\tfrac{m\pi}{L} x\right) a(x)\, dx}{\int_0^L \sin^2\left(\tfrac{m\pi}{L} x\right) dx}\,, \tag{22}$$

Notice the striking similarity between Eqs. (22) and (18) [especially when you recall that $a(x) = q(x,0)$]. Now, we can simplify Eq. (22) even a bit more. Using the fact that

$$\int_0^L \sin^2\left(\tfrac{m\pi}{L} x\right) dx = \frac{L}{2} \tag{23}$$

we have the result

$$\mathrm{Re}(a_m) = \frac{2}{L} \int_0^L \sin\left(\tfrac{m\pi}{L} x\right) a(x)\, dx\,. \tag{24}$$

So we now have $\mathrm{Re}(a_m)$ expressed in terms of the initial displacement $a(x)$. Similarly, the imaginary part of a_m can be expressed in terms of the initial velocity $b(x)$ as

$$\mathrm{Im}(a_m) = -\frac{2}{L\omega_m} \int_0^L \sin\left(\tfrac{m\pi}{L} x\right) b(x)\, dx\,, \tag{25}$$

As mentioned above when discussing the N-oscillator problem, you should think of Eqs. (24) and (25) as the inversions of Eqs. (8) and (9), respectively. Also note, there is nothing special about the index m in these last two equations. We could used any variable. In particular, when thought of as the inversion of Eqs. (8) and (9) we normally use the variable n to label the amplitudes and write these last two equations as

$$\mathrm{Re}(a_n) = \frac{2}{L}\int_0^L \sin\left(\tfrac{n\pi}{L}x\right)a(x)\,dx\,. \tag{26}$$

and

$$\mathrm{Im}(a_n) = -\frac{2}{L\omega_n}\int_0^L \sin\left(\tfrac{n\pi}{L}x\right)b(x)\,dx\,. \tag{27}$$

Summarizing, Eq. (7), which can be written as

$$q(x,t) = \sum_{n=1}^{\infty}\sin\left(\tfrac{n\pi}{L}x\right)\left[\mathrm{Re}(a_n)\cos\left(c\tfrac{n\pi}{L}t\right) - \mathrm{Im}(a_n)\sin\left(c\tfrac{n\pi}{L}t\right)\right], \tag{28}$$

along with Eqs. (26) and (27) are the complete solution to the initial value problem on the finite domain $0 \le x \le L$ for the bc's $q(0,t)=0$ and $q(L,t)=0$.

III. An Example Revisited

Let's consider the problem that we solved in the Lecture 9 Notes. Looking at this problem again will give us a good comparison of the two methods of solving the initial value problem.

The problem in the Lecture 9 notes has the initial conditions[2]

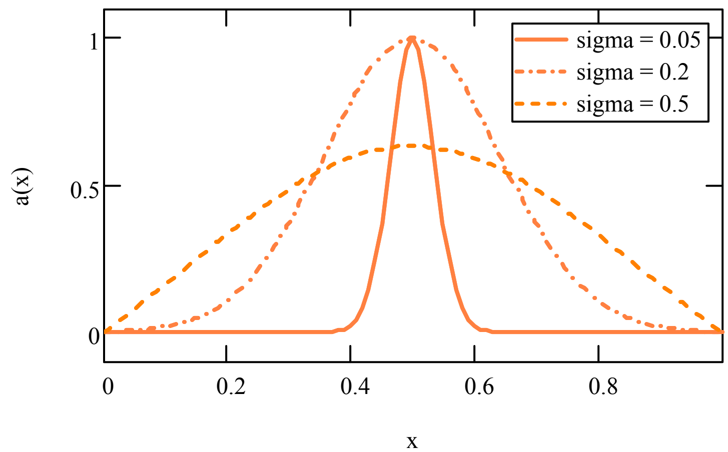
$$a(x) = A\left\{\exp\left[-\left(\frac{x-(L/2)}{\sigma}\right)^2\right] - \exp\left[-\left(\frac{(L/2)}{\sigma}\right)^2\right]\right\} \tag{29a}$$

and

$$b(x) = 0\,. \tag{29b}$$

[2] OK, so we now have another amplitude A, but it is not the same as the amplitudes A_n ($n = 1, 2, \ldots$) used earlier in Eq. (5).

Recall that $a(x)$ is a Gaussian peak that is (vertically) shifted so that the bc's are satisfied. The following figure plots $a(x)$ for the same values of the width parameter σ that we investigated in Lecture 9: $\sigma = 0.05$, $\sigma = 0.2$, and $\sigma = 0.5$.



The easy part of this particular problem is solving for $\mathrm{Im}(a_n)$. Using Eq. (27) we immediately see that $\mathrm{Im}(a_n) = 0$. Similarly (but not as simply!), using Eq. (26) we see that $\mathrm{Re}(a_n)$ is given by

$$\mathrm{Re}(a_n) = \frac{2A}{L}\int_0^L \sin\left(\tfrac{n\pi}{L}x\right)\left\{ \exp\left[-\left(\tfrac{x-(L/2)}{\sigma}\right)^2\right] - \exp\left[-\left(\tfrac{L/2}{\sigma}\right)^2\right]\right\} dx . \tag{30}$$

Unfortunately, the integral in Eq. (30) has no analytic solution.[3] Fortunately, a program such as Mathcad can numerically solve the integral. Unfortunately, as Eq. (28) indicates, we need an infinite number of a_n's! Fortunately, in most cases we only need to use a finite number of the a_n's in order to get a very good approximation to the exact solution. That is, in practice we typically use a truncated version of Eq. (28), which we can write as

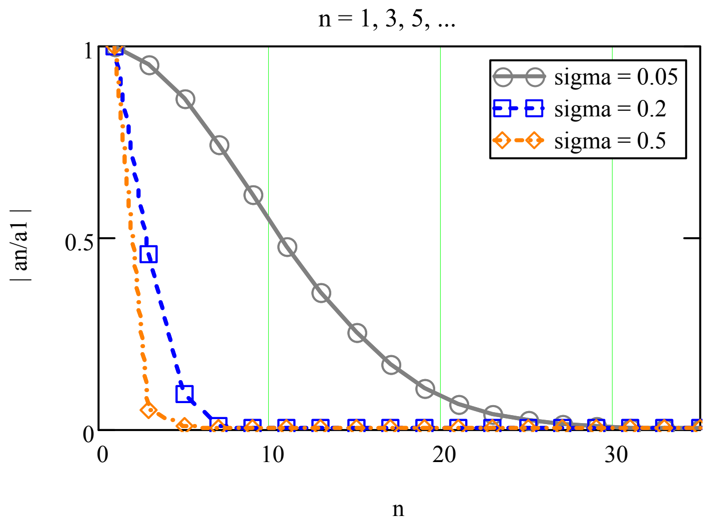
$$q(x,t)_M = \sum_{n=1}^{M} \sin\left(\tfrac{n\pi}{L}x\right)\left[\mathrm{Re}(a_n)\cos\left(c\tfrac{n\pi}{L}t\right) - \mathrm{Im}(a_n)\sin\left(c\tfrac{n\pi}{L}t\right)\right], \tag{31}$$

where $q(x,t)_M$ is the M-term approximation to $q(x,t)$. For the example at hand $\mathrm{Im}(a_n) = 0$ so we have

[3] At least as far as I know! Actually, it is not too difficult to show that $a_n = 0$ if n is even. But that still leaves the odd values of n to deal with.

$$q(x,t)_M = \sum_{n=1}^{M} \sin\left(\tfrac{n\pi}{L}x\right)\mathrm{Re}(a_n)\cos\left(c\tfrac{n\pi}{L}t\right), \tag{32}$$

So how many terms do we need to use? The answer, of course depends upon the accuracy that we require. But we can get a pretty good idea of the number of terms



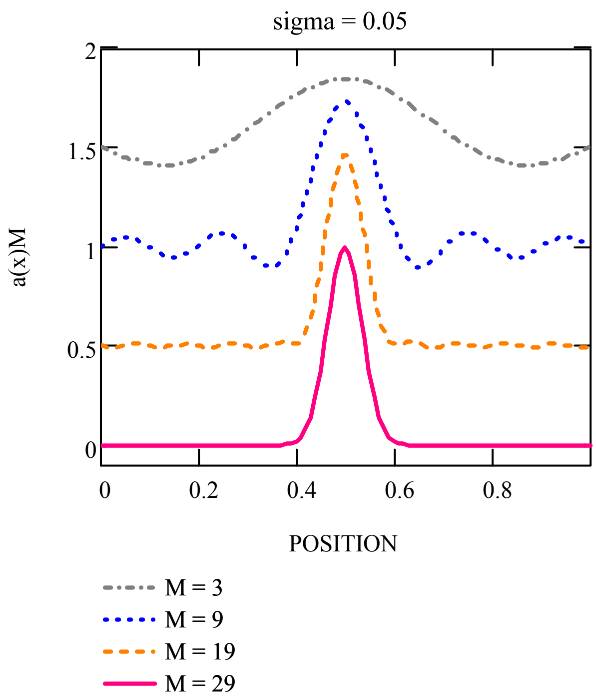
needed by plotting the absolute value of a_n vs n, as shown in the graph on the previous page. The three curves correspond to the three values of σ used in the previous graph. Note that the *y*-axis values have been normalized by $|a_1|$. Also, because $a_n = 0$ for even n, we have only plotted a_n for odd values of n.

Now this graph is very interesting: it shows that the more compact or sharper the wave (as indicated, in this case by a smaller value of σ), the more normal modes one must use to accurately describe the wave. From the above graph we see that for $\sigma = 0.05$, we need to use $M \approx 29$ to well-represent the wave, for $\sigma = 0.2$ we need $M \approx 5$, and for $\sigma = 0.5$ we need $M \approx 3$.

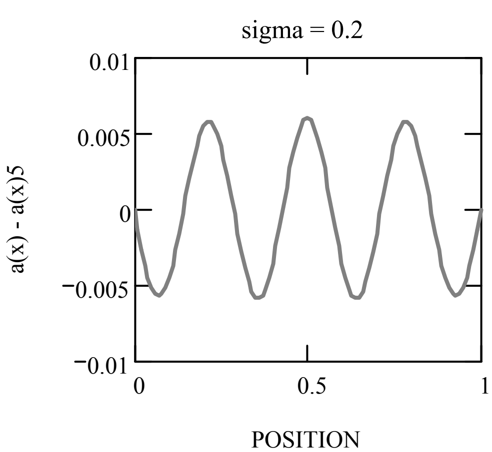
We can also ascertain the number of normal modes needed by comparing the initial condition $a(x)$ with the approximation $q(x,t)_M$ at $t = 0$. At $t = 0$ Eq. (32) becomes

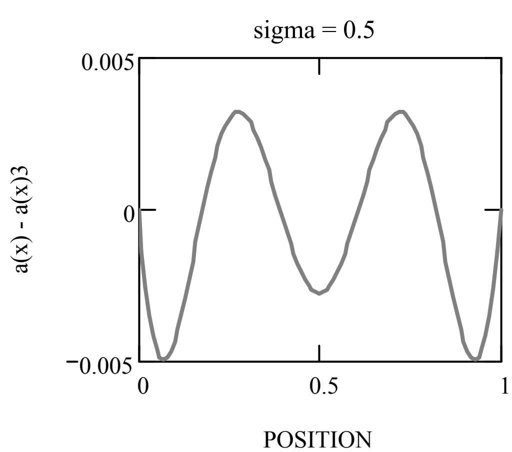
$$a(x)_M = \sum_{n=1}^{M} \sin\left(\tfrac{n\pi}{L}x\right)\mathrm{Re}(a_n), \tag{33}$$

where we have defined $a(x)_M = q(x,0)_M$. That $M \approx 29$ does a good job for $\sigma = 0.05$ is illustrated in the following figure, which plots $a_x(x)_M$ for several values of M. Notice that as M increases $a_x(x)_M$ more faithfully represents the function $a(x)$.

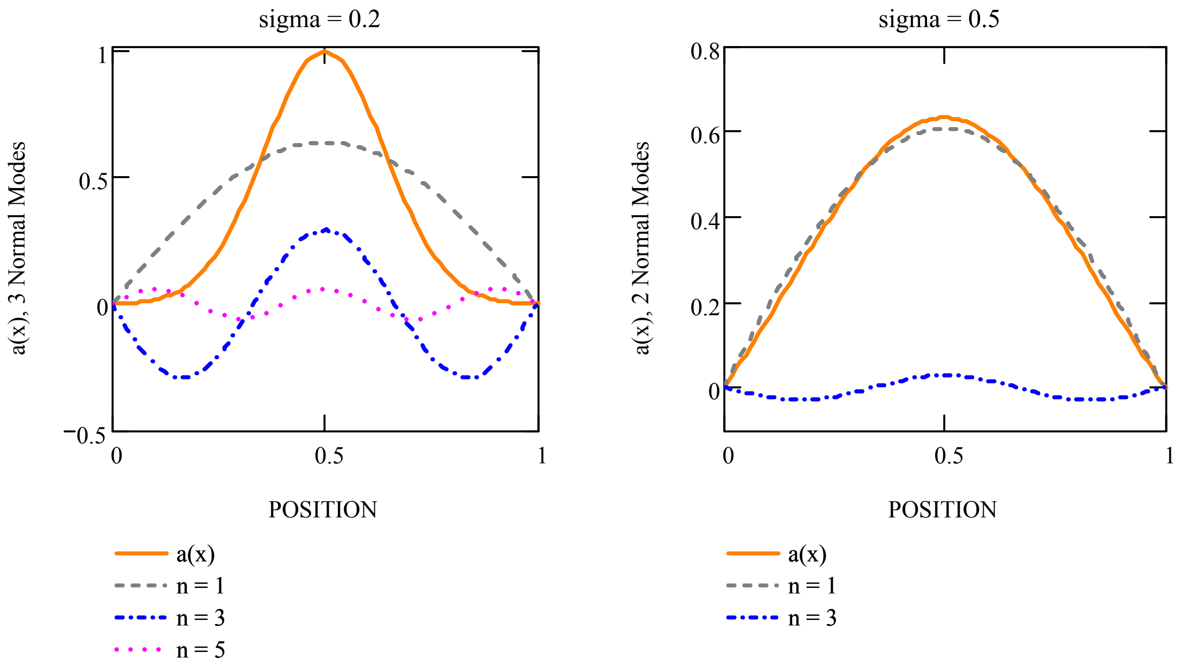


The next 2 figures show the same sort of thing for $\sigma = 0.2$ and $\sigma = 0.5$, but in a more direct fashion. This figures on the left plots the difference $a(x) - a(x)_5$ for $\sigma = 0.2$, while the figure on the right plots $a(x) - a(x)_3$ for $a_0 = 0.5$. In both cases, the error is < 0.06 for all values of x. Given the overall size of $a(x)$, these also seems like reasonable approximations.





The following two graphs are also somewhat illuminating. They plot the initial condition $a(x)$ along with the individual terms on the rhs of Eq. (27), $\sin\left(\frac{n\pi}{L}x\right)\mathrm{Re}(a_n)$, for $n = 1,\ldots,M$. From the graph for $\sigma = 0.5$ it is not hard to imagine that only the $n = 1$ and $n = 3$ are needed to accurately describe $a(x)$.



OK, so what about the time dependence? Now that we know how many normal modes we need, we can simply use Eq. (26) [where $\mathrm{Re}(a_n)$ is calculated with Eq. (24)] with the appropriate value of M. On the class web site there are videos of the resulting wave motion for all three values of σ discussed here. Both the $\sigma = 0.2$ and $\sigma = 0.5$ videos also show the motion of the individual normal modes that are used to produce the approximation $q(x,t)_M$.

Exercises

***10.1** Show that $\mathrm{Re}(a_n)\cos(\omega_n t) - \mathrm{Im}(a_n)\sin(\omega_n t)$ can be expressed more succinctly as $\mathrm{Re}\left(a_n e^{i\omega_n t}\right)$. Thus Eq. (6) can be alternatively expressed as $q_{(n)}(x,t) = \sin(k_n x)\mathrm{Re}\left(a_n e^{i\omega_n t}\right)$.

***10.2** In the notes it is stated that $\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$ is nonzero only if $m = n$ (where m and n are both integers). Using the trig identities for $\cos(x+y)$ and $\cos(x-y)$, do this integral and show that this statement is indeed true.

***10.3 An initial value problem**. Here we solve an initial value problem with the initial conditions

$$a(x) = 3\sin\left(\frac{2\pi x}{L}\right), \qquad b(x) = 0$$

(**a**) Graph $a(x)$. Does it satisfy Eq. (3), the boundary conditions?
(**b**) Find analytic solutions for $\mathrm{Re}(a_n)$ and $\mathrm{Im}(a_n)$ for this problem.
(**c**) Thus write down the solution $q(x,t)$ for this problem. What is special about this solution?

****10.4 Another initial value problem**. Here we solve an initial value problem with the initial conditions

$$a(x) = \begin{cases} \dfrac{4h}{L}x & 0 \le x < L/4 \\ \dfrac{4h}{3L}(L-x) & L/4 \le x \le L \end{cases}, \qquad b(x) = 0$$

(**a**) Carefully graph $a(x)$ (either by hand or using a computer program). Does it satisfy Eq. (3), the boundary conditions?
(**b**) Show that $\mathrm{Im}(a_n) = 0$ and $\mathrm{Re}(a_n) = \frac{32}{3}h\frac{\sin(n\pi/4)}{n^2\pi^2}$ (Either do the integrals by hand, consult an integral table, or use a program like Mathcad, and then simplify.)
(**c**) Thus write down the solution $q(x,t)$ for this initial-value problem.