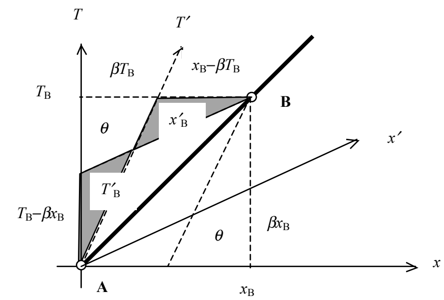
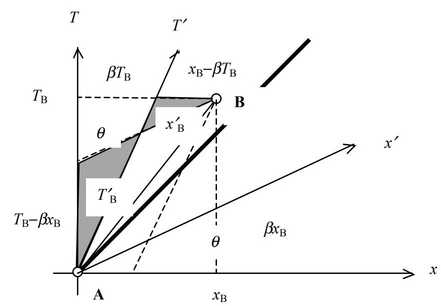
Special relativity, 2

Events not connected by light propagation

Previously, we considered two events (**A**, with s-t coordinates $\left(x_A, y_A, z_A, T_A\right) = \left(x'_A, y'_A, z'_A, T'_A\right) = (0,0,0,0)$ according to both O and O′ [moving relative to O with constant velocity β along the mutual x, x'-axes], and **B**, with coordinates $\left(x_B, 0, 0, T_B\right)$ and $\left(x'_B, 0, 0, T'_B\right))$ connected by a light pulse. In order for both observers to conclude that the speed of the pulse was 1, we had to modify the Newtonian reckoning of time, namely, that (contrary to Newton) $T'_B \neq T_B$. In particular, we found that by modifying the standard Newtonian s-t diagram as shown to the right we could argue that $\frac{T'_B}{T_B - \beta x_B} = \frac{x'_B}{x_B - \beta T_B}$ and consequently derive the successful velocity transformation rule $u'_{AB} = \frac{u_{AB} - \beta}{1 - \beta u_{AB}}$.



A moment's reflection shows that there is actually nothing special about **B** being connected to **A** by a light pulse. In the s-t diagram to the right a signal between **A** and **B** travels slower than light. The labels are all essentially the same as in the previous s-t diagram. The similar triangles we used to derive the ratios above are still there, just with altered space and time coordinate values. The same velocity transformation rule as above can now be applied to signals not traveling at speed = 1. You might well wonder how come we didn't know this rule instead of the Newtonian rule $u'_{AB} = u_{AB} - \beta$. It's because for the circumstances of Newtonian physics velocities relative to c are typically puny.



<u>Example</u>: The Juno Pluto probe is the fastest large object yet produced on Earth. It left Earth traveling at a relative speed of about 74 km/s. On the other hand, the Cosmic Microwave Background (CMB) is moving with a speed of about 370 km/s relative to Earth. Let's suppose observer O′ is fixed to the CMB and O is fixed to Earth. If the probe were heading in the same direction re the CMB as Earth, then according to Newton we would expect the probe *velocity* to be –296 km/s re the CMB. (Earth's *velocity* re the CMB is –370 km/s.) According to the Einstein picture, $\beta = 370/c$ =1.23x10^{-3} and $u = 74/c$ =2.5x10^{-4} (c = 300x10^{3} km/s). According to Einstein, what is u'? Using the velocity transform rule above, we find u' = –296*(1.0000003) km/s. No wonder we didn't know how to properly add velocities before we started thinking about light.

Incidentally, we can make little things (like electrons) go much faster than the Pluto probe. For a circumstance where a particle traveling with speed = ¾ relative to the laboratory emits another particle traveling relative to it, in the "forward" direction, with speed = ¾, the speed of the second particle relative to the laboratory is not ¾ + ¾ = 1.5, but rather 1.5/[1+(3/4)(3/4)] = 0.96.

If you examine the second s-t diagram you'll see that $\dfrac{T'_B}{T_B - \beta x_B} = \dfrac{x'_B}{x_B - \beta T_B} > 1$. Let's call this ratio γ. Thus, we can conclude that the rules of transformation between the x and T coordinates of O and O′ are

$x' = \gamma(x - \beta T)$ (as opposed to Newton's $x' = x - \beta T$)

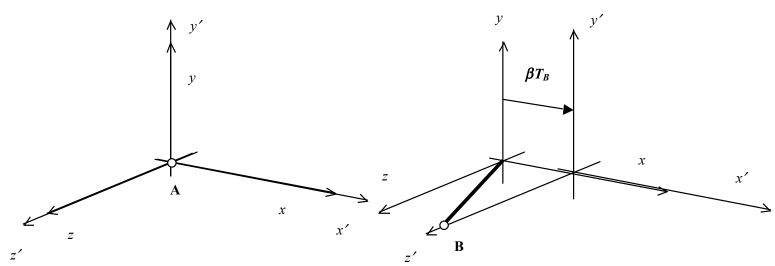
and

$T' = \gamma(T - \beta x)$ (as opposed to Newton's $T' = T$).

Notice, incidentally, the beautifully symmetric way in which the space and time coordinates (both measured as length) transform. This is the origin of the modern notion of "space-time" as opposed to Newton's more awkward "space AND time." If all we want to do is compute relative velocities along the *x*-axis we don't need to know what γ is, because γ cancels out when taking space to time ratios (velocities). But, γ has profound consequences for other kinematic and dynamic quantities of interest so evaluating it is essential. To do this, we make an appeal to symmetry: there is no difference between observers O and O′. According to O′, O is traveling in the $-x'$ direction and therefore O′ should relate the coordinates of events in the two frames as $x = \gamma(x' + \beta T')$ and $T = \gamma(T' + \beta x')$. Now, if we replace x' and T' in the latter equation by the expressions above we get $x = \gamma^2\left(1 - \beta^2\right)x$, which for arbitrary values of x implies

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \qquad (1)$$

Events in other directions: Lorentz transformations



It's important to keep in mind what events look like in space pictures as well as in s-t diagrams. The figure to the right shows the *spatial* coordinate frames of O and O′ at two times: when **A** occurs and later, when **B** occurs. The picture is drawn according to O. **A** is a flash of light from the two overlapping spatial origins at $T = T' = 0$. Later, the flash is detected at a point on the z'-axis (event **B**). The distance the origin of O′ has traveled between **A** and **B** along the x-axis is βT_B. Thus, O records **B** as taking place at $x_B = \beta T_B, z_B, T_B$, while O′ records it at $x'_B = 0,\ z'_B,\ T'_B$. Using the time transformation rule above, we have $T'_B = \gamma(1-\beta^2)T_B$. O′ records the speed of signal propagation from **A** to **B** to be $z'_B / T'_B = 1 = z'_B / [\gamma(1-\beta^2)T_B]$. O, on the other hand, records the speed of signal propagation to be $\sqrt{z_B^2 + \beta^2 T_B^2}\,/\,T_B = 1$. The latter relation can be solved for T_B: $T_B = z_B / \sqrt{1-\beta^2}$. If this expression is inserted into $1 = z'_B / [\gamma(1-\beta^2)T_B]$, we obtain $\gamma = \dfrac{z'_B}{z_B}\dfrac{1}{\sqrt{1-\beta^2}}$. Because $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, we conclude that $z'_B = z_B$. The same result would be obtained if we located **B** on the y'-axis. Thus, we conclude that **lengths perpendicular to the direction of relative motion don't change**.

We are now able to write down the correct and complete set of transformations for positions and times according to two observers in constant relative motion (whose origins coincide at $T = T' = 0$):

$$
\begin{aligned}
T' &= \gamma(T - \beta x) \\
x' &= \gamma(x - \beta T) \\
y' &= y \\
z' &= z
\end{aligned}
\qquad (2)
$$

Note that for two events separated by infinitesimal times and distances

$$
\begin{aligned}
dT' &= \gamma(dT - \beta dx) \\
dx' &= \gamma(dx - \beta dT) \\
dy' &= dy \\
dz' &= dz
\end{aligned}
$$

The latter lead immediately to the velocity transformations:

$$
\begin{aligned}
u'_x &= \frac{u_x - \beta}{1 - \beta u_x} \\
u'_y &= \frac{u_y}{\gamma(1 - \beta u_x)} \\
u'_z &= \frac{u_z}{\gamma(1 - \beta u_x)}
\end{aligned}
\qquad (3)
$$

where $u'_x = dx'/dT'$, $u_x = dx/dT$, and so forth. The coordinate transformations are called the *Lorentz transformations*. Together, the Lorentz [(2)] and velocity [(3)] transformations constitute Einstein's kinematic theory of *Special Relativity* ("special" because the relatively moving observers are both restricted to being inertial).