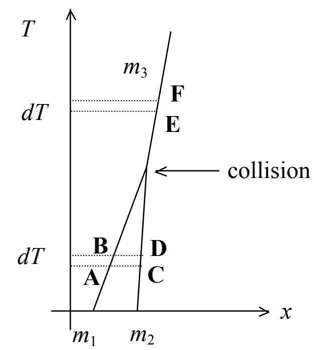
Special relativity, 5

More relativistic dynamics: Conservation of momentum



We have found that momentum will not be conserved for all observers using the classical definition of momentum. Let's examine conservation of classical momentum a little more closely. Assume that as recorded in frame O two masses, m_1 and m_2, with initial velocities u_1 and u_2 in the x-direction, collide and that after the collision one mass, m_3, emerges with velocity u_3. Make no assumption about m_3. Conservation of Newtonian momentum requires $m_1 \frac{dx_1}{dT} + m_2 \frac{dx_2}{dT} = m_3 \frac{dx_3}{dT}$ after substituting dx/dT for each u. An s-t diagram, according to O, of events leading up to and after the collision is shown to the right. In O, the positions of m_1 are recorded at events **A** and **B**, of m_2 at **C** and **D**, and of m_3 at **E** and **F**. Each pair of events is dT apart in time. The ratios of the respective dx s to dT are the corresponding u s.

Now, let's switch to frame O′ traveling with velocity β in the $+x$-direction. Newtonian relativity requires that velocities transform according to the rule $\frac{dx'}{dT'} = \frac{dx}{dT} - \beta$, because both O and O′ agree that $dT' = dT$. Thus, in O′ we have

$$m_1 \frac{dx_1'}{dT} + m_2 \frac{dx_2'}{dT} = m_1(\frac{dx_1}{dT} - \beta) + m_2(\frac{dx_2}{dT} - \beta) \textbf{ and } m_3 \frac{dx_3'}{dT} = m_3(\frac{dx_3}{dT} - \beta).$$

Momentum will then be conserved in O′, i.e., the "**and**" can be replaced by an = sign, provided that (a) momentum is conserved in O *and* (b) $m_1 + m_2 = m_3$—that is, provided *mass* is also conserved. The latter seems so obvious that it is rarely explicitly stated (this assumption is called *Newton's Zeroth Law of Motion* by Frank Wilczek in *The Lightness of Being*, Basic Books, 2008); indeed, in SR4 we just wrote down that $m_3 = 2m$ for the sticking collision of two equal masses. But, actually conservation of mass is a necessary complement to Newtonian momentum conservation. Though dx is different for different observers, dT is the same, and it's the dT' (which isn't the same as dT) in dx'/dT' that screws things up for special relativistic observers.

To address the latter problem we hypothesize that a better definition for momentum might be $p = m dx/d\tau$, where τ is the proper time between events along the world line of mass m. While dx is different for different observers, $d\tau$ is the same. What is $d\tau$? Two events separated by an infinitesimal time-like interval have an infinitesimal proper time difference defined by $d\tau = \sqrt{(dT)^2 - (dx)^2} = dT\sqrt{1-(dx/dT)^2} = dT\sqrt{1-u^2}$. Another observer will record different dx and dT values but will get the same $d\tau$. The stuff multiplying dT in this expression looks a lot like a factor of $1/\gamma$, which it exactly is if we switched to a frame at rest with respect to the traveling mass. But, of course, our second observer O′ need not be that frame. So we need to keep straight the γ associated with switching between O and O′ and the factor $1/\sqrt{1-u^2}$. Let's call the latter $\tilde{\gamma}$ ("gamma-twiddle"). Plugging $d\tau = dT/\tilde{\gamma}$ into our guess for p yields

$$p = m\tilde{\gamma}u. \quad (1)$$

Keep in mind that when measuring time in meters, velocity is dimensionless, so p has dimensions of mass. Of course, when $u << 1$, p just becomes the classical value, mu.

So, let's see what the new definition does for conserving momentum. Assume that in O $m_1 \frac{dx_1}{d\tau_1} + m_2 \frac{dx_2}{d\tau_2} = m_3 \frac{dx_3}{d\tau_3}$, where each $d\tau$ is calculated for pairs of events such as **A** and **B**, and so forth in the s-t diagram above. Now, switch to frame O′ moving as in our previous discussion. For momentum to be conserved in the collision according to O′ it must be that $m_1 \frac{dx_1'}{d\tau_1} + m_2 \frac{dx_2'}{d\tau_2} = m_3 \frac{dx_3'}{d\tau_3}$. To see if it is, use the Lorentz transformation: $dx' = \gamma(dx - \beta dT)$. (Here γ is the coefficient necessary to get from O to O′, not the mass's $\tilde{\gamma}$.) In O, dT is the same for all event pairs. Consequently,

$$m_1 \frac{dx_1'}{d\tau_1} + m_2 \frac{dx_2'}{d\tau_2} = \gamma[m_1 \frac{dx_1}{d\tau_1} + m_2 \frac{dx_2}{d\tau_2} - \beta(m_1 \frac{dT}{d\tau_1} + m_2 \frac{dT}{d\tau_2})] \textbf{ and } m_3 \frac{dx_3'}{d\tau_3} = \gamma(m_3 \frac{dx_3}{d\tau_3} - \beta m_3 \frac{dT}{d\tau_3})$$

So momentum is conserved according to O′ provided that (a) momentum is conserved according to O, *and* (b) $m_1 \frac{dT}{d\tau_1} + m_2 \frac{dT}{d\tau_2} = m_3 \frac{dT}{d\tau_3}$. Remembering that $d\tau = dT/\tilde{\gamma}$, the latter expression becomes

$$m_1\tilde{\gamma}_1 + m_2\tilde{\gamma}_2 = m_3\tilde{\gamma}_3.$$

This expression is similar to the requirement that mass has to be conserved for Newtonian momentum conservation to hold. But, note the presence of the gamma-twiddle factors; this is not simply conservation of mass—though it reduces to it when the gamma-twiddles are all close to 1. What is this new and extremely important piece of physics?

Force and the work-energy theorem

To answer this question, we explore what implication the new definition of momentum has for the concept of force. We would like to apply a force to m for a bit of time dt, then, as in Newtonian physics, set $F = \frac{dp}{dt}$, where p is the correct relativistic momentum measured in conventional units. Supposing this is valid (it certainly is when the speed of m is << c), we can ask, what work does F do when m is displaced, starting from rest, through $dx = vdt$? Thus, $W = \int_{x'=0}^{x} F\,dx' = \int_{x'=0}^{x} \frac{dp}{dt'} v dt' = c^2 \int_{u'=0}^{u} u' d(m\tilde{\gamma}' u') = mc^2(\tilde{\gamma} - 1)$ (in conventional units). Now, classically the *Work-Energy Theorem* says that the work done by a force acting on m in this way raises m's kinetic energy from 0 to K_C, where K_C is kinetic energy in *conventional* units (joules, for example). So, should we write $K_C = mc^2(\tilde{\gamma} - 1)$? Recall that we have previously argued that for small u, $\tilde{\gamma}-1=1\big/\sqrt{1-u^2}-1\approx\frac{1}{2}u^2$, so, remembering that $u=v/c$, this leads to $K_C\approx\frac{1}{2}mv^2$—the usual Newtonian form—when $v<<c$. Therefore, it is reasonable to state that *the relativistic kinetic energy is*

$$K_C = mc^2(\tilde{\gamma}-1)\,. \tag{2}$$

Equation (2) implies $K_C+mc^2=m\tilde{\gamma}c^2$, where mc^2 is defined as the mass's *rest energy* (in J; this is Einstein's famous mass-energy equivalence, "$E=mc^2$").

<u>Example</u>: In SI units the electron mass is 9.1x10^{-31} kg. The electron rest energy is (9.1x10^{-31} kg)x(3x10^{8} m/s)^2 = 8.2x10^{-14} J. It is often more useful to express this energy in electron volts (eV), where 1 eV = 1.6x10^{-19} J. In these units the electron rest energy is 5.11x10^{5} eV (about 0.5 MeV). Usually, when discussing properties of particles, such as electrons, one quotes mass in units of eV/c^2 (or MeV/c^2 (millions of eV) or even GeV/c^2 (billions of eV), or, more casually (and in the spirit of taking the speed of light to be 1), just (M or G) eV. Thus, the electron mass is often stated as about 0.5 MeV. (Neutrons and protons weigh almost 2000 times more than an electron so their mass is about 1 GeV.)

The quantity $m\tilde{\gamma}c^2$ is the freely moving mass's *total energy* (in J).

Note that when $u=1$, $\tilde{\gamma}=1\big/\sqrt{1-u^2}$ blows up. Photons travel at the speed of light and experiments show they carry finite kinetic energy and momentum: in conventional units, $E=hc/\lambda$ and $p=E/c$, where h is *Planck's constant* and hc = 1240 eV-nm (where 1 eV = 1.6x10^{-19} J). If photons had finite rest mass, both E and p would be infinite. Thus, it must be that m for a photon (or any particle traveling at the speed of light) is zero such that the product $m\tilde{\gamma}$ is E/c^2 (in conventional units, and just E when c = 1). Because $\tilde{\gamma}$ blows up as u approaches 1, no particle with rest mass can be accelerated to the speed of light: it would take an infinite amount of work. Thus, luxons remain luxons (until they are "destroyed"), and tardyons remain tardyons. (We've argued previously any tachyons probably can't interact with slowly moving matter. But tachyons have another problem: for them $\tilde{\gamma}$ is imaginary! Imaginary energy and momentum? Hmm.)

We can now interpret the extra condition, $m_1\tilde{\gamma}_1+m_2\tilde{\gamma}_2=m_3\tilde{\gamma}_3$, for all observers to agree on momentum conservation. Multiply through by c^2 and find that **momentum will be conserved in all frames if the <u>total</u> energy of the freely moving particles is as well**! In other words, conservation of momentum and conservation of energy are intimately part of the same general law: conservation of energy-momentum. So *relativity* not only mixes space and time, it also *mixes energy and momentum*. All of this results from the fact that all inertial observers measure the speed of light to be the same. This is the second example of how symmetry and dynamics are intertwined.

<u>Example</u>: Suppose m traveling with initial velocity u_0 collides with a second m, initially at rest. After the collision, the two stick together and travel away with velocity u_f. We write

$$m\tilde{\gamma}_0 u_0 = M\tilde{\gamma}_f u_f$$

for conservation of momentum and

$$m\tilde{\gamma}_0 + m = M\tilde{\gamma}_f$$

for conservation of energy. Combining the two equations yields

$$u_f = u_0 \frac{\tilde{\gamma}_0}{\tilde{\gamma}_0 + 1}$$

$$M = m\frac{\tilde{\gamma}_0 u_0}{\tilde{\gamma}_f u_f}$$

For example, suppose u_0 = 0.8. Then $\tilde{\gamma}_0$ = 1.667, u_f = 0.5, $\tilde{\gamma}_f$ = 1.155, and M = 2.309 m. In other words, u_f/u_0 = 0.625, *NOT* the Newtonian value of 0.5, while M > 2 m.

How come, in the example, M is greater than 2 m? Well, the initial kinetic energy in the example is $m(\tilde{\gamma}_0 - 1)$ = 0.667 m, while the final kinetic energy is $M(\tilde{\gamma}_f - 1)$ = 0.358 m, that is, 0.309 m is lost. But, that's just the difference between M and 2 m. *The lost kinetic energy shows up as an effective gain in the system's mass.* This is consistent with

$$m_{body} = \sum \left(K + U + m_{particles}\right) \tag{3a}$$

or in conventional units

$$m_{body}c^2 = \sum (K_C + U_C + mc^2)\,, \tag{3b}$$

The sum in (3a) or (3b) is over all of the body's internal particles and includes their (internal) energies of motion and interaction (i.e., U or U_C), as well as their rest energies. The mass of a composite body is *not* simply the sum of the rest masses of the particles from which it is made.

Example: The mass of a proton is 1.0078 (in atomic mass units), the mass of a neutron is 1.0087, but the mass of a deuteron (one proton bound to one neutron) is 2.0141 *not* 1.0078+1.0087 = 2.0165 as when the neutron and proton are far apart and presumably not interacting. The fact that the deuteron is less massive than its constituent particles means that the sum of $K + U$ for the constituents is < 0 when the neutron and proton are close. While K is always positive, U can have either sign—in this case, apparently negative.

Contrast this situation with that of the proton—also a composite body. A proton consists of two *up* quarks and one *down* quark. The estimated masses of the individual quarks are about 0.004 and 0.008, respectively, meaning that **the rest energies of the constituent quarks give rise to a just a few percent of the observed proton mass**. It must be that the sum of $K + U$ for the constituents is > 0 when the quarks are close, and even more positive when they are farther apart—otherwise they wouldn't stay close! Unlike neutrons and protons, quarks actually interact more strongly the farther apart they are. That's an interesting story for a bit later.