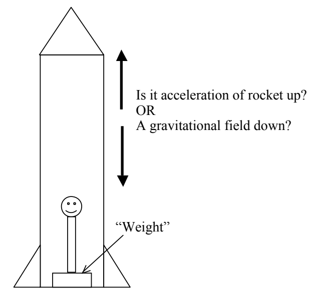
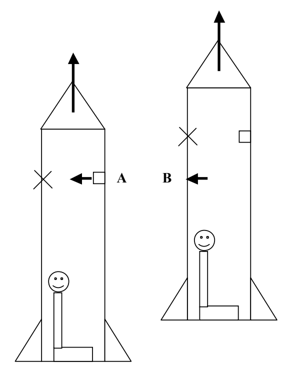
General relativity, 3

Gravity as geometry: part II

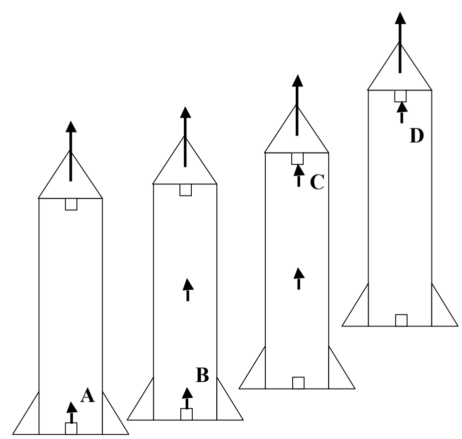
Even in a region of space-time that is so small that tidal effects cannot be detected, gravity still seems to produce curvature. The argument for this point of view starts with the recognition that, for mechanical systems, it is impossible to distinguish a frame of reference with a uniform gravitational field from a uniformly accelerating frame that has no gravity. Thus, for example, in a (small) rocket ship with no windows it is not possible to determine whether the weight one reads standing on a scale at the tail of the rocket is due to the rocket being stationary in a g-field or the rocket having acceleration g in the opposite direction.

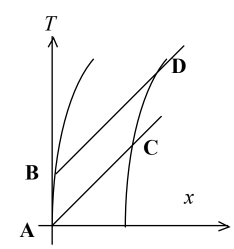


Einstein proposed that it wasn't just mechanics that got confused by acceleration, but all physical processes. This is Einstein's **Principle of Equivalence**: *no physical experiment can distinguish a uniformly accelerating frame from a frame with a uniform gravitational field.* One immediate consequence of the Equivalence Principle is that *light must fall in a gravitational field*. In the figure to the right, a pulse of light is emitted at event A in a rocket accelerating upward. If the rocket were not accelerating the light would hit the wall at the “X,” but by the time the pulse reaches the wall at event B the “X” has moved up. Thus, for an observer fixed to the rocket, the pulse appears to fall downward toward the tail. If the Equivalence Principle is correct, then light should fall in a gravitational field.



In addition, suppose a sequence of pulses is emitted from the tail of the rocket toward the nose as in the figure to the right. Event A is the emission of the first pulse; B is the emission of the second; C is the detection of the first pulse; D is the detection of the second. Suppose that the time between A and B is T_0 according to a clock at the tail. What is the time between C and D at the nose? If the tail was moving upward but the nose was not moving (i.e., if the rocket was getting shorter) the emitter would be traveling toward the detector and there would be a Doppler shift: the interval C-D would be shorter than the interval A-B. In this case, light emitted from the tail and received at the nose would be “blue-shifted.” On the other hand, if the nose was moving and the tail not, the received interval would be longer and the light would be “red-shifted.” You might think that these two shifts would cancel out, but that's not correct. Consider the s-t diagram to the right. The observer, O, is an inertial observer outside the rocket. The rocket accelerates in the $+x$-direction according to O. Events A and B happen at the tail of the rocket, C and D at the nose. Because of the rocket's motion, A and B have different x-values, as do C and D, according to O. Moreover, because the motion is accelerated the average speed of the tail between A and B is less than the average speed of the nose between C and D. Thus, the blueshift due to the emitter's motion between A and B is less than the redshift due to the detector's motion between C and D. If the Equivalence Principle is correct, *light traveling “upward” in a gravitational field should be red-shifted*.



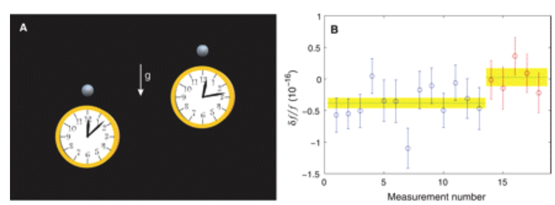


The predicted gravitational red-shift of light was first measured by Rebka and Pound in 1960, in an experiment involving sending gamma rays of a well-defined frequency up a 22.5 m tower at Harvard University. The detector at the top of the tower—tuned to the exact emission frequency—was set into slight downward motion so that its (easily measured) motional blue-shift exactly compensated for the gravitational red-shift. The detailed prediction of general relativity (we'll get to that later) was confirmed in this experiment with an uncertainty of about 1%. More recently (2010), the uncertainty in this measurement has been reduced to about 1 part in 10^8. There is little doubt that light, despite having no mass, gravitates. Such effects have an important implication for maintaining high resolution positioning via the Global Positioning System (where the rates of atomic clocks, in orbit and on Earth at two different heights in Earth's gravitational field, have to be constantly reconciled).

To see quantitatively how gravitational redshift works, recall the Doppler shift formula (SR3) $\lambda_d = \sqrt{\frac{1+\beta}{1-\beta}}\lambda_e$, where λ_e is the wavelength of radiation at the emitter and λ_d the wavelength measured at the detector. If $\beta > 0$ (< 0), the detector is traveling away from (toward) the emitter and $\lambda_d > \lambda_e$ ($\lambda_d < \lambda_e$). If both emitter and detector are in motion then $\lambda_d = \sqrt{\frac{1+\beta_d}{1-\beta_d}}\sqrt{\frac{1-\beta_e}{1+\beta_e}}\lambda_e$. For the s-t diagram of the accelerating rocket, we can replace β_e by the average velocity of the tail from A to B and β_d by the average velocity of the nose from C to D. For accelerations, g, similar to Earth gravity, and time between the respective events the respective periods of the light emitted and detected, it is reasonable to assume that $\beta_{d,e} << 1$ and that $\beta_d >> \beta_e$. Consequently, the wavelength shift is approximately $\lambda_d \approx [1+\beta_d]\lambda_e$. Taking g to be constant, we can write $\beta_e = g\left(\left(T_C - T_A\right)+\left(T_D - T_B\right)\right)/2c^2$ (where T is in m and g is in 1/m). Both time intervals in this expression are approximately equal to L, the length of the rocket. Thus, finally

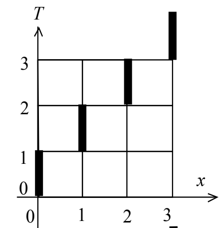
$$\lambda_d = \left(1 + gL/c^2\right)\lambda_e .$$

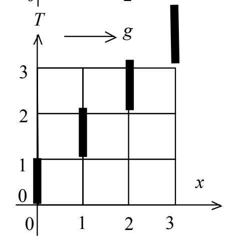
The wavelength measured by the detector gets larger the greater is the vertical distance (L) from the emitter to the detector. If the detector is expecting "blue" light, it actually measures "red." That's the phenomenon of gravitational redshift. Moreover, when time is in m, wavelength equals period so that the time between crests at the detector is longer than that at the emitter. To the detector it seems as if the time has slowed down at the emitter. Using the same trapped-ion clocks mentioned in SR3, the gravitational effect on clocks separated by a vertical distance of just 33 cm has been directly measured. See the figure to the right. (Science **329**, 1630-1633 (2010).)



<u>Example</u>: For g = 10 m/s^2 and L = 3x10^5 m (similar to low-Earth orbit conditions), $gL/c^2 \approx$ 3x10^{-11}. The period of a satellite in low-Earth orbit is about 5000 s; so gravitational redshift causes orbiting and Earth-bound clocks to disagree by about 0.2 μs per orbit. After many orbits this can build up to a large discrepancy. As such a discrepancy corresponds to 60 m of light travel, GPS has to take gravitational redshift into account.

So how can gravitational clock-shift be viewed as geometry? In the first s-t diagram to the right, in an non-accelerated frame, we can imagine four clocks at spatial coordinates 0, 1, 2, and 3, "ticking" in synchrony. The bold lines represent the time difference between a light crest emitted at $x=0, T=0$ and a second emitted at $x=0, T=1$, detected sequentially at $x=1, x=2, x=3$, respectively. These differences are all equal. The second s-t diagram shows the same initial set-up, but now this frame rigidly accelerates to the left starting at T = 0. Because of the Equivalence Principle, this is the same as "turning on" a gravitational field pointing in the $+x$-direction at T = 0. While the time difference between crests is 1 at $x=0$, it's longer at $x=1$, longer at $x=2$, and longer still at $x=3$. The clocks record different times between the same two crests at different "heights" in the gravitational field. The bold world lines on the diagram depict the de-synchronization caused by acceleration/gravity. Gravity "mangles" both space and time.





Einstein's theory of gravity

Einstein's version of gravity says that gravitational effects can be understood as curvature of space and time. A test particle feely falling through space-time travels on the "straightest" world line—a "geodesic"—that it can; it feels no gravitational force. In special relativity, of all the possible worldlines connecting two events (involving acceleration) the one with *maximum proper time* along it is a *straight line*. The same criterion is used for geodesics. *Of all the worldlines in curved spacetime connecting two events the one with maximum proper time is the geodesic (free-fall) worldline.*

Einstein's gravitational field equations are similar in many respects to Maxwell's equations for electromagnetism. Like Maxwell's equations, they are second order partial differential equations. Solutions to Maxwell's equations are electric and magnetic fields; solutions to Einstein's equations are gravitational fields (though disguised as components of the metric, $g_{\mu\nu}$). Both Maxwell and Einstein relate their respective fields to "sources." For Maxwell the sources are stationary and moving charges (currents). For Einstein the sources *include* stationary and *moving* masses and—because of the mass-energy equivalence—all forms of energy. In both theories, moving sources make fields—that is, a kind of "magnetism." In both theories, *accelerating* sources make fields that propagate—that is, *electromagnetic and gravitational radiation*.

But there is also a *very significant difference* between Maxwell and Einstein. Electromagnetic fields do not carry electric charge—the stuff that makes them. When two electromagnetic fields overlap, they linearly superpose—that is, you just add the two fields as vectors to see what happens. Because of linear superposition, after passing through each other, electromagnetic waves are identical to before they superposed. For Einstein, however, the source of gravity is ALL forms of energy, and as gravitational fields also carry energy they make more of themselves. Gravitational fields do not linearly superpose; they can modify each other when they overlap.

The nonlinearity of Einstein's theory of gravity makes it a complicated business. But fortunately we only need a couple of formal results to be able to say a lot about several important physical situations: the gravitational fields we will consider are (a) outside and (b) inside uniform, nonrotating, spherically symmetric sources.

The Schwarzschild Metric

Outside **a stationary, spherically symmetric mass**, M, the most general solution of Einstein's equations is due to Karl Schwarzschild (published in 1915, one month after Einstein's first paper on general relativity); it can be expressed in terms of proper time as

$$(d\tau)^2 = \left(1-\frac{r_S}{r}\right)(dT)^2 - \left(1-\frac{r_S}{r}\right)^{-1}(dr)^2 - r^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right], \tag{1}$$

where

$$r_S = \frac{2GM}{c^2}. \tag{2}$$

Spherical coordinates are employed because the gravitational field of a spherically symmetric mass is also spherically symmetric. The quantity r_S is the *Schwarzschild radius* of the mass M. Note that the **Schwarzschild radius only depends on the mass** and not its physical extent (i.e., its physical radius); every mass has a unique Schwarzschild radius.

Example: Because G and $1/c^2$ are both small, the Schwarzschild radius of a spherical mass is *generally* much smaller than its physical radius. For Sun, for example, M = 2x10^{30} kg, so r_S is about 3 km, compared with the physical radius of about 7x10^5 km. For Earth, M = 6x10^{24} kg, so r_S = 9 mm (versus 6.4x10^3 km). Thus for orbits near, but outside, Earth or Sun $r_S/r \ll 1$ for typical values of r. Thus, the Schwarzschild proper time is very nearly equal to the special relativistic proper time for these situations.

In contrast, many galaxies are believed to have enormous central bodies consisting of perhaps a billion solar masses; the Schwarzschild radius of such a body would be about 10^9 km. Interestingly, if the central body were as dense as Sun, say, its physical radius would be about 10^3 that of Sun's, roughly the same as its Schwarzschild radius. Incidentally, for the visible universe, $M \approx$ (a few)x10^{53} kg (maybe), so $r_S \approx$ (a few)x10^{23} km $\approx$ (a few)x10^{10} light years—which is on the order of the radius of the visible universe.

The center of M in this coordinate system is at r = 0 for all time. The spatial coordinates are fixed to M; they are not "free-fall" coordinates. If we let $M \to 0$ or $r \to \infty$ (i.e., far from M), we obtain the special relativistic (i.e., force-free) proper time in spherical coordinates:

$$(d\tau)^2 = (dT)^2 - (dr)^2 - r^2\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right].$$

The spacetime of special relativity is flat. In the Schwarzschild proper time, the factor multiplying the term $(dT)^2$ corresponds to the gravitational redshift and the factor multiplying $(dr)^2$ corresponds to an extra contribution due to tidal effects. They reflect the fact that the Schwarzschild spacetime is curved. As we will see, the Schwarzschild metric harbors a number of interesting consequences for objects in free fall near M.

As we will be interested in freely falling test bodies (negligible masses), it is useful to note that a spherically symmetric, inward pointing gravitational field cannot exert a torque about the center of M. Recall that torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$ ($\vec{\tau}$ is torque, not proper time), and if $\vec{F}$ is parallel to $-\vec{r}$, $\vec{\tau}$ must be zero. But torque is the time rate of change of angular momentum, $\vec{\tau} = d\vec{L}/dt$, so objects in free fall have *constant angular momenta* (about the center of *M*)—both in magnitude and direction. Because $\vec{L} = \vec{r} \times \vec{p}$, $\vec{L}$ is perpendicular to a plane containing both $\vec{r}$ and $\vec{p}$; and because $\vec{L}$ is constant, $\vec{r}$ and $\vec{p}$ must always be in the same plane. In other words, free fall trajectories lie in a single plane. As a result, we might as well set $\theta = 90^\circ$, in which case *the Schwarzschild proper time simplifies to*

$$\left(d\tau\right)^2 = \left(1 - \frac{r_S}{r}\right)\left(dT\right)^2 - \left(1 - \frac{r_S}{r}\right)^{-1}\left(dr\right)^2 - r^2\left(d\phi\right)^2. \tag{3}$$

<u>Example</u>: An important application of Equation (3) is the GPS system. In this application Earth is the source body. Clocks on Earth record a proper time interval between when a GPS satellite is directly overhead to start an orbit and when it is overhead again one orbital pass later. These clocks are at the same respective radial positions (relative to the center of Earth). The clocks move with Earth's rotation at a dimensionless speed $u_E = R_E\, d\phi/dT$, where R_E is Earth's physical radius. Consequently, between these two events $\Delta\tau_E = \left(1 - \frac{r_S}{R_E} - u_E^2\right)^{1/2} \Delta T$, where ΔT is the corresponding time measured on clocks at rest a very large distance from Earth. In the GPS satellite clocks would read $\Delta\tau_{GPS} = \left(1 - \frac{r_S}{r_{GPS}} - u_{GPS}^2\right)^{1/2} \Delta T$ between these events. Earth rotates through an angle of 2π in a day, leading to $u_E \approx 1.5\text{x}10^{-6}$ (calculate it). In addition, $r_S/R_E = 1.5\text{x}10^{-9}$. The orbital radius of a GPS satellite with orbital period of 12 h is about $4.2R_E$, yielding $r_S/r_{GPS} = 3.6\text{x}10^{-10}$; with an orbital speed of 4 km/s, $u_{GPS} = 1.3\text{x}10^{-5}$. As the various contributions to 1 in both expressions are small, the binomial expansion can be used (yet again) to find $\Delta\tau_{GPS}/\Delta\tau_E \cong 1 + \frac{1}{2}(\frac{r_S}{R_E} - \frac{r_S}{r_{GPS}}) - \frac{1}{2}(u_{GPS}^2 - u_E^2)$. The first contribution is due to the gravitational redshift (clocks higher in the gravitational field run faster), the second due to special relativistic time dilation (faster traveling clocks run slower). The value of the gravitational contribution is about +46 μs per orbit, the second is about –7 μs per orbit; i.e., the general relativistic effect is opposite in sign to the special relativistic one, and about 6 times greater in magnitude. Both have to be accounted for in order to make GPS work. (At launch, the clocks on the satellite are set to keep time slower than clocks left on Earth at a rate of about 39 μs per orbit.)