Structure of matter, 1

In the hot early universe, prior to the epoch of nucleosynthesis, even the most primitive nuclear material—i.e., protons and neutrons—could not have existed. Earlier than 10^{-5} s or so after $t = 0$, the universe would have been a hot soup consisting of the most elementary of particles—photons, electrons, positrons, neutrinos, quarks, and gluons. We now turn to the," "Standard Model of Particle Physics," our current understanding of these elementary building blocks and their interactions. The Standard Model of Particle Physics (SMPP), developed in fits and starts over the past 50 years, is a quantitatively predictive theory of subatomic matter. It accurately describes the structure of matter at the smallest length scales yet probed, just as the Standard Model of Cosmology (SMC)—the FLWR spacetime (including cold, dark matter and vacuum energy)—accurately describes the structure of matter at the largest observed length scales. The SMPP and SMC are not only complementary in scale, they are also complementary in "force": the SMC is only about gravity, while the SMPP says nothing about gravity at all.

The building blocks: the elementary particles

The SMPP postulates that all of the matter in the universe (with the possible exception of dark matter) consists of combinations of a small number of different kinds of "elementary" particles, that is, those that appear to have no internal structure (at least not at the energy scales currently available to probe "inside"). These particles are distinguished by their intrinsic angular momentum, i.e., their "spin." Those with half integer spin are "fermions"; those with zero or one unit are "bosons." In its "minimal" form, the SMPP consists of 12 massive fermions, all of which carry a property of matter called "weak charge." Six of these particle "flavors" (the quarks) also carry another property called "color charge," while the remaining six "flavors" (the leptons) do not. In each case, the flavors are organized into three "families." The SMPP also contains 6 bosons. Five of these have one unit of spin, while the 6th has zero spin. The five unit-spin bosons are associated with the SMPP's two fundamental interactions: gluons "mediate" the color interaction; and the $W^{\pm}, Z$, and photon mediate the electroweak interaction. The final, spin-zero, particle is the much-discussed Higgs boson. A tabular representation of the SMPP follows.

Spin 1/2 fermions

Quarks

Q	Flavor	Mass	Flavor	Mass	Flavor	Mass
+2/3	u	2	c	1,270	t	170,000
–1/3	d	4.8	s	100	b	4,200

Leptons

Q	Flavor	Mass	Flavor	Mass	Flavor	Mass
0	ν_e	<2x10^{-6}	ν_μ	<0.2	ν_τ	<20
–1	e	0.51	μ	106	τ	1,777

Spin 1 bosons

Name	Interaction	Q	Mass
gluon	color	0	0
photon	electroweak	0	0
W^+	electroweak	+1	80,400
W^-	electroweak	–1	80,400
Z	electroweak	0	91,200

Spin 0 boson

Name	Q	Mass
Higgs	0	125,000

In the tables, Q is the particle's electric charge in units of $e =$ 1.609x10^{-19} C, and mass in units of MeV/c^2. (Thus, the top quark mass is 170 GeV/c^2, and so on.) Though the wide range of mass values is weird, the SMPP "periodic tables" appear to be remarkably simple. All particles, except for the massless gluons and photons, carry weak charge. The color charge of the quarks has three values: "red, green, and blue." Gluons also carry color charge–actually, two kinds: a color and an "anti-color." If one wanted to treat all of the color combinations as different particles, there would be 18 different quarks and 8 different gluons. Moreover, every particle has a twin anti-particle, so if these are counted separately the number of particles swells to over 60.

The SMPP is formulated in the fundamental framework of **quantum field theory** (QFT). Special relativity is based on a single, well-verified fact: *all inertial observers record the same speed for light traveling in vacuum.* General relativity is based on special relativity plus a single, well-verified fact: *inertial and gravitational masses are identical*. Quantum mechanics is similarly based on a single, well-verified fact: *all quantum entities have simultaneous particle and wave properties*—the latter being associated with probabilities of particle events occurring. The marriage of quantum mechanics and special relativity begets a formalism called quantum field theory. (Note: to date, quantum mechanics *has not* yet been married to *general* relativity.) A quantum field is built from a classical field—which is a smoothly continuous function of position and time (i.e., a "wave"). The classical field part of the quantum field is a solution to a "field or wave equation." The difference between a classical and quantum field is that the amplitude of the former is just a (possibly complex) set of numbers while the amplitude of the latter is an operator that can "create" or "annihilate" particles—thus, interchanging energy and mass.

Nonrelativistic quantum mechanics: the Schrödinger field

In order to account for wave-particle duality, Schrödinger's version of nonrelativistic (i.e., low energy) quantum mechanics is based on field equation where an operator $E_{op} = K_{op} + U_{op}$ operating on a classical field $\psi(\vec{r}, t)$, "measures" a classical particle's mechanical energy, $E = K + U$. The energy operator is $i\hbar \frac{\partial}{\partial t}$. The classical kinetic energy is $\frac{p^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$ and the momentum operator in the x-direction is $-i\hbar \frac{\partial}{\partial x}$ (with similar derivatives for the other directions). Thus, the Schrödinger field equation is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + U\psi \, ; \qquad (1)$$

in (1), $U_{op}\psi$ is just U times ψ. **Because equation (1) describes electrons in computers and lasers it is responsible for a significant fraction of the world's economy**. In Schrödinger quantum mechanics there *is* a particle, but its position and properties are unknown (and unknowable) until a measurement is made. The solution to the Schrödinger Equation is a classical field, a smoothly continuous *complex* function of position and time. The quantity $|\psi|^2 = \psi^* \psi$ (where ψ^* is the complex conjugate of the complex field ψ) is interpreted as the probability per unit volume (probability density) of finding the particle at position $\vec{r}$ at time t. Because there *is* a particle, $\int\limits_{all\ space} |\psi|^2 dxdydz = 1$, i.e., the probability the particle can be detected somewhere is 1. In such problems, there is no need for creation and annihilation: the particle just *is*. In relativity, where energy and mass are interconvertible, there are situations where it is impossible to state how many particles there *are*. This ambiguity is even worse for massless particles, such as photons. When particles can come and go a new description is necessary. That is why quantum fields were invented.

Relativistic quantum mechanics: the Dirac field

In special relativity the energy-momentum relation for a massive, *free* particle is $E^2 = c^2 p^2 + m^2 c^4$ (in conventional units). Following the successful strategy of nonrelativistic quantum mechanics, it is possible to replace E and $\vec{p}$ by derivatives to make a wave equation. In operator language, the appropriate wave equation should be $E_{op}^2 \psi = (c^2 p_{op}^2 + m^2 c^4)\psi$ (where the mass-squared "operator" just consists of multiplying ψ by mass-squared). In terms of partial derivatives, this equation is

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) - \frac{m^2 c^4}{\hbar^2}\psi \ . \qquad (2)$$

Equation (2) was actually formulated by Schrödinger (in 1926) *before* he hit on his now famous equation (1). (For curious historical reasons, (2) is known as the Klein-Gordon (K-G) Equation. When m is set to zero, (2) becomes the Maxwell wave equation describing spin-1, massless particles traveling at the speed of light—i.e., photons.)

One problem with (2) is there is no potential energy (i.e., no forces). To rectify this, **P.A.M. Dirac** tried something nutty that—as nutty things sometimes do—turned out to have incredible consequences. First, he set $E = \sqrt{c^2 p^2 + m^2 c^4} + U$ then boldly proposed a new wave equation, $E_{op}\psi = \left(\sqrt{c^2 p_{op}^2 + m^2 c^4} + U\right)\psi = \left(c\vec{\alpha} \bullet \vec{p}_{op} + \eta mc^2 + U\right)\psi$, in which the square root is replaced by the sum of a momentum dependent part and a rest energy dependent part. A particle's momentum in 3+1 space-time has three components—thus, $\vec{p}_{op} = \left(p_{op,x}, p_{op,y}, p_{op,z}\right)$—and each has its own α multiplier, i.e., $\vec{\alpha} = \left(\alpha_x, \alpha_y, \alpha_z\right)$.

Solutions of the "energy" equation should also be solutions of the "energy squared" equation. Thus, for a free particle (with $U = 0$) it must be that $E_{op}^2 \psi = (c^2 p_{op}^2 + m^2 c^4)\psi$, namely, that you get the K-G equation back when you square the Dirac operator. With this requirement we find that (a) $\alpha_i \alpha_j + \alpha_j \alpha_i = \mathbf{0}$, where $i \neq j$, i and j can be $x, y,$ or z; (b) $\alpha_i \eta + \eta \alpha_i = \mathbf{0}$, for any i; (c) $\alpha_i^2 = \mathbf{1}$, for any i; and (d) $\eta^2 = \mathbf{1}$. A moment's thought shows that the last three equations cannot be simultaneously true if the α s and η are simple numbers. *But* they can be if the α s and η are *matrices*. This means ψ must also be a matrix, in fact, a column matrix. In these expressions, the bold **0** signifies a matrix with 0 everywhere while the bold **1** signifies a matrix with 1 along the diagonal and 0 everywhere else. Because ψ is a matrix, η is also a matrix.

Example: Suppose $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, $N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $(MN)_{11} = M_{11}N_{11} + M_{12}N_{21}$, $(MN)_{12} = M_{11}N_{12} + M_{12}N_{22}$, and so on. The result (show it) is $MN = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On the other hand, $NM = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So, $MN + NM = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$. Similarly, $M^2 = N^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$.

That ψ is a column matrix means there is a "cost" for going from a field equation with second order derivatives, such as the Klein-Gordon equation, to a field equation with only first order derivatives (such as Dirac): more fields (the components of the matrix) are required. Dirac found that the smallest matrices that would do the trick he wanted had to have 4 rows and 4 columns (hence, *bigger* than the matrices in the example); he also found that ψ had to be a column matrix consisting of 4 rows and 1 column. The collection of matrices can be expressed as $\vec{\alpha} = \begin{pmatrix} \mathbf{0} & \vec{\boldsymbol{\sigma}} \\ \vec{\boldsymbol{\sigma}} & \mathbf{0} \end{pmatrix}$, where **0** is a 2x2 matrix of all zeroes and each component of $\vec{\boldsymbol{\sigma}}$ is also a 2x2 matrix, and whose square is the 2x2 matrix **1**; the matrix η is $\eta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$. If we let $\psi = \begin{pmatrix} \psi_U \\ \psi_L \end{pmatrix}$, where ψ_U and ψ_L are both 2x1 vectors (i.e., $\psi_U = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \psi_L = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$), the Dirac Equation becomes two coupled equations

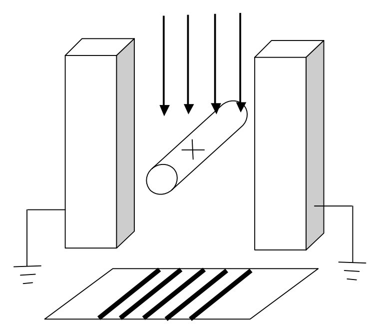
$$\begin{aligned} E_{op}\psi_U &= c\vec{\sigma} \bullet \vec{p}_{op}\psi_L + mc^2\psi_U + U\psi_U \\ E_{op}\psi_L &= c\vec{\sigma} \bullet \vec{p}_{op}\psi_U - mc^2\psi_L + U\psi_L \end{aligned} \qquad (3)$$

For zero momentum, free particle ($U = 0$) states, we see that ψ_U has the positive rest energy $+mc^2$ and ψ_L has the *negative* rest energy $-mc^2$!

Although it is not immediately obvious, it turns out that the "orbital" angular momentum, $\vec{L}$, of a freely translating Dirac mass relative to some point is *not* conserved; however, the quantity $\vec{L} + \frac{\hbar}{2}\vec{\sigma}$ *is* conserved. Thus, $\frac{\hbar}{2}\vec{\sigma}$ acts like angular momentum, but not one associated with orbital motion. Moreover, for a freely translating particle $\vec{\sigma}\bullet\vec{p}$ has two possible values, > 0 when $\vec{\sigma}$ has a component parallel to $\vec{p}$ and < 0 when it has a component antiparallel to $\vec{p}$. (These correspond to the two components of each of ψ_U and ψ_L.) These properties suggest that a free Dirac mass has spin 1/2, with its "z-component" either along the particle momentum or opposite it—states of **right-handed and left-handed helicity**, respectively. *As we will see, the Dirac Equation describes the dynamics of all quarks and leptons* (see the table above).

Now, suppose that the potential energy in equations (3) is due to an electrostatic interaction: $U = q_E \phi_E$, where q_E is the electric charge and ϕ_E is the electrostatic potential (in volts). It is straightforward to convert the negative mass equation in (3) into a positive mass equation by just multiplying both sides by –1. The negative mass equation then becomes $E'_{op}\psi_L = -c\vec{\sigma} \bullet \vec{p}_{op}\psi_U + mc^2\psi_L - q_E\phi_E\psi_L$ (where $E'_{op}\psi_L$ gives a positive energy value when $\phi_E = 0$). This equation describes a positive energy, positive rest energy particle but with opposite sign charge (and opposite helicity from ψ_U). If the original positive energy ψ_U describes an electron with charge $-e$, the second ψ_L describes a particle with the same spin magnitude and mass as the electron but with charge $+e$. The latter are called *positrons*; positrons are said to be anti-electrons. **The Dirac Equation, therefore, automatically has in it not only spin, but also *antimatter*!** Carl Anderson confirmed the existence of the positron in 1932 using a cloud chamber to investigate particles produced by high-energy cosmic rays. Anderson wrote later that he vaguely knew of Dirac's work, but didn't consciously think that the apparently positively charged electrons he was occasionally observing had anything to do with it. In short, Dirac's magical inspiration to linearize $\sqrt{p^2c^2 + m^2c^4}$ by using matrices (just a formal mathematical exercise) produced the totally unexpected physics of spin-1/2 and antiparticles. *Sometimes*, at least, the magic works!

Appendix: Wave-particle duality



An electron "double slit" apparatus—depicted to the right—provides a prototypical demonstration that quantum entities can exhibit both particle and wave properties. In the apparatus, electrons, with a known momentum *p*, are sent down a tube with grounded walls. Near the bottom of the tube is a positively charged wire and below that a charged coupled device (CCD) detector. Electrons hitting the CCD deposit their momentum at a point, just as would be expected for a particle. When the experiment is run one electron at a time, the positions of the hits are unpredictable: they appear to be random. When the positions of many hits are superposed, however, a pattern of stripes, where many hits occur separated by regions where almost none occur, appears. The pattern is the same as the interference pattern that is observed when light passes through the two slits in an opaque plate. Measuring the angle (from the wire) each stripe of maximum hits makes with respect to the straight-on direction allows one to infer that, like light, the electrons have a wavelength (λ). When the electron momentum is varied the pattern shifts, but in all cases $\lambda = h/p$, where h is Planck's constant. Thus, when an electron hits the plate it delivers momentum (and energy) just like a particle, but before hitting the plate it exhibits interference just like a wave. That's wave-particle duality. Sounds crazy, but it's true.