

Special relativity, 6

Energy-momentum vector

The total momentum and total energy of an isolated system of particles are simultaneously conserved, *provided* the single particle momentum and total energy are defined as $p = m\tilde{\gamma}u$ and $E = m\tilde{\gamma}$, respectively:

$$p = m\tilde{\gamma}u = m \frac{dx}{d\tau}$$

$$E = m\tilde{\gamma} = m \frac{dT}{d\tau}$$

A second inertial observer will record different momentum and energy values, but by inserting the Lorentz coordinate transformations and remembering that proper time is invariant under such transformations, will record

$$p' = m \frac{dx'}{d\tau} = m \frac{d}{d\tau} \gamma(x - \beta T) = \gamma(p - \beta E)$$

$$E' = m \frac{dT'}{d\tau} = m \frac{d}{d\tau} \gamma(T - \beta x) = \gamma(E - \beta p)$$

In other words, momentum and energy (measured in units of mass) transform in exactly the same beautifully symmetric way as space and time (measured in units of length) under a Lorentz transformation.

If we think of (x, T) as the space and time components of a “position” vector in 1+1 dimensional space-time, then (p, E) are the space and time components of a *momentum-energy vector* that transforms precisely the same way as the position vector. (The generalization to 3+1 dimensions in both cases should be straightforward.) We can think of a Lorentz transformation as a kind of generalized “rotation.” In ordinary geometry, rotations leave the lengths of position vectors invariant. Lorentz transformations leave proper times invariant, so we define the “length” squared of the 1+1 dimensional, s-t position vector (x, T) to be the invariant proper time squared: $\tau^2 = T^2 - x^2$. (Remember, the proper time squared can be $>$, $=$, or $<$ 0!) In analogy, the “length” squared of the 1+1 momentum-energy vector (p, E) is the invariant quantity: $E^2 - p^2$. What is this quantity that all inertial observers agree on? To see, substitute for p and E :

$$E^2 - p^2 = (m\tilde{\gamma})^2 - (m\tilde{\gamma}u)^2 = (m\tilde{\gamma})^2(1 - u^2) = m^2;$$

the invariant “length” (squared) is *the square of the particle’s mass*. In conventional units this relation can be re-expressed as

$$E_c^2 = p_c^2 c^2 + m^2 c^4. \tag{1}$$

Mass-energy and nuclear physics

The most important applications of Einstein’s mass-energy equivalence are found in nuclear physics. Atomic nuclei are composite systems consisting of collections of *nucleons*—neutrons and protons. Nucleons, in turn, are also composite systems consisting of “quarks.” (It is useful to think of nucleons as “quark atoms” and nuclei as quark “molecules.”) The fundamental relation underlying a large number of nuclear phenomena is (Equation (4) SR5, in conventional units)

$$m_{\text{nucleus}} c^2 = \sum (K + U + m_{\text{nucleons}} c^2).$$

The Heisenberg Uncertainty Principle requires that the nucleons in a nucleus, because they are confined to a small region of space, have high kinetic energies. Protons have equal electric charge and therefore repel each other—the more so the closer they are packed together. At very small separations, identical nucleons (which are fermions) repel one another because of the Pauli Exclusion Principle. The effects of Heisenberg, Coulomb, and Pauli all try to blow a nucleus apart. If there were no stable nuclei, there would be no atoms, no molecules, no *us*. Fortunately, there's a kind of glue that holds nucleons together: when they are sufficiently close, nucleons attract one another via a "strong nuclear force" (about which we'll have a lot more to say later). Thus a stably bound nucleus is a delicate balancing act involving microscopic fly-apart and the stick-together dynamics.

Though nuclei are extraordinarily smaller than a beaker of gas, they nonetheless follow the same kind of "thermodynamic" rules: the likelihood of finding a collection of nucleons in some coarse macroscopic (laboratory level) state depends on how many microscopic (internal) states can produce that coarse state and how often such states are actually visited. Thus, suppose S_1 and S_2 are two different macroscopic states of nucleons with equal total energies, $E(S_1) = E(S_2)$, but with very unequal numbers of microscopic states, $N(S_2) \gg N(S_1)$. If the microscopic dynamics allows both S_1 and S_2 to be observed a representative fraction of the time, it is much more likely to observe S_2 than S_1 .

An important example is the neutron itself. In this case, S_1 = neutron and S_2 = proton+stuff (here "stuff" is an electron plus an "electron anti-neutrino"—again, we'll come back to this fundamental process and its consequences later). The microscopic dynamics (involving constituent quarks) permits transitions between S_1 and S_2 , and, though S_1 and S_2 have the same energy, S_2 has many more possible states (because it consists of several particles flying about). The average rate for a neutron to go to the more likely configuration of a proton+stuff is determined by the microscopic interactions between its constituent quarks. If we started with a large number of free neutrons, then after about ten minutes (614 seconds, actually) on average (a time determined by the microscopic dynamics) they will equally likely be observed as a proton+stuff—that is, half of them would have undergone a transformation. Because a free neutron changes over time into more likely macroscopic states, it is said to be "unstable" and "decay." In slightly scary historical terms it is said to be "radioactive." The mean time for half of the unstable nuclei of a given type to decay is called that nucleus' "half-life."

The reason the decay products have more states than does the neutron has to do with energy. A neutron's rest energy is 939.57 MeV (million electron volts) while a proton's is 938.28 MeV. When a neutron decays into proton+stuff there is about 1.29 MeV of energy to divvy up between the resulting particles and much of this shows up as kinetic energy. Thus, the decay products can be moving about in a huge number of possible states of motion. You might wonder why there are any neutrons at all, given their proclivity to decay. The answer is again a matter of energy. The total energy of a nucleus includes Pauli Exclusion and proton repulsion contributions. When a neutron decays into a proton in a nucleus the exclusion "pressure" of the neutrons goes down, the exclusion pressure of the protons goes up, and the electric repulsion also goes up. If there is more "up" than "down," the decay of the neutron produces a spontaneous net increase in energy, a violation of energy conservation. Therefore the neutron

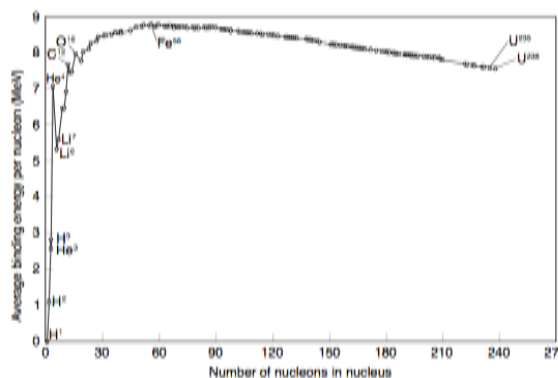
decay, in this case, is prohibited. In other words, while free neutrons always eventually decay, **neutrons in a nucleus can be stabilized by the need to conserve energy.**

It is useful to define the potential energy of interaction between the nucleons of a nucleus to be zero when large distances separate the nucleons. Suppose that the nucleons are also at rest in this separated-state. Then the total energy of the collection is simply the sum $\sum m_{\text{nucleons}} c^2$. When the nucleons are brought together in a compact arrangement, they interact and they move (because of the Heisenberg Uncertainty Principle). If the decrease in potential energy is more than the increase in kinetic energy, i.e., if

$$m_{\text{nucleus}} c^2 = \sum (K + U + m_{\text{nucleons}} c^2) < \sum m_{\text{nucleons}} c^2,$$

then by shedding the difference $\sum m_{\text{nucleons}} c^2 - m_{\text{nucleus}} c^2$, somehow, the nucleons will form a “bound” configuration—namely, the nucleus. This energy difference is also the minimum amount of energy necessary to completely separate the nucleons; it is therefore called the “binding energy” of the nucleus. Even though a nucleus is bound for a long time, it need not be stable. If the microscopic internal dynamics can produce another arrangement of nucleons that has more states—while conserving energy—then a transition to the more likely nucleonic states will eventually occur. Like the free neutron, such a nucleus is “radioactive.”

There are 252 known varieties—“isotopes”—of stable nuclei made of different numbers of neutrons and protons, and a large number more that are unstable but with very long “lifetimes.” The *binding energy per nucleon* (B/A) of some long-lived nuclei is as shown in the figure to the right as a function of *total nucleon number* (A). The isotopes are expressed as chemical symbols with nucleon number in a preceding superscript. Per nucleon, ^{62}Ni (a nickel nucleus with 28 protons and 34 neutrons) is the most stable isotope (although “iron-56”, ^{56}Fe , is pretty close).



Note that if a nucleus with $A = 200$ could be split into two nuclei each with $A = 100$, there would be a net gain in binding energy: $(8.5 - 7.8 \text{ MeV/nucleon}) \times 200 \text{ nucleons} = 140 \text{ MeV}$. This energy difference would show up as kinetic energy of the smaller nuclei. Processes similar to this, where heavy nuclei are made to split into lighter ones yielding kinetic energy of the fragments, are called **nuclear fission**. Fission is the basis for the commercial nuclear power industry; in fission reactors the kinetic energy produced is used to boil water, which is then used to generate electricity.

Similarly, if two ^2H nuclei could be made to form a ^4He nucleus, about $6 \text{ MeV/nucleon} \times 4 \text{ nucleons} = 24 \text{ MeV}$ would be released (as kinetic energy of the He nucleus). Processes similar to this are called **nuclear fusion**. Though this example only yields about 1/6 as much energy as the previous example, there are only 1/50 the number of nucleons involved. Thus, per nucleon, fusion is roughly 8 times more efficient, per unit mass, than fission in producing energy. Fusion takes place in stars. The energy released in stellar fusion events is what makes them glow. By creating sunlight, nuclear fusion is responsible for life on Earth. (On the other hand, whether it will ever be achieved in commercial reactors on Earth to make hot water and electricity is still very much in doubt. But that is another story.)