

## Structure of matter, 2

### Quantum electrodynamics

Quantum electrodynamics (QED) is the quantitatively best physical theory yet developed. Where it has been tested, it agrees with experiment to at least a few parts in  $10^{11}$ ! It is a theory of how charged particles and photons interact. It starts with electrons, for example, described by the Dirac Equation, interacting with photons described by a suitable electromagnetic potential energy. The latter cannot be due only to electrostatics because an electric field in one frame of reference is an electric *and* a magnetic field in a second frame moving relative to the first; so  $U = q_E \phi_E$  is inadequate. If we write  $U = q_E \phi_E + U'$  the Dirac Equation become  $E_{op} \psi = (c \vec{\alpha} \cdot \vec{p}_{op} + \eta m c^2 + q_E \phi_E + U') \psi$ . This is the same as  $(E_{op} - q_E \phi_E) \psi = (c \vec{\alpha} \cdot \vec{p}_{op} + \eta m c^2 + U') \psi$ . In this form, it looks like the energy is being transformed by subtracting the electric potential energy from it. But special relativity treats energy and momentum on the same footing, so  $U'$  should be incorporated into  $\vec{p}_{op}$ , for example, as  $c \vec{\alpha} \cdot (\vec{p}_{op} - q_E \vec{A}_E)$ , that is,

$$(E_{op} - q_E \phi_E) \psi = [c \vec{\alpha} \cdot (\vec{p}_{op} - q_E \vec{A}_E) + \eta m c^2] \psi. \quad (1)$$

In electromagnetism, the field  $\vec{A}_E$  is called the electromagnetic “vector potential.” (Formally, the magnetic field is derived from  $\vec{A}_E$  by taking its curl:  $\vec{B} = \text{curl}(\vec{A}_E)$  (where  $\text{curl}(\vec{A}_E) = \nabla \times \vec{A}_E$ ). The electric field is  $\vec{E}_E = -\text{grad}(\phi_E) - \frac{\partial \vec{A}_E}{\partial t}$  (where  $\text{grad}(\phi_E) = \nabla \phi_E$ .) When the Maxwell Equations are written in terms of the scalar ( $\phi_E$ ) and vector potentials ( $\vec{A}_E$ ), we find

$$E_{op}^2 \phi_E - c^2 p_{op}^2 \phi_E = -(\hbar c)^2 \rho_E / \epsilon_0, \quad E_{op}^2 \vec{A}_E - c^2 p_{op}^2 \vec{A}_E = -(\hbar c)^2 \mu_0 \vec{J}_E. \quad (2)$$

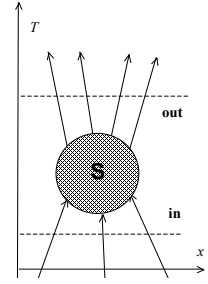
The left hand sides of both of the latter equations correspond to field equations derived from the relativistic energy-momentum relation for (spin-1) massless free particles; the right hand sides are the sources of the potentials, namely, the charge and current densities associated with the  $\psi$  fields in (1). These densities, in turn, are related to the square of the  $\psi$  field. In other words, Equations (1) and (2) are a set of coupled, *nonlinear* partial differential equations. As you might guess, finding exact solutions for general conditions is impossible. We have to be content with approximate solutions.

Example: Electrons in atoms obey (1), which contains both relativistic and magnetic effects. To first order where the electromagnetic part is approximately just the electrostatic Coulomb interaction, (1) predicts only smallish relativistic corrections to the Schrödinger description. In hydrogen, the corrections lead to an about 1% lowering in the energy levels. In heavier atoms, however, the corrections are more significant. In Pb, for example, the energy levels are lowered by several eV. Indeed, it has been argued that 80% of the potential difference generated in lead-acid batteries is due to relativistic corrections to the Schrödinger predictions. In this sense, your car starts because of relativistic quantum mechanics.

## Quantum field theory

QED was historically the first true quantum field theory. *Quantum field theory is all about creating and annihilating particle states.* A *particle state* can be thought of as a vector whose components are the numbers of particles with all of the values of energy, momentum, and spin that the associated field equations (e.g., (1) and (2)) allow. These vectors are not like electromagnetic fields; for example, they don't "live" in spacetime. Instead, they live in an abstract "particle number space." Quantum field theory describes how to transform one particle state into another. Quantum field theoretic calculations are mathematically challenging, so we won't do any—we'll only talk about the results.

The s-t diagram to the right depicts the usual formulation of QFT. At time  $T_{in}$  there are  $N$  (3 in the diagram) free (noninteracting) particles each with well-defined masses, spins, energies, and momenta, and at time  $T_{out}$  there are  $N'$  (4 in the diagram) free particles each with their own well-defined masses, spins, energies, and momenta. (Particle number need not be conserved, because pure energy can be converted into mass [and vice versa]—as long as various other conservation laws are obeyed.) In between, in the s-t blob designated **S**, the particles interact, possibly changing types and dynamical properties. The "in state" and "out state" are abstract vectors, denoted  $|in\ state\rangle$  and  $|out\ state\rangle$ , respectively, that enumerate how many particles of each type there are and what their intrinsic (i.e., mass and spin) and extrinsic (i.e., energy and momentum) properties are. The transition from "in" to "out" is formally expressed as  $|out\ state\rangle = \mathbf{S} |in\ state\rangle$ , where **S** is the so-called "scattering-matrix" (or more compactly, "S-matrix").



All of the details of the interactions are contained in **S** through the definition:

$$\mathbf{S} \equiv \exp(\mathbf{INT}) = I + \mathbf{INT} + \frac{1}{2!}\mathbf{INT}^2 + \frac{1}{3!}\mathbf{INT}^3 + \dots + \frac{1}{n!}\mathbf{INT}^n + \dots ;$$

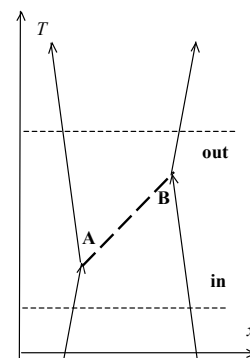
This is a generalization of the expression  $\exp(x) = 1 + x + \frac{1}{2!}x^2 + \dots$ , where  $x$  is just a number. In this expression  $I$  is the identity matrix (1s along the diagonal, 0s everywhere else) and **INT** is a matrix that describes how particle states interact. For example, electrons interact electromagnetically, so, in this case, **INT** will describe the coupling of electron and electromagnetic fields. Each successive term in this expansion involves interactions of ever-greater complexity. In general, the in- and out-vectors can have many components (i.e., states) each, representing many input conditions (states) with corresponding possible outputs. The quantity  $\mathbf{S}_{fi}$  is the element of the S matrix that connects the specific in-state  $i$  to the specific out-state  $f$ . It is a complex number called the "transition amplitude." The real number  $|\mathbf{S}_{fi}|^2$  is the *transition probability* for the process  $i \rightarrow f$ . In addition to probability, the transition amplitude is used to calculate all of the experimentally measurable quantities. Note that if there are no interactions, the in- and out-states are identical. If they are not identical, then the lead term,  $I$ , in the expansion above contributes nothing to  $\mathbf{S}_{fi}$ . The term  $\mathbf{INT}_{fi}$  corresponds to a *direct* interaction between the particles in the in- and out-states. All of the interactions in the standard model are *mediated* by other particles; that is, there are always *intermediate* states between  $i$  and  $f$ . Consequently,  $\mathbf{INT}_{fi} = 0$ . On the other hand, the term  $\mathbf{INT}_{fi}^2$  does not vanish. In fact,  $\mathbf{INT}_{fi}^2 = \mathbf{INT}_{fs} \cdot \mathbf{INT}_{si}$ , where  $\mathbf{INT}_{si}$  transforms the in-state into all possible intermediate states  $s$  and  $\mathbf{INT}_{fs}$  transforms  $s$  into the out-state. The "all possible" part here requires summing (integrating) over all

energies, momenta, and spins allowed by conservation laws for the intermediary states. Higher order terms in  $\mathbf{S}_I$  involve additional intermediary states.

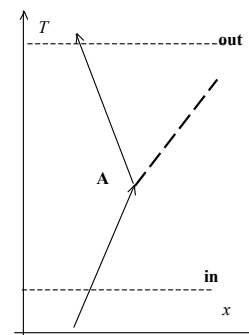
### Back to QED

The field interactions underlying QED involve products of charged fields and electromagnetic potentials: charged fields only interact through electromagnetism and, because they carry no electric charge, electromagnetic fields do not interact with one another. **S**-matrix calculations require evaluating integrals involving *many* such field product interactions. In the late 1940s, **Richard Feynman** introduced an important calculational tool that vastly simplifies QED calculations. The tool is a pictorial method for assessing the most important processes involving "real" and "virtual" charged particles and photons and for quantitatively determining the associated probability and energy of the electromagnetic phenomena.

A QED *Feynman diagram* is a kind of cartoon version of a space-time diagram on which solid lines represent the world lines of charged particles and dotted lines represent the world lines of photons, such as in the figure to the right. In QFT language, the in-state consists of two electrons (in the example), one with energy  $E$ , momentum  $\vec{p}$ , and spin component  $s$ , and the other with energy  $E'$ , momentum  $\vec{p}'$ , and spin component  $s'$ . The out-state consists of one electron with energy  $E''$ , momentum  $\vec{p}''$ , and spin component  $s''$ , and a second with energy  $E'''$ , momentum  $\vec{p}'''$ , and spin component  $s'''$ . As depicted, at event **A** the leftmost in-electron is "annihilated," while the leftmost out-electron is "created," and in addition, a photon is also created. Event **A** is called a *vertex*; that's where all of the electromagnetic interaction occurs. All QED vertices involve 2 charged particles and 1 photon. At **B**, a second interaction occurs: the photon created at **A** is annihilated as is the rightmost in-electron, and, in addition, the rightmost out-electron is created. That is, overall the in-electrons "interact" by "exchanging" a photon, becoming as a result the out-electrons. The intermediary photon is not detected; only the in- and out-states are detected. In this sense, the electrons are "real," but the photon is "virtual." This diagram is associated with the  $\text{INT}^2$  term in the **S** matrix expansion.



In the diagram above, total energy, momentum, and spin, in- and out-, are all conserved. This implies that Feynman's virtual exchange particles are curious. Consider the figure to the right, for example, which is part of the previous figure. By itself, the events depicted cannot occur because energy and momentum cannot be simultaneously conserved. (Suppose you are in the rest frame of the incoming electron. In this frame, before **A** there is zero kinetic energy; the total energy is only the electron's rest energy. After **A**, there is still the electron rest energy, but now there are also electron and photon kinetic energies [required to conserve momentum].) In quantum field theory energy, momentum, and spin are rigorously conserved at all times. As a result, virtual particles must carry weird, unphysical energies (including masses) and momenta. *Virtual particles do not really exist; they are only parts of the Feynman algorithm for assigning exact values to each term in the **S** matrix expansion.*



Feynman diagrams are not drawn precisely. Thus, charged particle world lines don't have to deviate when photons come and go, and virtual photons don't have to travel at the speed of

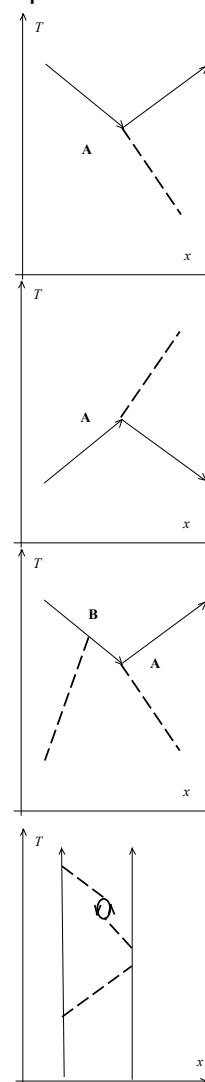
light—they can just go from one charge to another instantaneously! For each possible diagram, Feynman’s rules assign an “amplitude,” a complex-valued function of energy, momentum, and spin that describes the nature of the charged particle-photon interaction. The absolute square of the amplitude is used in integrals over all virtual particle energies, momenta, and spins to produce the **INT** terms in the **S** matrix.

In drawing Feynman diagrams for QED there are some rules. (1) A massive particle (e.g., an electron or a quark) has a world line with arrow pointing upward in time. (2) A massive anti-particle (e.g., a positron or an anti-quark) has a world line with arrow pointing *downward* in time. (This doesn’t mean that anti-particles come out of the future, it’s just a convention to keep straight what’s happening.) (3) When a photon interacts with a charged particle it does so at a vertex consisting of 3 prongs, consisting of 2 charged particle world lines and 1 photon world line (this is necessary to guarantee charge conservation). This is called a *minimal interaction vertex*. (4) Diagrams showing an allowed process can be rotated or flipped to show other allowed processes.

To see how these rules work let’s take the second diagram above and rotate it by 90° to the right. We get something like the first figure to the right. This shows a photon annihilating and spontaneously creating a positron (to the left) and an electron (to the right) at **A**. (Note that the positron ages in the upward *T* direction even though its world line arrow points down.) This process is called “virtual pair creation.” If that figure is flipped over (a “time reversal” transformation), we get the second figure on the right. There, a positron (on the right) annihilates with an electron (on the left) at **A** to create a virtual photon. To produce either *real* pair-creation or *real* pair-annihilation requires at least two real photons. But according to the rules stated above, vertices can only have three lines each. Thus, an acceptable pair creation process might be as shown to the right below. In that figure, the rightmost photon creates a real electron and a virtual positron at **A**, after which the virtual positron absorbs the leftmost photon at **B**, and “becomes” real. Thus, there are *two* real photons in the in-state and a real electron-positron pair in the out-state—and, overall, momentum and energy are conserved. Note that the net effect of all of this is that two photons (of total energy equal to at least 1.02 MeV, i.e., the sum of the out-particles masses) can effectively “collide” to create a pair of particles. *This is not expected from Maxwell’s theory of electromagnetism*, which predicts that two beams of light just pass through each other without alteration. Inasmuch as pair creation via two-photon collision is now well established experimentally, it is clear that *Maxwell’s equations are only an approximation to electromagnetic phenomena*.

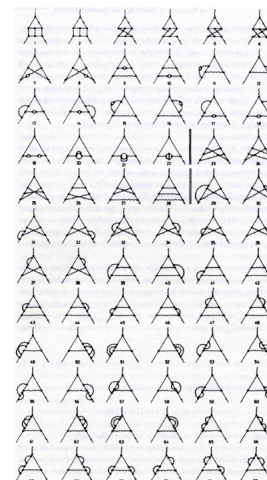
Feynman diagrams can have lots of adornments, such as the ones in the figure to the right. Note that along one of the virtual photon world lines shown, a virtual positron-electron pair suddenly appears, then quickly disappears. Adornments like the ones shown are less and less likely the more vertices are involved. The QED amplitude at each vertex has a weighting factor that is proportional to the square root of the “fine structure constant”

$\alpha_E = k_E e^2 / \hbar c \approx 1/137$  (where  $k_E = 1/4\pi\epsilon_0$  is the Coulomb force constant and  $e$  is the electronic charge). This dimensionless ratio measures the intrinsic strength of electromagnetic interactions; it shows up, for example, in all calculations involving electrons in atoms. Because the **INT** terms are also squared in calculating rates and probabilities, a diagram with two vertices contributes about  $10^4$  times more to a QED calculation than one with four



vertices. Relatively speaking, calculations in QED converge rapidly with increasing diagram complexity.

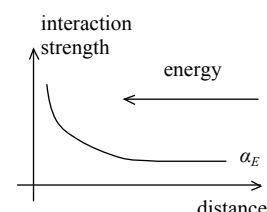
The statement that QED calculations rapidly converge is a bit glib. It turns out that the energy-momentum integrals in QED calculations involving closed loops, such as the Feynman diagram above, are infinite! Feynman found that you could get rid of all these embarrassing infinities by cutting off the integrations over energy, momentum, and spin and substituting the measured values of electric charge and electronic mass whenever you had to. A theory of this kind, where infinities can be made finite by the substitution of a small number of measured quantities, is called *renormalizable*. For example, QED can be made into a nice, tame theory by the proper substitutions of particle charge and mass. Among the many triumphs of the renormalization trick probably the most impressive is the calculation of the magnetic moment of the electron. Dirac's relativistic electron wave equation predicts that the magnetic



moment of the electron should be  $\mu_{spin} = g \left( \frac{e}{2m_e} \right) \left( \frac{\hbar}{2} \right)$ , where  $m_e$  is the electron's mass, and  $g$ ,

“the gyromagnetic ratio,” equals 2 exactly. QED, on the other hand, predicts that  $g$  is modified by all kinds of virtual particles (because of the higher order terms in the **S** matrix expansion), and that when these are taken into account  $g = 2.00231930444$ . Extraordinarily precise measurements of  $g$  (made possible by extraordinarily precise frequency measurements) give 2.00231930438 with an uncertainty of 8 in the last decimal place. This is an agreement to a few parts in  $10^{11}$ ! Never before in human history has there been such an amazing agreement between theory and experiment. QED makes many other predictions, most notably in the realm of atomic and laser physics. In all cases it has been demonstrated to be correct. Despite its quirky (and, to this day, somewhat puzzling) renormalization rule, QED must really be correct. Incidentally, among the diagrams that had to be included to get this level of precision are those shown to the right (where time runs to the right, instead of up). Each has 7 vertices, and the whole calculation took one person two years to complete (before such calculations were automated).

QED calculations for electron-electron scattering reveal a surprise: the measured value of electric charge has to depend on the total energy of the real in-particles. At higher incident energies the two electrons can get closer to one another. The figure to the right represents the results of two-electron scattering. At low collision energies, the strength of the interaction is just  $\alpha_E$ , as above. As the



collision energy increases to large values, and the two electrons can get closer to one another, QED predicts that the strength of interaction also increases. It's as if  $\alpha_E$  (and, hence, electronic charge) increases with energy. The highest collision energies reached in particle colliders is about  $10^{12}$  eV, corresponding to interaction distances of about  $10^{-10}$  nm ( $= 10^{-19}$  m). For such collisions QED appears to be correct. In other words, the Coulomb force law with constant electron charge isn't correct at extremely high energies and short distances.

We conclude this brief introduction to QED, by noting that QED *appears* to pose an incredible dilemma for the FLWR cosmology. Recall that the expansion of the universe is accelerating, and that this can be explained by a “vacuum energy density,”  $\rho_v \cong 7 \times 10^{-27}$  kg/m<sup>3</sup>. It turns out that quantum field theory seems to predict that there should indeed be a “vacuum energy density.” Remember, quantum field theory starts with a classical field that determines the allowed

energies, momenta, and spins of the particles associated with it. The quantum field then creates and annihilates such particles. The quantum field has a formal operator that measures the energy of any particle state. For electromagnetism, its form is  $\text{energy operator} = \hbar\omega(\text{number operator} + 1/2)$ . The *number operator* part first annihilates a particle state then recreates it, while multiplying the state by the number of particles with the requisite energy, momentum, and spin. That is, the *number operator* “measures” the number of particles of a given kind in a particle state. If the starting particle state is the “vacuum” of the electromagnetic field, it contains no particles (photons) to annihilate, so the action of the *energy operator* acting on the vacuum effectively reduces the *energy operator* to just  $\hbar\omega/2$  for each value of energy, momentum, and spin. The “vacuum energy density,” then, is the sum of *all* allowed  $\hbar\omega/2$  values, which is infinite.

Nature might not abhor a vacuum, but it certainly abhors infinity. As in the renormalization trick discussed above, the sum probably should be cut off somehow. One argument goes like this. The Schwarzschild radius is  $r_s = 2GM / c^2$ . Suppose the  $M$  here is  $\Delta E / c^2$  and that  $\Delta E \approx \hbar c / 2r_s$  (by suspiciously invoking the Heisenberg Uncertainty Principle). This leads to the result  $r_s = \sqrt{G\hbar c / c^4} \approx 1.6 \times 10^{-35} \text{ m}$ , a distance known as the *Planck length* (the putative distance where general relativity is no longer a proper description of spacetime) and an associated energy  $\approx 6 \times 10^{27} \text{ eV}$ . At these scales there is no sensible physical theory (yet), so at this energy there is no longer any meaning to QED. Adding up all of the allowed energies below this cutoff gives a value  $\rho_{QED} \cong 10^{94} \text{ kg/m}^3$ ! Thus, if the vacuum energy density of the FLWR universe is attributed to

both the vacuum energy of QED *and* a cosmological constant—that is,  $\rho_v = \frac{\Lambda c^2}{8\pi G} + \rho_{QED}$ —the cosmological constant  $\Lambda$  has to be negative and has to reduce  $10^{94} \text{ kg/m}^3$  to about  $10^{-26} \text{ kg/m}^3$ , an exact subtraction trick of around 120 digits!!!! The discrepancy of a factor  $10^{120}$  between  $\rho_v$  and  $\rho_{QED}$  is one of the worst predictions in human intellectual history. Its resolution awaits a new theory combining general relativity and quantum mechanics, requiring, perhaps, a whole new way of thinking about the world (e.g., string theory?).