Special relativity, 3

A few kinematic consequences of the Lorentz transformations

How big is gamma? The Lorentz transformations depend on the factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, where $\beta = V/c$. For macroscopic objects, $\beta << 1$, so $\gamma \approx 1$. It's only for situations where V approaches *c* that γ begins to take on values significantly different from 1. Here's a brief table (you should check the γ values) indicating how this works:

β	γ
0	1
0.1	1.005
0.5	1.155
0.75	1.512
0.9	2.294
0.99	7.089
0.999	22.366

For small β, γ can usefully be approximated using the binomial expansion: $\gamma = \left(1-\beta^2\right)^{-1/2} \approx 1 + (-1/2)(-\beta^2) = 1 + \frac{1}{2}\beta^2$ (see the Appendix). For example, the error in the approximation is less than 3% even for β = 0.5, and still only about 15% for β = 0.75. (You should check these also.) The Lorentz transformations

$$T' = \gamma(T - \beta x)$$
$$x' = \gamma(x - \beta T)$$
$$y' = y$$
$$z' = z$$

reduce to Newtonian relativity when β is small because (a) $\gamma \approx 1$, (b) the term βx becomes small (as long as x is not very large), and (c) the term $\beta T = \frac{V}{c} ct = Vt$ stays finite.

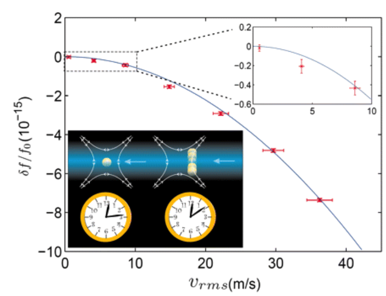
Simultaneity: Note that the Lorentz transformations can equally well be written in terms of space and time *differences*:

$$\Delta T' = \gamma(\Delta T - \beta \Delta x)$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta T)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

In Newtonian relativity, if one inertial observer records two events occurring at the same time, so will any other inertial observer. *In Newtonian physics, simultaneity is absolute*. But, that's *not true in Einsteinian special relativity*. If O records two events occurring at the same time, $\Delta T = 0$, but at different places along the x-axis, $\Delta x \neq 0$, O′ records them as occurring at different times, $\Delta T' = -\gamma\beta\Delta x$. Note that if $\Delta x > 0$ then according to O′ *the event at the greater x occurs earlier than the event at the lesser x.*

Time dilation: Consider the time $\Delta T'$ between two events that happen at the same spatial place according to O′ ($\Delta x' = \Delta y' = \Delta z' = 0$). This time interval can be thought of as the ticking of a clock at rest in the frame of O′, for example, or, if you think of the clock as an organism, the "biological time" between the two events. It is termed the *proper time along a straight worldline connecting the two events* in question. Using the Lorentz transformations we see that $\Delta x' = \Delta y' = \Delta z' = 0$ implies $\Delta x = \beta \Delta T$ and therefore $\Delta T' = \gamma(1-\beta^2)\Delta T = \Delta T / \gamma$. Thus, according to O the time that passes between the two events is $\gamma \Delta T'$, which, because $\gamma > 1$, is bigger than $\Delta T'$. In other words, *O interprets the clocks of O′ to be running slowly*.

Example: Muons are unstable particles that are produced at an altitude of about 20 km above the Earth by collisions between atmospheric atoms and cosmic rays. Unstable particles decay at random times but have an average *half-life*—the time necessary for their probability of survival to decrease by a factor of ½. Every half-life that passes decreases the survival probability by an additional factor of ½. The half-life of a muon to an observer "attached" to it (i.e., it's "rest frame") is about 2.2×10^{-6} s. Even if the average muon were traveling at the speed of light it would take about 20×10^3 m/3×10^8 m/s = 6.7×10^{-5} s to reach Earth from where it was produced; that's about 30.4 half-lives, corresponding to a probability (1/2)(1/2)…(1/2) = $2^{-30.4}$ = 7×10^{-10} of reaching Earth. In other words, exceedingly few such muons would reach the surface of the Earth. On the other hand, muons *are* detected at sea level with fair frequency. The explanation is time dilation. Muons travel with a speed of about 0.99*c* relative to the Earth, so to an observer attached to Earth a muon's clock runs about 7.1 (from the table) times too slow. That means that the dilated half-life (according to Earth) is about 15.6×10^{-6} s and that the probability of reaching Earth is about $2^{-4.3}$, or about 5%.



You might think that measurement of time dilation effects requires relative motions close to the speed of light, but time dilation for slowly moving clocks can be measured provided the clocks used are sufficiently accurate. Recently, time dilation has been measured using new, trapped-ion clocks whose "ticks" are determined to about 1 part in 10^{17}. The figure to the right shows experimental results comparing the clicking rate of a "stationary" clock with the reduced rate of one moving relative to it at speeds of a few m/s! The solid curve is the prediction of special relativity. (From: C.W. Chou, et al., Science **329**, 1630-1633 (2010).)

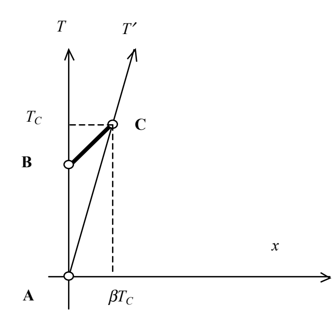
Length contraction: Suppose in the frame of O′ there's a stick lined up along the x'-axis that's L_0 long (according to O′). To correctly measure the length of the stick in O, the positions of the ends of the stick must be recorded at the same instant as the stick flies by. These two events are simultaneous in O (but not in O′) with $\Delta T = 0$ and $\Delta x = L$. Using the Lorentz transformations we can write $\Delta x' = \gamma \Delta x = \gamma L$. Because the events happen at the ends of the stick according to O′, $\Delta x' = L_0$. Thus, the length of the stick according to O is $L = L_0/\gamma$. Again, $\gamma > 1$, so *the stick's length as measured by O is shorter than its rest length*. (O′ records that $\Delta T' = -\beta\gamma L$, i.e., that the more positive end of the stick is marked *before* the less positive one.) The length L_0 is called the *proper length along a straight world line connecting the two events* in question.

Example: You might be puzzled by the previous example: a muon has to travel 20 km to get to Earth, but that would take over 30 half-lives, and so it appears to have very little chance of surviving the trip; on the other hand, according to the Earth observer, it only takes about 4 lifetimes to get to Earth so chances of survival are not vanishingly small. Which is it? The answer lies in the fact that to the muon observer the distance Earth is from the muon when the muon is created is length contracted by a factor of $1/\gamma$ = 1/7.1. So instead of Earth having to travel 20 km to get to the muon, it only has to travel about 2.8 km (according to the muon observer). Note that, to the muon observer Earth is traveling at a speed of 0.99*c*, so it only takes about 9.5x10^{-6} s to reach the muon (i.e., about 4.3 half-lives, thus finding the muon still alive with a probability of about 5% when it does so).

Proper time is invariant: Consider the combination of space and time intervals given by $(\Delta\tau)^2 = (\Delta T)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]$. This looks kind of like the Pythagorean length of a vector in four dimensions, **except** for the minus sign in front of $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$. Unlike Pythagorean length—which is always positive, $(\Delta\tau)^2$ can be positive (in which case it is said to be "time-like"), negative ("space-like"), or zero ("light-like"). When $(\Delta\tau)^2$ equals zero, the two events can be connected by a light signal. The Lorentz transformations were established to preserve the constancy of the speed of light for all observers. Thus, two events with $(\Delta\tau)^2 = 0$ in one inertial frame will have $(\Delta\tau)^2 = 0$ in any other. It turns out that it doesn't matter what $(\Delta\tau)^2$ is, its value is invariant under Lorentz transformation. When $(\Delta\tau)^2$ > 0, its value is the square of the proper time along a straight world line connecting the two events because that is the value in a frame where the two events occur at the same place (i.e., where $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 0$). When $(\Delta\tau)^2$ < 0, $(\Delta s)^2 = -(\Delta\tau)^2$ is the square of the proper length along a straight world line connecting the two events.

Example: In the previous example, we found that the muon observer reckons the time lapse between events **A** (muon creation) and **B** (muon and Earth coincide) to be about 9.5x10^{-6} s. Because **the two events happen at the same place in space (i.e., *at the muon*) according to the muon observer**, 9.5x10^{-6} s is the proper time between the events. For an observer fixed to Earth, however, event **A** occurs at an altitude of 20.0 km, while event **B** is the muon (traveling at 0.99c) reaching Earth. To the Earth observer this occurs 6.73x10^{-5} s later. The latter corresponds to 20.2x10^3 m, as measured in meters. Thus, according to the Earth observer the proper time between **A** and **B** is $(\Delta\tau)^2 = (20.2\text{x}10^3\,\text{m})^2 - (20\text{x}10^3\,\text{m})^2 = 8.1\text{x}10^6\,\text{m}^2$. The interval between **A** and **B** is time-like. The proper time according to the Earth observer is the square root of $(\Delta\tau)^2$, which, after dividing by *c* yields about 9.5x10^{-6} s. This is exactly the proper time (in s) the muon observer records for the same two events.

Doppler shift for light: Suppose that light is continuously emitted from the origin of O and that event **A** corresponds to the emission of a crest in the associated electric field. The next crest is emitted at event **B**, T_B later, according to O. The second crest reaches O′ at event **C**, whose space and time coordinates are $(\beta T_C, T_C)$ according to O. These facts are summarized in the s-t diagram to the right. Since the light from **B** to **C** covers the same distance as time, $T_C = T_B + \beta T_C$ or $T_C = T_B/(1-\beta)$. Using the Lorentz transformation we can write



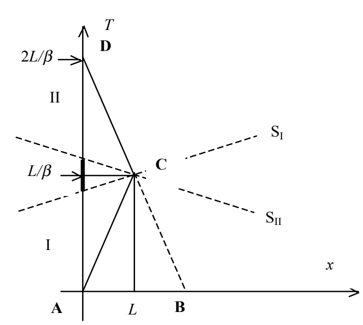
$T'_C = \gamma(T_C - \beta x_C) = \gamma(\frac{T_B}{1-\beta} - \beta^2 \frac{T_B}{1-\beta}) = \gamma \frac{1-\beta^2}{1-\beta} T_B = \sqrt{\frac{1+\beta}{1-\beta}} T_B$. Note that when time is measured in meters, the relation $c = \lambda f$ becomes $\lambda = T$, where here T is the period of the light wave (in meters). Because T'_C is the period of the light according to O′ (the "detector") and T_B is the period according to O ("the emitter") we find $\lambda_d = \sqrt{\frac{1+\beta}{1-\beta}} \lambda_e$. Since O′ is moving away from the source ($\beta > 0$), O′ observes the light to have a longer wavelength than does O; that is, O′ observes the light to be "red-shifted." This red shift is essentially due to time dilation, and is called the *Doppler shift*. If O′ were moving toward the source the sign of β would be flipped, and the light would appear to be "blue-shifted."

Example: Suppose a red (650 nm) laser beam is directed from Earth toward the muon in the previous examples. What "color" will the muon detect? Since Earth approaches the muon with a velocity

$\beta = -0.99$, $\lambda_d = \sqrt{\frac{1-0.99}{1+0.99}}(650\text{nm}) = 46\text{nm}$. The muon observes the laser emitting soft X-rays.

Incidentally, this is a cautionary tale for space travelers who would like to travel near the speed of light: even the long wave cosmic microwave photons would be blue-shifted to dangerously short wavelength gamma rays!

The "twin paradox": The story associated with this famous puzzle is: twin X leaves Earth, travels at speed β until reaching Alpha Centauri, then immediately turns around and comes back home, again at speed β. On return, twin Y, who stays on Earth, says that because X was moving at high speed, X is younger than Y's recorded time by a factor of $1/\gamma$ (due to time dilation). X counters by saying, No, it was Y who was moving, so Y *is younger than X's recorded time* by a factor of $1/\gamma$ (due to time dilation). This apparent paradox is often "resolved" by saying that X is not attached to a single inertial observer (X accelerates at the turn around point) and therefore is not entitled to use the Lorentz transformations. Consequently, only Y gets to do the calculation and it must be that X really is the younger twin. While this turns out to be true, a more careful treatment of this situation produces a new and important piece of physics.



First, let's resolve the paradox the "proper" way. The s-t diagram to the right shows: (a) X leaves Earth at event **A** in frame O′; (b) another frame, O″, heads toward Earth leaving from event **B** at the same time, according to O, as event **A**; (c) O′ gets to Alpha Centauri at event **C** and X immediately switches to frame O″ (suffering in the process a harrowing acceleration); (d) O″ arrives at Earth at event **D**. Suppose the velocity of O′ relative to Earth is β and of O″ relative to Earth is $-\beta$. Suppose also that the distance from Earth to Alpha Centauri, according to O, is L. According to O, it takes X an amount of time equal to L/β to get to **C**. Y's biological time between events **A** and **D** is therefore $2L/\beta$, the proper time along the straight world line from **A** to **D**. Because s-t intervals are invariant, O′ and O″ must deduce the same proper time for that world line. The biological time for X, on the other hand, can be calculated by adding two proper times: O reckons the proper time from **A** to

C to be $\sqrt{\left(\frac{L}{\beta}\right)^2 - L^2} = \frac{L}{\beta\gamma}$ (the same as O′) and $\sqrt{\left(\frac{L}{\beta}\right)^2 - L^2} = \frac{L}{\beta\gamma}$ from **C** to **D** (the same as O′′). Thus, X's biological time from **A** to **C** to **D** must be $2L/\beta\gamma$, less than Y's by a factor of $1/\gamma$, and all observers agree on that. Incidentally, this raises an interesting point: ***the proper time between two events depends on the sequence of straight world lines connecting them, with the maximum elapsed proper time being along a single straight, connecting world line***.

But this isn't the new piece of physics, yet. To see what that is, let's consider X's story. According to X, Alpha Centauri is only L/γ away from Earth (because of length contraction) and it initially travels toward X at speed β; consequently the time elapsed for X to get to from **A** to **C** must be $L/(\beta\gamma)$. On the other hand, X knows Y's clock is running slowly by a factor of $1/\gamma$, so the time Y records when X is at **C** must be $L/(\beta\gamma^2)$, according to X. That's the time denoted by I in the s-t diagram above. Since it takes the same amount to get back home from **C**, X concludes to have aged by $2L/(\beta\gamma)$ between **A** and **D**, but, X asks, shouldn't Y have aged by even less, namely, $2L/(\beta\gamma^2)$ (i.e., I + II). (That's the supposed "paradox.") Y, a faithful believer in s-t diagrams, responds, "X, you knucklehead, you've left out the bold interval of time on the diagram. I (that is, Y) obviously aged (by an amount $2L/\beta - 2L/(\beta\gamma^2) = 2L\beta$) while you transferred from O′ to O′′!" In fact, that's right. The new piece of physics is this: *the abrupt acceleration of X at* **C** *makes a piece of spacetime unobservable to X!*

Appendix: The binomial expansion

A binomial is a sum of two terms raised to a power: $(X+Y)^p$. If we set $Z = X+Y$ and $f(Z) = (X+Y)^p$, then a Taylor expansion of $f(Z)$ around $Y=0$ yields $f(Z) = f(X) + \frac{1}{1!}Y \cdot f'(X) + \frac{1}{2!}Y^2 \cdot f''(X) + ... + \frac{1}{n!}Y^n \cdot f^{[n]}(X) + ...$ (the f' s are derivatives of $f(Z)$ evaluated at $Z = X$). This is equivalent to $(X+Y)^p = X^p + pYX^{p-1} + \frac{1}{2}p(p-1)Y^2X^{p-2} + ...$. This expansion is very useful for obtaining approximate values when $X >> Y$ and often only the first two terms are necessary, namely, $(X+Y)^p \approx X^p + pYX^{p-1}$.