

General relativity, 5

Black Holes

The proper time for the Schwarzschild solution of Einstein's equations,

$$(d\tau)^2 = \left(1 - \frac{r_s}{r}\right) (dT)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} (dr)^2 - \left(\frac{L}{mr}\right)^2 (d\tau)^2, \quad (1)$$

appears to have a fatal flaw, namely, the factor $\left(1 - \frac{r_s}{r}\right)^{-1}$ blows up when $r = r_s$. So, is this a problem?

First answer: Remember, (1) is only valid for r outside M . For objects like Earth and Sun, $r \gg r_s$ so there's no problem using (1) for gravitational phenomena outside of them.

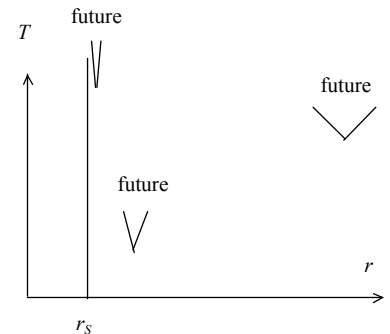
Inside the objects, (1) isn't valid so the blowing up of $\left(1 - \frac{r_s}{r}\right)^{-1}$ is not relevant. Second answer:

Calculation of the curvature of the Schwarzschild space-time shows no catastrophe at $r = r_s$, so apparently this anomaly is an *artifact* of the choice of the (T, r, ϕ) coordinates.

On the other hand, there are some interesting predicted phenomena when M is *inside* r_s —a situation that might actually occur as a result of extreme *stellar collapse*. Such condensed objects are **black holes**. To begin to see what is involved, consider light emitted in a *radial* direction in the Schwarzschild s-t. In the proper time above, set L to zero (radial motion, no angular momentum). Proper time is zero along the world line of the light, so (1)

indicates that a photon's radial velocity at any r is $\frac{dr}{dT} = \pm \left(1 - \frac{r_s}{r}\right)$. According to the

Schwarzschild observer (SO), far from M light travels at speed 1 (as it should). (Far from M there is essentially no gravity, so the SO is just an ordinary special relativistic observer at large r .) Far from M , every event in space-time has attached to it an ordinary looking light cone (i.e., one with sides having slopes = ± 1). As one nears M from far away, however, the magnitude of the speed of a photon *appears* to the SO to decrease. That means that light cones closer to M get skinnier, as shown to the right. A funky thing happens to light at the Schwarzschild radius. According to the SO, the radial velocity of light goes to zero at $r = r_s$. Thus, according to this scenario, a light source at r_s would not be able to emit light outward.



There's another aspect of the Schwarzschild proper time that supports the idea that near r_s "things get dark." Suppose light is emitted radially outward from a "stationary" source at $r > r_s$ and detected by a stationary receiver at $r \gg r_s$. We can use (1) for both emitter and

detector to derive the gravitational red shift: $(d\tau_e)^2 = \left(1 - \frac{r_s}{r}\right) (dT_e)^2$, $(d\tau_d)^2 = (dT_d)^2$. The

quantities $d\tau$ are the times recorded locally between crests of a light wave, for example, while the dT s are coordinate times between crests—for example, one tick each. In $c = 1$ units, the

time between crests is the same as the wavelength of the light, so $\lambda_d = \lambda_e / \sqrt{1 - \frac{r_s}{r}}$. The closer r is to r_s , the greater is the red shift. For $r = r_s$, $\lambda_d = \infty$: **the emitted light is completely black** to the detector!

What an infalling particle observes

Now let's "ride with" a massive particle as it falls in from infinity toward the source mass. We can use (1) above, where $r(\tau)$ is interpreted as the position coordinate the particle is passing at its proper "biological" time τ . As for radially propagating light, L is zero, though for a mass $d\tau > 0$. As in GR 4, to make sure the mass is actually falling on the correct trajectory, we have to invoke the conservation condition $\left(1 - \frac{r_s}{r}\right) \frac{dT}{d\tau} = e$. Plugging this into (1) leads to

$\left(1 - \frac{r_s}{r}\right) = e^2 - \left(\frac{dr}{d\tau}\right)^2$. For a particle falling in from $r \gg r_s$ starting out almost at rest, with $dr/d\tau \approx 0$, $e^2 = 1$ (i.e., to the particle the energy is all rest energy). Thus, for such a particle how r changes with time is determined by $\frac{dr}{d\tau} = -\sqrt{\frac{r_s}{r}}$. Integrating, we get $\frac{2}{3}r^{3/2} = -r_s^{1/2}\tau + K$,

where K is an integration constant. This expression is a statement that the particle's radial coordinate and its elapsed proper time from some starting point are simply related. Let's start the particle's clock at $\tau = 0$ when $r = r_0$. Then, the elapsed time on a clock fixed to the particle to fall from r_0 to r is $\tau = \frac{2}{3}r_s^{-1/2}(r_0^{3/2} - r^{3/2})$ (measured in meters). Thus, starting at any r_0 it takes a finite time on the particle's clock to get to $r = 0$. **To the particle there's nothing special about r_s** . This is consistent with the claim that the blowing up of $\left(1 - \frac{r_s}{r}\right)^{-1}$ at r_s is a mathematical artifact.

What a fixed observer observes

Let's switch to the perspective of an observer outside the falling particle. How much time does it take for the particle to fall some distance according to this observer (not falling with the particle)? To answer that, we differentiate both sides of $\tau = \frac{2}{3}r_s^{-1/2}(r_0^{3/2} - r^{3/2})$ with respect to T and plug $dT/d\tau$ into the constraint equation $\left(1 - \frac{r_s}{r}\right) \frac{dT}{d\tau} = e = 1$. The result is

$\frac{dr}{dT} = -\left(\frac{r_s}{r}\right)^{1/2} \left(1 - \frac{r_s}{r}\right)$. This can be integrated with some technical help (e.g., Mathematica) to yield $T = r_s \left[-2\left(\frac{r}{r_s}\right)^{1/2} - \frac{2}{3}\left(\frac{r}{r_s}\right)^{3/2} + \tanh^{-1}\left(\frac{r}{r_s}\right)^{1/2} \right] + K$, where, again, K is a constant of

integration. According to this result, regardless of where the particle starts outside of r_s , the time necessary to get to r_s involves $\tanh^{-1}(1)$, which is ∞ . Thus, **to an outside observer, a particle falling in toward the Schwarzschild radius never actually gets there!** To an

observer far from r_s an object falling into the Schwarzschild radius gets darker and darker, and it takes an infinite length of time for it to get there. Of course, the particle reckons it takes a finite time to get to r_s . Interestingly, if the particle starts at r_s it also takes an infinite amount of time to fall to $r = 0$ according to the external observer—while for the particle this trip takes just $\tau_s = \frac{2}{3}r_s$. (For a solar mass, τ_s would be about 10^{-5} s, which is somewhat less than infinity!)

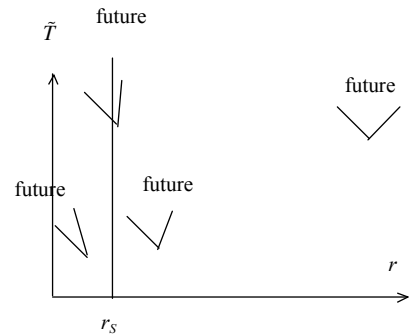
An even funkier thing happens when $r < r_s$. In this region, $1 - \frac{r_s}{r}$ is negative, going to $-\infty$ as r approaches zero. That means that the term in the Schwarzschild proper time that is proportional to $(dT)^2$ becomes negative while the term proportional to $(dr)^2$ becomes positive. In other words, the roles of these two terms in contributing to the proper time are switched for r inside r_s . Inside r_s , the coordinate T becomes a space-like coordinate, while r becomes a time-like coordinate. The schizophrenic nature of the Schwarzschild proper time is due to the particular coordinates chosen. There are better sets of coordinates that make clear what happens inside r_s . In one of these, the proper time becomes

$$(d\tau)^2 = \left(1 - \frac{r_s}{r}\right)(d\tilde{T})^2 - \left(1 + \frac{r_s}{r}\right)(dr)^2 - \frac{2r_s}{r}d\tilde{T}dr - r^2(d\phi)^2. \quad (2)$$

Now, in these coordinates nothing blows up at r_s , though there's still the nasty blowup at $r = 0$. The latter is perfectly reasonable, however. If r is always outside M no matter how small r is, M must be a point mass—a mass of infinite density. Such an object surely must make trouble! Note also that for $r \gg r_s$, (2) just becomes the usual flat space proper time. For large r , \tilde{T} is exactly T , as we would expect it to be. Consider radial light rays for (2):

$$0 = \left(1 - \frac{r_s}{r}\right)(d\tilde{T})^2 - \left(1 + \frac{r_s}{r}\right)(dr)^2 - \frac{2r_s}{r}d\tilde{T}dr.$$

Solving for $dr/d\tilde{T}$ yields two possibilities: -1 and $(1 - r_s/r)/(1 + r_s/r)$. The former implies that light falling in toward $r = 0$, travels at the speed of light according to the external observer. The latter, however, implies that outward propagating light has different apparent speeds depending on the point of emission. If r is large, the outward propagating light travels with velocity $= +1$, as expected. Closer to r_s the outward speed slows, reaching zero at r_s . When $r < r_s$, the latter velocity becomes negative, meaning that light cannot travel outward once the source is inside the Schwarzschild radius. This is shown in the figure to the right. Inside r_s the “future” always points toward $r = 0$. As light is the fastest signal, the fate of *all* signals inside r_s is to wind up smacking into the *singularity* at the spatial origin. Once a particle enters the interior region it must join all the other mass crunched down to a point.

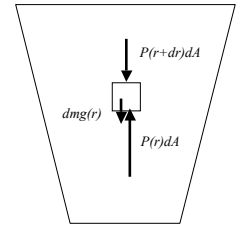


Stellar collapse

The equilibrium of an extended massive object requires the balance of various competing processes. In a solid body on Earth, for example, the kinetic energy of atoms plus the Pauli Exclusion of atomic electrons, which would tend to blow the body apart, is balanced by attractive electrical forces, producing a structure that from the outside appears to be static. A

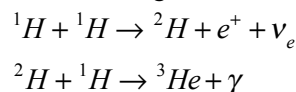
similar balancing act takes place in large objects such as planets, stars, and galaxies. In large objects gravity often supplies the glue to hold the object together. As we've seen in the Schwarzschild solution to Einstein's equations, gravity can sometimes be very strong. In that case, if a body is in equilibrium the question is what keeps it from collapsing under its own gravity.

A useful model of the internal equilibrium of a massive object borrows from the simple picture of hydrostatic equilibrium. To the right, we see a wedge of material in a large body and, within it, a small lump of material acted on by pressure forces and gravity. The lump is at a radial distance r from the center of the body; the pressure forces have to be greater at the bottom of the lump than at the top so that their difference balances the pull of gravity. The situation is summarized by the equation: $dP / dr = -\rho g$. This condition for static equilibrium masks several complexities. Among these are (a) P , g , and ρ are all functions of r , (b) P and ρ depend on temperature which, in turn, depends on r and possibly time (because of chemical and/or nuclear reactions and radiative cooling), and (c) P and ρ are related to each other through some kind of "equation of state."

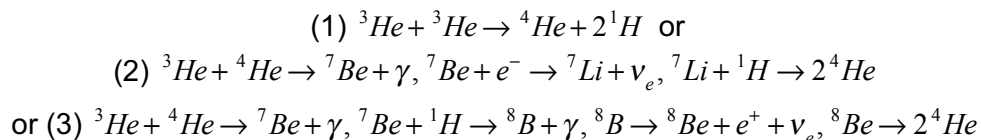


To determine the exact equilibrium structure of a massive object requires knowing reliable functional forms for all of these variables—knowledge that is often incomplete. Nonetheless, some broad qualitative ideas can be discussed. A star starts out as a gas cloud (with density— 10^{-3} gm/cc, say—much less than liquid water). As the cloud condenses under its own gravity, the atoms and molecules of the cloud gain kinetic energy. Thus, the cloud gets hotter and its internal pressure increases. Further condensation requires shedding this increased kinetic energy, typically in the form of electromagnetic radiation. If that is possible (sometimes it isn't) the cloud continues to condense and ultimately nuclear fusion begins.

The easiest (least energy required) achievable fusion reaction involves hydrogen, so a "successful" star must start out as a cloud consisting of primarily hydrogen. The most elementary fusion sequence is the so-called **proton-proton chain**. In it, four hydrogen nuclei eventually become one helium-4 nucleus (2 protons + 2 neutrons). There are several possible scenarios by which this might happen, all starting with the steps:



In the first step, two protons fuse, producing a deuteron (${}^2_1\text{H} = 1$ proton + 1 neutron), a positron (the anti-electron), and an electron neutrino. This step involves "inverse beta decay," the inverse of the usual decay of a neutron, and is a result of what's known as the "weak interaction." (We wouldn't be here without the weak interaction!) Also, the released positron usually quickly encounters an electron leading to $e^- + e^+ \rightarrow 2\gamma$. In step 2, the deuteron fuses with another proton, producing a helium-3 nucleus (2 protons + 1 neutron) and a high-energy photon (γ). After these first steps the possibilities include:



In each of these cases, we get an overall $4{}^1_1\text{H} \rightarrow {}^4_2\text{He} + \text{light stuff}$. In all cases this reaction is the primary step in *stellar nucleosynthesis*, the formation of heavier nuclei in stars. In Sun, the

first of these scenarios is about 6 times more likely than the second, and the third hardly happens at all. The difference in mass between 4 protons and 1 helium-4 is 26.73 MeV, so that's more-or-less how much energy is carried away in each case by the "light stuff." In each scenario the 26.73 MeV is partitioned differently among the photons and neutrinos. In a star like our Sun that isn't very massive, the interior is heated by the released photons. The neutrinos escape, taking their energy with them.

During the time fusion is occurring the crush of gravity is balanced in large part by the high temperature and pressure produced by fusion-energy release. (Sun is an example of this thermal pressure/gravity-balancing act; the resulting solar density is about that of liquid water, 1 gm/cc.) In fact, because the particles in the star have a distribution of velocities, at high temperature some will be able to escape the star's gravitational pull. This outstreaming of high-energy particles is called the "stellar wind" and the result is a gradual loss of stellar mass.

As helium builds up from the fusion process it tends to "sink" to the star's interior forming a helium-rich core. Depending on the star's mass it can experience one of several possible fates. For a relatively low mass star like Sun, the outer shells of low-density hydrogen can expand (and cool) forming a "red giant," the radius of which might exceed the Sun-Earth distance. The cooling outer atmosphere of the star can recollapse producing a small, cold remnant, held up by Pauli Exclusion of its free electrons (a "white dwarf"), or possibly reignite other fusion processes (such as a "helium flash"). Electron exclusion can only support a star against its own gravity if the stellar mass is less than about 1.4 solar masses. If the mass of the dying star is larger (up to 2 solar masses, say), gravity can squeeze the electrons and the nuclear protons in the star together to form neutrons, with the result being that the heavy nuclei gradually "dissolve." Eventually, such an object will be made to a large extent by neutrons—it will be a "neutron star" with density similar to that of an atomic nucleus, about 10^{15} gm/cc. Neutron stars are giant nuclei and often have giant magnetic fields. As these magnets rotate, they radiate electromagnetic energy at all kinds of wavelengths, producing the phenomenon of periodically pulsed light and radio waves. Thus, rotating neutron stars are also known as "pulsars." (In 1967 **Jocelyn Bell**, then a graduate student, discovered the first [and second] pulsar[s]. In 1974 her thesis adviser, Anthony Hewish, received the Nobel Prize for this discovery—but not her. Later she said, "I am not myself upset about it—after all, I am in good company, am I not!" Subsequently, Bell has received seven international awards, including the 2018 \$3M Special Breakthrough Prize of the Milner Foundation. [She gave the \$3M to the Institute of Physics to help female aspiring physicists.]

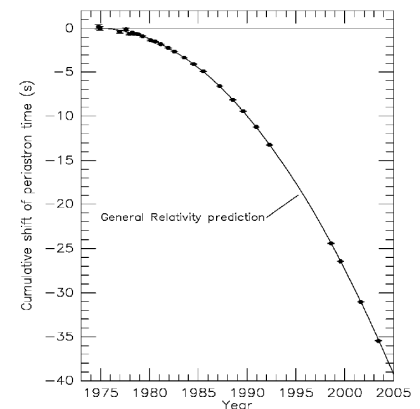
The radiation from pulsars tends to be collimated in a beam, so pulsar detection requires the beam to sweep across Earth (probably only a rare occurrence). Neutrons Pauli-exclude also, so the resulting (non-thermal) pressure opposes the condensation due to gravity. If the neutron star is not too massive, it will be stable. If the mass exceeds 2-3 solar masses, however, gravity will win out and condensation will continue. One possible more condensed configuration is a "quark star," held up because quarks (more about them later) also Pauli exclude. The gravity of more massive stars can even squeeze their quarks together; at this point, there is no known mechanism to prevent further compaction and ultimately the star turns into a black hole.

For reasons that are still not well understood, some very large stars (5-20 solar masses) can undergo a cataclysmic condensation that results in a very rapid increase in nuclear fusion, involving even some processes that would not normally occur. In fact, such a sudden compaction is believed to be one of the only means by which nuclei heavier than iron and nickel can be synthesized. (The merger of two neutron stars is another.) Associated with this sudden

condensation and energy build-up is a large mass ejection, an event that is called a “supernova.” All of the elements heavier than iron and nickel on Earth must have originated in a supernova explosion before our solar system developed. A supernova can release enormous amounts of energy in a very short time. Such sudden flare-ups in brightness have been observed even in very distant galaxies. Fortunately for us, no really close supernova has occurred during our species’ existence. On the other hand, supernovae are also essential to our existence: the heavy nuclei ejected from a supernova event constitute the material from which planets like Earth (as well as organisms that might populate them) are built. The exact mechanism for the explosive ejection of mass from a supernova is still controversial, but there is some evidence that neutrino pressure might be important. If that turns out to be true, we will again owe our existence to the weak interaction. The remnants of supernovae often are highly condensed objects, sometimes neutron stars, sometimes black holes.

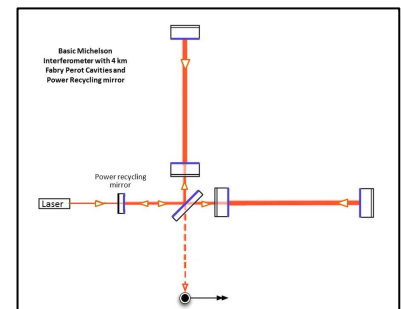
Condensed binaries

Because of their small physical extent, highly condensed stellar remnants are typically difficult to detect. An exception is a pulsar, provided its radiation happens to sweep by Earth. As mentioned in GR4, Taylor and Hulse discovered a remarkable pulsar, one whose radiation Doppler shifts back and forth in a way that can only be due to orbital motion. This pulsar orbits a second condensed stellar remnant once every 7.75 h, suggesting they are very close to one another; each of the partners has a mass of about 1.4 solar masses (and are therefore both probably neutron stars). The orbital period of the system has been observed to be getting systematically shorter. This suggests the system is losing energy and the partners are “falling in” to each other. The general relativistic explanation for this is **gravitational radiation**. Just as Maxwell’s field theory predicts that accelerating charge makes electromagnetic radiation, Einstein’s gravitational field theory says accelerating masses (of the right arrangement) create ripples in space-time that travel away from the mass at the speed of light. It is possible to calculate the power radiated by a rotating dumbbell, and, using that as a model of an orbiting binary, compare it with the rate of decay of the orbit of the Hulse-Taylor system; the comparison is shown to the right (taken from arXiv:astro-ph/0407149v1). The difference between observation and prediction is about 0.1%. For their discovery of this extraordinary general relativistic laboratory, Taylor and Hulse were awarded the 1993 physics Nobel Prize. More recently (2003) another pulsar binary has been discovered in which *both* partners are pulsars with beams sweeping by Earth. These two very precise clocks (with about 1.3 solar masses apiece) orbit a common center-of-mass every 2 h, and therefore will eventually provide even more rigorous tests of general relativity than Hulse-Taylor.



LIGO

The decrease of the orbital period of the Hulse-Taylor binary is an indirect measurement of gravitational radiation. We now have direct measurements from the **LIGO** collaboration. The **L**aser **I**nterferometer **G**ravitational-wave **O**bservatory is actually two interferometers, each with two 4 km long arms, a high intensity laser, a series of mirrors, and a photodetector (see figure to the right, from <https://www.ligo.caltech.edu/page/ligos-ifo>). One interferometer is located in Washington state, the other in Louisiana, a separation of



3000 km. The mirrors (a) split the initial beam, (b) reflect and recombine the two split beams, and (c) “recycle” photons. The latter mirrors effectively increase the arm length from 4 km to 1120 km, and effectively increase the laser power from 200 W to 750 kW. To reduce scattering from atmospheric molecules, the tubes the light travels through are evacuated to about 10^{-12} atm. The mirrors are suspended in such a way as to eliminate almost all vibrations due to seismic noise. When a gravitational wave passes through the interferometers it can temporarily change the lengths of the arms by a very small amount. The precision of the interferometers allows detection of arm-length changes to about 10^{-19} m (a small fraction of the diameter of a proton!). Detection of a gravitational wave passing by Earth requires the two interferometers to simultaneously record the same changes in their interference patterns.

The first such coincident event was observed in September 2015; a second was recorded in December 2015 and a third in January 2017. The characteristics of the first are consistent with the merger of two orbiting black holes, having masses equal to 35 and 30 solar masses (M_{\odot}). The resulting merged black hole appears to have a mass of $62 M_{\odot}$, meaning that the energy carried away by the gravitational radiation is equivalent to $3 M_{\odot}c^2$. To date, there is no explanation for how such large black holes could have formed; stellar collapse puts an upper limit on mass of about $20 M_{\odot}$. The second event was more consistent with black hole formation from stellar collapse. The two initial masses were 14 and $7.5 M_{\odot}$, and their merger released $0.7 M_{\odot}c^2$.

LIGO has been joined by the Virgo gravitational wave observatory in Italy. In August 2017, this new collaboration recorded both gravitational and electromagnetic waves produced by the merger of two neutron stars. This (“multi-messenger”) observation established that gravitational waves also travel at the speed of light (to within 1 part in 10^{15})—long thought to be true but not verified until this event. This observation places very severe constraints on alternative theories to general relativity with a cosmological constant. Those who don’t like the cosmological constant (or, equivalently, vacuum energy) will have to work harder to come up with a theoretical idea that fits the new data. Soon, observatories in Japan, and India will come online with the LIGO collaboration, making gravitational wave detection much more precise and, in the future, perhaps even commonplace. According to one theoretical estimate, LIGO+ should begin to see 5-ish gravitational wave events per year, thus providing a whole new kind of astronomy and a whole new way to test general relativity.

Note: While LIGO and Virgo are currently undergoing upgrades, old data have been reexamined. As of December 2018 (the most recent data release) 11 gravitational wave events have been detected.