Center of Mass and Motion

1. Interesting Motion

A universe that consisted of purely stationary mass would be pretty boring. Time requires change—both for its definition and for its measurement. Thus, in a static universe time would have no meaning. In such a universe there certainly would be no life, no intelligent thought, no social behavior. But motion alone doesn't automatically ensure that things will be interesting. For example, a universe in which some of the mass moved at a constant speed in one direction would be only slightly more complicated than a purely static universe. There still wouldn't be any critters walking around or music being played. In order for there to be genuinely complex behavior, there has to be *change in motion*—matter speeding up and slowing down and going around corners. When matter speeds up, slows down, or goes around corners it is said to be **accelerating**. *Complexity requires acceleration*.

Maybe you can convince yourself that acceleration is interesting but unchanging motion isn't by recalling how it feels to ride in a car traveling along a straight, flat highway that has recently been resurfaced. If the car's speedometer is fixed at a constant reading you can close your eyes and not know you are moving at all, no matter how fast the speedometer says you are moving. Of course, roads aren't straight and flat for very long stretches. You feel clues that you are moving from the little bumps and turns the car makes. Riding in an elevator may be a better example. Once the elevator gets going, only the flashing floor numbers give any hint that anything is happening – no matter how fast the elevator is traveling or whether you are going up or down. When the elevator starts up or slows to a stop, you feel that. But you do not feel the constant-speed parts of the motion. Unchanging motion feels exactly like standing still. It's boring. Unchanging motion doesn't demand much attention. But, interesting, changing motion does. As we shall se when we study Newton's laws of motion, *acceleration always has a cause; it doesn't happen spontaneously.*

2. Center of Mass

Our goal here is to begin to study the motion of real-world objects like an *E. Coli* swimming in a person's gut or a jaguar dashing after a gazelle or a satellite being launched into orbit from the cargo bay of the Space Shuttle. Each of these examples involves bodies that are made of huge numbers of atoms and occupy an extended region of space. *Each somehow involves motion of the whole as well as motion of parts* (flagella wiggling, tails snapping, antennae rotating). In addition, if we were able to look on a microscopic level, we'd see that all of the atoms in each of the bodies would also be jiggling furiously – independently of their overall macroscopic motion. The description of all of these various motions can be extremely complicated. To make progress we need a simple starting point. That's provided by the **center-of-mass** (CM) of the body of interest. Center-of-mass is a measure of where the "average" mass is.

To get some idea of what the center-of-mass means, let's think a bit about how you calculate an average. For example, suppose you've scored 70 on one exam and 100 on a second. If the two exams are of equal importance—that is, of *equal weight*—the average is (70+100)/2 = 85, a value that lies exactly midway between 70 and 100. On the other hand, suppose the exam on which you scored 100 is twice as important as the other exam; it has *twice the weight* of the other exam. What would the average be then? That situation would be equivalent to having taken *three* exams, *one* with a score of 70 and *two* with scores of 100. The average would then be $(1 \times 70 + 2 \times 100)/3 = 90$. Note that $(1 \times 70 + 2 \times 100)/3$ is the same as ((1/3)70+(2/3)100). That is, the 100 point score is 2/3 of the total weight of the scores and the 70 point score is 1/3 of the total weight. So the average can be calculated by multiplying 70 by its fraction of the total weight (1/3), 100 by its fraction of the total weight (2/3), and adding the results. Note also that the difference between 100 and 70 is 30; the average value 90 is 10 points below 100 and 20 points above 70.

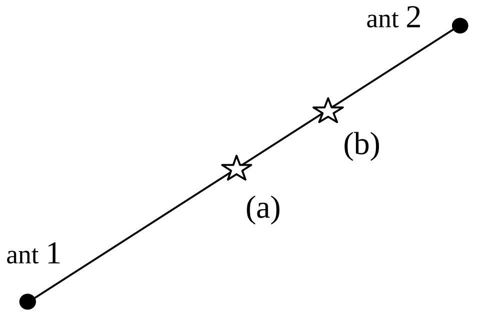

FIG. 1: The center-of-mass (CM) of two ants when the mass of ant 2 is (a) the same as the mass of ant 1 and (b) twice as big as the mass of ant 1. The CM of two ants has an equivalent mass that is equal to the sum of the masses of the two ants.

That is, 90 is 1/3 the total difference separating the scores away from the score whose weight is 2/3, and 2/3 the total difference away from the score whose weight is 1/3.

Center-of-mass is reckoned similarly. Suppose we are looking down at two ants walking on the ground. From our perspective, the ants appear as small dots—for all intents and purposes what one might call "point masses." The CM of the two ants is defined as a point on the line connecting them. That point will be closer to the more massive of the ants and farther from the less massive one—in just the same way that the weighted average of two scores is positioned between the two. Thus, if the ants have the same mass, the CM is at the point exactly halfway on the line connecting them (Figure 1 (a)). On the other hand, if one ant has twice the mass of the second, the CM is the point on the line connecting them that is 1/3 away from the ant whose mass is 2/3 the total mass and 2/3 away from the ant whose mass is 1/3 the total (Figure 1 (b)).

Example 1 Suppose one ant has a mass of 22 mg and the second has a mass of 3 mg. At one instant the two ants are 10 cm apart. Where is the CM at that instant?

Solution: The two ants' total mass is 25 mg, so the fraction of the total mass associated with the more massive ant is $22/25 = 0.88$ and with the less massive $3/25 = 0.12$. Consequently, the CM is the point on the line connecting the ants that is $(3/25)10$ cm $= 1.2$ cm away from the more massive ant and $(22/25)10$ cm $= 8.8$ cm away from the less massive ant.

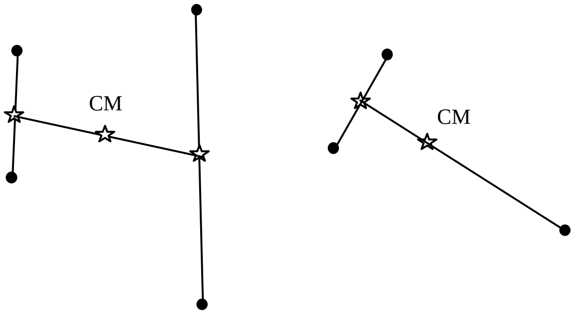

FIG. 2: The CM of a four-ant cluster (left) and a three-ant cluster (right). In each case the masses of the individual ants are equal.

The center-of-mass of a collection of more than two ants (or any point masses) can be determined geometrically in the following way. Imagine connecting the ants to each other in pairs. An ant can belong to one pair only. Thus, if you have an even number, N, of ants there will be half as many pairs, $N/2$. If there is an odd number of ants then there will be one ant left unpaired. Next, find the CM for each pair. Treat each CM as if it were an ant with the total mass of the two it came from. Then repeat the process again and again until there is only one CM.

Figure 2 shows such a construction for four ants and for three ants, all of equal mass. In each case, the CM of a pair is assigned the sum of the masses of the pair members. That's why in the three-ant example the final CM is closer to the CM of the first paired ants than the single unpaired ant. You should prove to yourself that it doesn't matter what pairs you choose to start with by taking different pairs from the ones shown in the figure; you always get the same final CM

3. CM Motion and Relative Motion

Now, as the ants move about, the CM of the collection of ants moves, too. Imagine recording the positions of the ants for a few seconds, say, by using a video camcorder. Such a recording is actually a sequence of still frames. For

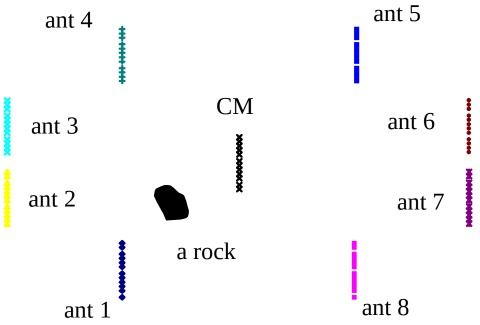

FIG. 3: Eight identical ants marching directly north at the same speed. Their CM also travels north at the same speed.

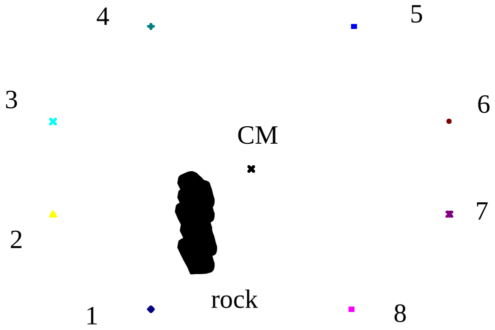

FIG. 4: The positions of the 8 ants relative to an observer moving along with the CM for the rigid translation shown in Fig. 3. Note that none of the ants' positions change.

standard American video the frames are 1/30th of a second apart. At first we assume that the camera is held fixed. Figure 3 is a cartoon of 12 successive frames superposed on top of each other-i.e., it's a "time lapse photograph." What is shown is a "platoon" of eight ants, all marching directly north at the same, constant speed. The streaks are the sequential positions of the respective ants. There are also 12 pictures of a rock. But since the rock and the camera are not moving relative to each other all 12 rock pictures are in the same place.

Let's assume that the ants all have the same mass. The CM of the platoon is sketched on each frame of the superposition shown in Figure 3. At every instant the CM is in the middle of the platoon. Thus, as the platoon marches north the CM moves north at the same rate as the ants. We give the motion displayed in Figure 3 a special name: **rigid translation**. A rigid translation is a motion involving a collection of elementary units (e.g., the ants) in which each moves in the same direction by the same amount in a given time.

It is interesting to note what an "observer" moving along with the CM would see for the case of a rigid translation. This can be accomplished by moving the camera northward at exactly the same rate that the platoon is moving. A superposition of 12 successive frames made in such a way is shown in Figure 4. In Figure 3 an observer moving with the CM takes one step north at exactly the same moment that all eight of the ants do so. Thus all eight ants remain in place relative to this observer ant, and thus their positions in this moving **frame of reference** are fixed. The rock, on the other hand, is seen to move southward relative to the camera. The main point of this little thought experiment is that *to describe a rigid translation one need only describe the motion of the CM.*

No collection of real ants moves as rigidly in lockstep as depicted in Figure 3. A more realistic pattern of ant motion might involve each of the individual ants moving north at the same *average* speed as in Figure 3 but also performing little idiosyncratic departures from the constant speed northward trend. An ant might speed up or slow down a little from time-to-time, and also possibly dart east and west a bit. Let's assume that these variations are done without rhyme or reason (i.e., they are "random"), but are always small. Figure 5 shows how the platoon and the CM might move under these conditions. Note that the motion of each individual ant is quite complicated and somewhat erratic. Nonetheless, the CM progresses northward almost as steadily as in Figure 3. As far as the motion of the CM is concerned, it's as if the erratic part of the ants' motions cancel out. Indeed the motion of the CM becomes smoother and more regular as the number of members of the collection increases.

Again, it is instructive to ask what an observer moving with the CM sees for the motion of Figure 5. In this case, an observer moving along with the CM will *not* see the eight surrounding ants remain in place. Instead, from the perspective of an observer moving with the

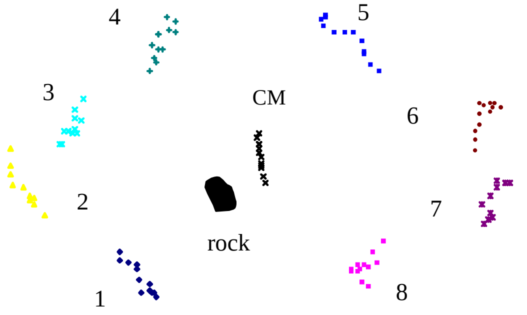

FIG. 5: Eight identical ants moving north with the same average speed as in Fig. 3, but with very small, random changes in motion. The CM moves to the north almost as steadily as in Fig. 3.

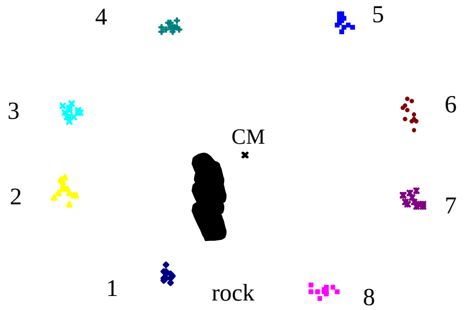

FIG. 6: The motion of the 8 ants of Fig. 5 as seen by an observer moving along with the CM.

CM each of the surrounding ants performs a little random wiggle-north-south, east-west. This is shown in Figure 6.

One can summarize the results of Figures 3 - 6 by saying that *the motion of a collection of elementary units is reducible to (1) the motion of the collection's CM and (2) the motions of each elementary unit relative to the CM.*

4. Interesting Motion and Forces

This last statement is an important key to our study of motion. But before we understand why, we need another important statement about motion: *interesting (accelerated) motion of an object always arises from the forces on that object.* When we study Newton's laws of motion we will see exactly how this works.

But for now we shall focus on the following aspect of forces and motion, which is that the motion of the CM of an object is only due to **external forces**, i.e., forces that come from some other object. This implies that the forces among the constituents of an object (**internal forces**) cannot change the motion of the CM of the object.

As an example consider a drop of water out in space somewhere. There are many forces between the atoms in the drop that cause the atoms within the drop to be constantly changing their motion as they move past each other. However, these internal forces between the atoms in the drop do not change the motion of the CM of the drop. If the drop is far enough away from any other objects so that there are no external forces on the drop, then the CM of the drop will either be stationary or moving at constant velocity (both of which are fairly uninteresting motions). It will not accelerate.

In the first half of this course we will be mostly interested in CM motion. We will thus be able to ignore the internal forces among the constituents of an object. This is a very good thing for it would be impossible to keep track of the motion all of the constituents (i.e., atoms) of a macroscopic object. In the second half of this course we will be interested in the motion of the atoms that make up an object. However, because there are so many of them, we can only describe their motion in average terms: this is the domain of the subject of thermodynamics.