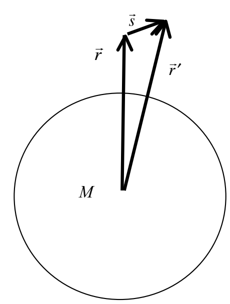
General relativity, 2

Newton's law of gravitostatics is incompatible with special relativity. To see this, suppose at time t in frame O m_1 is at $x_1(t)$ and m_2 is at $x_2(t)$. Newton's gravitational force law says $F_{1\,on\,2}(t) = Gm_1m_2 / \left[x_2(t) - x_1(t)\right]^2$ and relativistic dynamics says $dp_2/dt = F_{1\,on\,2}$. Transforming to another frame O′ moving relative to O leads to $dp_2'/dt' = F_{1\,on\,2}'$. But what is $F_{1\,on\,2}'$? If $x_1(t)$ and $x_2(t)$ are the simultaneous positions of m_1 and m_2 in O, the transformed positions x_1' and x_2' are *not* simultaneous in O′. Obviously, Newtonian gravity has to be modified to make it relativistically correct. The same is true for the electrostatic Coulomb force. But, for electric charges there's also magnetism, and that helps save the day. Clearly, what's needed for gravity is an analog of magnetism. To create such an analog, Einstein appealed to geometry.

Gravity as geometry: part I

Though Earth's gravity is approximately constant everywhere near the surface of Earth, variations in other bodies' gravity at Earth turn out to have important, noticeable effects. In particular, the daily oceanic high and low tides are caused by the gravitational variations of our moon and Sun.



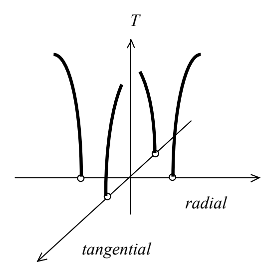
In the figure to the right we see a large source of gravity, M. A small reference mass is at $\vec{r}$ and a small test mass is at $\vec{r}'$. The position of the test mass relative to the reference mass is $\vec{s} = \vec{r}' - \vec{r}$ and the acceleration of the test mass is $\frac{d^2\vec{s}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} - \frac{d^2\vec{r}}{dt^2}$. Suppose the reference and test masses do not interact and both freely fall in M's gravitational field: that is, $\frac{d^2\vec{r}}{dt^2} = \vec{g}(\vec{r}) = -G\frac{M}{r^3}\vec{r}$ and $\frac{d^2\vec{r}'}{dt^2} = \vec{g}(\vec{r}') = -G\frac{M}{r'^3}\vec{r}'$. The position $\vec{r}'$ can differ from $\vec{r}$ both in length and direction. Let $\vec{s} = s_r\hat{r} + s_n\hat{n}$, where $\hat{n}$ is a unit vector perpendicular ("normal") to $\hat{r}$. Relative to the center of the laboratory, s_r is positive if the box is "above" it and negative "below;" s_n is positive if the box is to the "right" of center and negative to the "left." Assuming that $s \ll r$, it is straightforward to show that $\frac{\vec{r}'}{r'^3} \approx \frac{\vec{r}}{r^3} + \hat{r}(-2\frac{s_r}{r^3}) + \hat{n}\frac{s_n}{r^3}$, so that $\frac{d^2\vec{s}}{dt^2} = -\frac{GM}{r^3}\left(-2s_r\hat{r} + s_n\hat{n}\right)$ or

$$\frac{d^2s_r}{dt^2} = \frac{2GM}{r^3}s_r$$
$$\frac{d^2s_n}{dt^2} = -\frac{GM}{r^3}s_n \qquad (1)$$

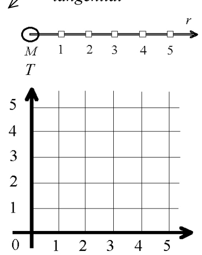
Example: Suppose $\vec{r}$ is the center of the International Space Station, where $r = 6.8\text{x}10^6\,\text{m}$ and a pencil is 1 m from the center of the ISS. The pencil will accelerate toward or away from the center with an acceleration whose magnitude is about g_E x 1 m/6.8x10^6 m—a few x 10^{-7} g_E. That's why NASA speaks about "microgravity" aboard the Space Station.

The accelerations in (1) result from "tidal" effects—differences in g due to the source M; they are *not* due to forces between the reference and test masses. That is, ***tidal forces tend to pull things apart along the radial direction and squeeze them together in the tangential direction***. This happens on Earth due to Sun and Moon. Relative to Earth's center (the reference) "particles" of ocean water are pulled in the radial directions toward Sun and Moon and squeezed together in the respective tangential directions. As Earth rotates under these bulges and dips large bodies of water seem to rise and fall. You've probably heard that the ocean tides on Earth are mostly caused by Moon. If Moon's pull at Earth is so much less than Sun's how can that be? The answer has to do with how rapidly g falls off with r.

<u>Example</u>**:** Sun has a mass = $2\text{x}10^{30}$ kg and a radius = $0.7\text{x}10^{6}$ km = $0.7\text{x}10^{9}$ m. Sun's gravitational field at its surface is therefore g_S = 272 m/s^2 (calculate it), about 28 times greater than g_E. Earth is about $1.5\text{x}10^{11}$ m from Sun so g due to Sun at Earth is a factor of about $(0.7\text{x}10^9/1.5\text{x}10^{11})^2 = 2\text{x}10^{-5}$ less than at Sun's surface, or about $6\text{x}10^{-4}\, g_E$. Earth's moon has a mass = $7.4\text{x}10^{22}$ kg and is $3.85\text{x}10^{8}$ m away. So, g of Moon at Earth is about $3.4\text{x}10^{-6}\, g_E$ (calculate it)—much less than g of Sun. But, $(g_{moon}/r_{moon})/(g_{sun}/r_{sun}) \approx 2.2$, so Moon is more important.

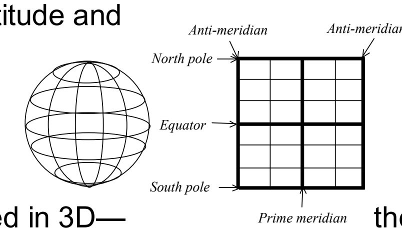


A space-time diagram for an observer fixed to the reference point recording the worldlines of two test bodies each 1 m above and below and two test bodies each 1 m to the right and left looks like the figure to the right. The world lines shown are for bodies that, starting from rest, are in free fall—that is, *feel no external force*. In the absence of tidal effects, all of these world lines would be straight and parallel to the T-axis. Because of the tidal effects, world lines that start out parallel to one another either converge or diverge.



It is quite reasonable to interpret this converging and diverging of freely falling world lines as a result of geometry, as Einstein did in his 1915 paper on general relativity. To see how this might be, consider the figure to the right. At the top, we see a large source of gravity, M, and five test masses arrayed initially along a single radial line. The position of the first test mass is labeled by r_1 relative to the center of M, r_2 for the second and so on. If each test mass in the array is simultaneously released from rest and free falls toward M, each retains its *position label* (not its position), but the physical distance between each pair of masses grows with time. That is, the tidal force of M causes the masses to spread out in the radial direction. This continues until the falling masses crash into M. In terms of position labels (also called "free fall" coordinates), r, the $r\text{-}T$ spacetime diagram looks like the square grid beneath the starting configuration. This seems like an ordinary spacetime coordinate system for special relativity, but there's something new. As stated, the physical distance between successive free fall coordinates depends on time. That is not at all like the space and time coordinates in special relativity with no gravity. It is, however, analogous to what happens if two airplanes fly at the same constant speed from the South Pole heading north along two different longitude routes. Though the longitude coordinates of the two airplanes remain constant, the physical distance between them will constantly increase until they get to the Equator, then will decrease thereafter. This is obviously because the surface of Earth that the planes are flying over is curved.

A Mercator Projection is a planar representation of the latitude and longitude coordinates on Earth that identifies any point on the surface. It is an "embedding" of the Earth's 2D surface in a 2-dimensional plane, as in the (approximate) figure to the right. In the projection, the Earth's latitude and longitude coordinate system certainly looks like a nice, flat, rectangular grid. On the other hand, when viewed from "outside"—that is, when embedded in 3D—the Earth's surface is clearly curved. So if we were 2D bugs confined to crawl around on the surface of the Earth would we say our world was intrinsically curved or flat? It seems like *intrinsic curvature* shouldn't depend on what coordinate systems are used.



There are several ways of measuring intrinsic curvature. All involve taking second derivatives of some function in its many independent variable "directions." The definition of space-time curvature used in general relativity ("Riemannian curvature") focuses on how proper time intervals are measured in some system of time and space coordinates. In special relativity, the proper time between two nearby events is $\left(d\tau\right)^2 = \left(dT\right)^2 - [\left(dx\right)^2 + \left(dy\right)^2 + \left(dz\right)^2]$, when Cartesian spatial coordinates (x, y, z) are used. The same proper time is $\left(d\tau\right)^2 = \left(dT\right)^2 - \{\left(dr\right)^2 + r^2\left[\left(d\theta\right)^2 + \sin^2\theta\left(d\phi\right)^2\right]\}$, when spherical spatial coordinates (r, θ, ϕ) are used. In general, $\left(d\tau\right)^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dq^\mu dq^\nu$, where q^μ is a general s-t coordinate with the superscript taking on one of four possible values, 0 (for time) through 3 (1, 2, 3 for space). The 16 quantities $g_{\mu\nu}$ together constitute the so-called "metric." (A geeky bit: the metric is a tensor of rank two—that is, it has two subscripts, as opposed to a vector, which has one, or a scalar, which has none.) The Riemannian curvature is a linear combination of the various first and second derivatives of $g_{\mu\nu}$.

<u>Example</u>**:** For the proper time $\left(d\tau\right)^2 = \left(dT\right)^2 - [\left(dx\right)^2 + \left(dy\right)^2 + \left(dz\right)^2]$, the s-t coordinates can be taken to be $q^0 = T, q^1 = x, q^2 = y, q^3 = z$, and the metric components are then $g_{TT} = 1, g_{xx} = -1, g_{yy} = -1, g_{zz} = -1$, with all others being = 0. For the proper time $\left(d\tau\right)^2 = \left(dT\right)^2 - \{\left(dr\right)^2 + r^2\left[\left(d\theta\right)^2 + \sin^2\theta\left(d\phi\right)^2\right]\}$, the s-t coordinates can be taken to be $q^0 = T, q^1 = r, q^2 = \theta, q^3 = \phi$ and $g_{TT} = 1, g_{rr} = -1, g_{\theta\theta} = -r^2, g_{\phi\phi} = -r^2\sin^2\theta$, with all others being = 0. It's clear that the first of these metrics has all zeroes for its first and second derivatives and therefore corresponds to zero curvature. The second metric has some first and second derivatives that don't vanish. On the other hand, though it's a bit of a mess to do, combining these derivatives in the definition of the Riemannian curvature also produces zero. Thus, **even though the proper time for a special relativistic inertial observer in spherical coordinates looks like there should be curvature, there isn't any**. Contrast this conclusion with the so-called "Schwarzschild proper time":

$$\left(d\tau\right)^2 = \left(1 - \frac{r_S}{r}\right)\left(dT\right)^2 - \left(1 - \frac{r_S}{r}\right)^{-1}\left(dr\right)^2 - r^2\left[\left(d\theta\right)^2 + \sin^2\theta\left(d\phi\right)^2\right]$$

The factors multiplying $(dT)^2$ and $(dr)^2$ cause the Riemann curvature to not be zero everywhere. We'll have a lot more to say about this space-time presently.