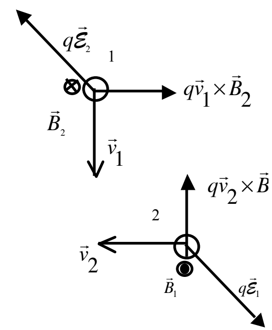
Special relativity, 7

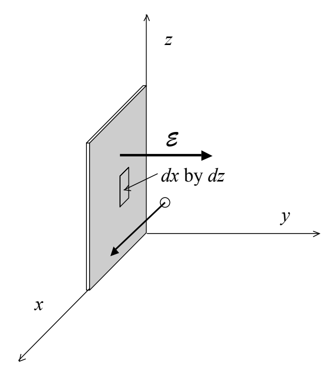
Relativistic electromagnetism

Let's return to the problem posed at the end of BK2. A charged particle traveling in a magnetic field feels a magnetic force given by $\vec{F} = q\vec{v} \times \vec{B}$, an incorrect Newtonian force, not $\vec{F} = q(\vec{v} - \vec{v}_{ether}) \times \vec{B}$, a correct Newtonian force. (But as noted in SR1, Earth doesn't appear to move relative to an ether.) As argued there, this force makes the particle veer off its original course toward some marker. On the other hand, if you were riding along in a frame in which the particle was instantaneously at rest it would have *no* velocity, and therefore feel *no* magnetic force. So does the particle veer toward the marker or not?



Here's another magnetic puzzle. Two positive charges interact as shown in the figure to the right. Particle 1 travels at the moment shown vertically downward with speed v_1. Particle 2 travels to the left with speed v_2. Particle 1 creates a *B*-field at the site of particle 2 that points out of the page. Particle 2 creates a *B*-field at the site of particle 1 that points into the page. If we assume the usual electric and magnetic fields and forces, we find that the two charges repel due to Coulomb interaction, as shown, and that the two "q v cross B" forces point as shown. If we add the two forces on particle 1 and the two on particle 2, the two net forces don't point in opposite directions. In other words, the force of 1 on 2 is not equal and opposite to the force of 2 on 1. This is a violation of Newton's Third Law, and, hence, the momentum of the two-particle system is not conserved—despite there being no external force. Don't you hate it when that happens?

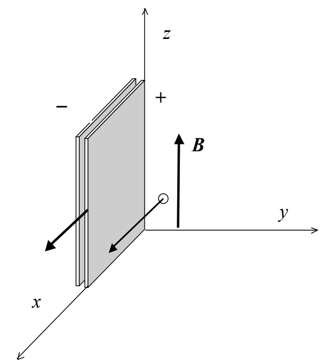
It turns out that relativity comes to the rescue in both cases.



The first puzzle has to do with how electric and magnetic fields transform from frame to frame. Let's start with a pure electric field in one frame and see what happens to it in another frame moving relative to the first. To make a nice uniform electric field we imagine an infinite sheet of charge with uniform positive charge density σ_0 (coulombs/square meter in conventional units) as measured by observer O at rest with respect to the sheet. Suppose the sheet is in the x-z plane, as in the figure to the right. In O, the electric field is $\mathcal{E}_y = \sigma_0 / 2\varepsilon_0$ (the other components equal zero). A particle flies along the $+x$-axis parallel to the sheet with speed β according to O, as in the figure. In a little square patch of size $dxdz$ there's a little charge $dq = \sigma_0 dxdz$. An observer (O′) riding along with the particle observes that lengths in the x-direction are contracted by a factor of $1/\gamma$, where $\gamma = 1\big/\sqrt{1-\beta^2}$. To conserve charge and compensate for the area decrease, O′ must measure the charge density to be $\gamma\sigma_0$. Consequently, in O′ the electric field is $\mathcal{E}'_y = \gamma\sigma_0 / 2\varepsilon_0 = \gamma\mathcal{E}_y$.

O′ also observes a magnetic field, although O does not. This is because according to O′ the charges are moving with speed β. Surface charge density times velocity is a *current* density: $K = \sigma V = \sigma c\beta$ (where K is measured in amps/meter in conventional units). Just as the infinite sheets of charge produce uniform electric fields, infinite sheets of current produce uniform magnetic fields. For the situation depicted, $B'_z = -\mu_0 K'/2$ (the other components equal zero). According to O′, $\sigma' = \gamma\sigma_0$, so $B'_z = -\mu_0\gamma\sigma_0 c\beta/2$. As there is no magnetic field in O, the magnetic field in O′ comes about as a transformation of the *electric* field in O. Because $\mathcal{E}_y = \sigma_0/2\varepsilon_0$, the two fields are related by $B'_z = -\mu_0\gamma(2\varepsilon_0\mathcal{E}_y)c\beta/2 = -\gamma\beta\mathcal{E}_y/c$. (Note that the direction of $\vec{B}'$ is along z while the direction of $\vec{\beta}$ is along x and the direction of $\vec{\mathcal{E}}$ is along y. This sounds like a vector cross product of some kind.) We thus conclude that *an electric field in one frame transforms into electric **and** magnetic fields in another*.

There is (as you might imagine) an analogous relation for magnetic fields. We can produce a uniform magnetic field in O by, for example, sliding the positive sheet of charge relative to O in the $+x$-direction with speed u. Moreover, we can get rid of the electric field in O by placing a uniformly charged negative sheet immediately behind the positive one (as in the figure to the right). Because the positive sheet is moving relative to O, O measures a higher positive charge density than in the situation above: i.e., $\sigma_+ = \gamma_u\sigma_0$, where $\gamma_u = 1/\sqrt{1-u^2}$. For this case, the electric field in O due to the positive sheet is $\mathcal{E}_y = \gamma_u\sigma_0/2\varepsilon_0$. To cancel it, the negative sheet has to have a stationary negative charge density $\sigma_- = -\gamma_u\sigma_0$. The current density measured by O is $K = \gamma_u\sigma_0 cu$ and the associated magnetic field is $B_z = \mu_0\gamma_u\sigma_0 cu/2$.



The charges on the positive sheet in O′ have a velocity equal to $u' = (u-\beta)/(1-u\beta)$, while those on the negative sheet have a velocity $-\beta$ in the x-direction. The positive and negative charge densities in O′ are $\sigma'_+ = \gamma'\sigma_0$ (that's because the positive charges are already moving in O and velocities don't simply add in SR), where $\gamma' = 1/\sqrt{1-u'^2}$, and $\sigma_- = -\gamma\gamma_u\sigma_0$. The associated current densities are $K'_+ = \gamma'\sigma_0 cu'$ and $K'_- = \gamma\gamma_u\sigma_0 c\beta$. Substituting for u' in K'_+ and adding the currents yields $B'_z = \mu_0\sigma_0\gamma\gamma_u cu/2 = \gamma B_z$. So the magnetic field in O′ is just γ times greater than the magnetic field in O. (That's like what happened to the electric fields in the previous situation.) *But* there's also an electric field in O′. That's because $\sigma'_+ \neq \sigma'_-$. The negative charges are more contracted than the positive ones in O′. Adding the two charge densities leads us to $\mathcal{E}'_y = -\gamma\gamma_u\sigma_0\beta u/2\varepsilon_0$ or, in terms of B_z, $\mathcal{E}'_y = -\gamma\beta cB_z$. In other words, *a magnetic field in one frame transforms into magnetic **and** electric fields in another*. This fact is the resolution of the apparent dilemma we started all of this with: while the particle is deflected in the negative y-direction by a magnetic field in O, its deflection in O′ is due to the electric field produced by the transformation. Life is good.

One might imagine that it would be possible to identify the electric and magnetic fields with time and space components of a 1+1 s-t vector. But that's an illusion created by the fact that in the discussion here, we have only allowed the fields to have one component each. When all six components are present the transformation laws mix them in complicated ways. This complexity actually becomes beautiful again provided the components of $\vec{\mathcal{E}}$ and $\vec{B}$ are put together in what is called the "electromagnetic field tensor." That discussion will have to wait for another course, but for here, the take-home lesson is that relativity truly unifies electric and magnetic effects just as it truly unifies space and time and momentum and energy.

The second puzzle described at the outset is yet another example of how electromagnetism is at odds with Newtonian physics—only now it doesn't look like the speed of light has anything to do with the problem. That's actually wrong. The resolution of the problem has everything to do with the speed of light; it's just that the way it works is exceedingly subtle. To short circuit the full story, photons carry momentum as well as energy (as we found in SR6). In essence, this is how the conservation of momentum for the system in the figure is re-established. There's more momentum than just carried by the charges: electromagnetic fields carry momentum, too. When carefully accounted for, momentum of charges + momentum of fields *is* conserved. Thank goodness!

The subtle story of how electric and magnetic effects are unified by special relativity is a third example of how symmetry (i.e., the invariance of the speed of light to all inertial observers) and dynamics are intertwined.