

General relativity, 1

In special relativity, events occur in the arena of space-time which may be coordinatized differently by different observers, but which is otherwise immutable. Adding gravity to relativity provides an amazing result: space-time becomes “organic,” taking its form from the matter and energy it contains. This is Einstein’s *general theory of relativity* and it has the capacity to tell us about the past and future of the universe. Embedded in the history book of the cosmos are several chapters on the origins of matter. As a result, *relativity + gravity* unites the structures of matter on the largest and smallest scales.

Newtonian gravitostatics

To begin the story of matter in the universe, we need to recall a few aspects of the classical (static) theory of gravity due to Newton. Newton’s theory of gravity states that any two point-like masses attract one another with a force that is proportional to the product of the masses and inversely proportional to the square of their distance of separation. Thus, this force

of attraction can be expressed as $\vec{F}_{1 \text{ on } 2} = -G \frac{m_1 m_2}{r^2} \hat{r}$. In this equation, r is the distance

between the masses, m_1 and m_2 , and \hat{r} is a unit vector pointing *from* m_1 *to* m_2 . The quantity G (the “universal gravitational constant”) is the intrinsic strength of the gravitational interaction (i.e., independent of which bodies are interacting). In conventional SI units

$G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$ —that is, a 1 kg mass separated by 1 m from another 1 kg mass pulls on the second mass with an almost nonexistent gravitational force of $6.67 \times 10^{-11} \text{ N}$. Gravity is a puny force. By way of calibration, two protons repel each other with an electrical force $k_E q^2 / r^2$, where $k_E = 9 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2$ and $q = 1.6 \times 10^{-19} \text{ C}$, and attract each other with a gravitational force Gm^2 / r^2 , where $m = 1.67 \times 10^{-27} \text{ kg}$. The ratio of these forces does not depend on r since both forces $\propto 1/r^2$; the numerical value of the ratio $F_{\text{grav}}/F_{\text{elec}}$ is 8×10^{-37} ! Thus, compared with electrical interactions, gravity is phenomenally weak. Wherever electrical interactions are essential—as in atomic, molecular, and solid-state physics—gravity is always ignorable.

On the other hand, there are about as many electrons as protons in bulk matter and all the various electrical pulls and pushes from charges inside a bulk object on charges outside it tend to cancel out. Such cancellation doesn’t occur for gravity, however. That’s because there is (apparently) just one kind of mass. So gravitational pulls outside a bulk body get increasingly stronger the more massive the body is. Indeed, planets, stars, galaxies, and the whole universe are held together by gravity; on such scales it is the electrical force that is insignificant.

The modern view of gravity (and all other interactions) is that force arises from a field. In the language of fields, the Newtonian force law is rewritten as $\vec{F}_{1 \text{ on } 2} = m_2 \vec{g}_1$, where

$\vec{g}_1 = -G \frac{m_1}{r^2} \hat{r}$. Here, the force is parsed into (a) m_1 making a field \vec{g}_1 everywhere around it (though getting weaker the farther away from m_1 one goes) and (b) m_2 “coupling” to \vec{g}_1 . The result is that m_2 feels the force $\vec{F}_{1 \text{ on } 2}$. In this view, m_2 doesn’t care gravitationally about m_1 ; it just cares about the field m_1 makes. There is a reciprocity arising from how the gravitational

force is defined: $m_2 \vec{g}_1 = -m_1 \vec{g}_2$, that is, m_2 makes a field that m_1 couples to. Note that this reciprocity is consistent with Newton's Third Law: $\vec{F}_{1 \text{ on } 2} = -\vec{F}_{2 \text{ on } 1}$.

Spherically symmetric distributions of mass create gravitational fields that, outside of them, are identical to what a point particle of the same mass would produce if it were located at the center of the distribution. Thus, the magnitude of Earth's gravitational field is $g(r) = G \frac{M_E}{r^2}$, where M_E is Earth's mass and $r > R_E$ (Earth's radius). At the surface, $g(R_E) = g_E$ is approximately 9.8 N/kg. Because $R_E = 6.4 \times 10^3 \text{ km} = 6.4 \times 10^6 \text{ m}$, we can "weigh" Earth: $M_E = 6 \times 10^{24} \text{ kg}$.

Example: Often, we are interested in "near-Earth" situations, such as satellites in "low-Earth" orbit. In these cases, distance is often reckoned as an *altitude* above Earth's surface: $r = R_E + h$. If $h \ll R_E$ then, using the binomial expansion,

$$g(h) = G \frac{M_E}{(R_E + h)^2} \approx GM_E \left(R_E^{-2} - 2R_E^{-3}h \right) = g_E (1 - 2h / R_E).$$

The altitude of the Hubble Telescope, for example, is about 560 km so $h/R_E = 560/6.4 \times 10^3 = 0.09$, and the magnitude of Earth's gravitational field at Hubble is (only) about 18% less than at Earth's surface.

Note that mass plays three roles in Newtonian gravity. Mass has an active gravitational role: it makes field. It has a passive gravitational role: it couples to field. And it has the role of inertia: force equals *inertial mass* times acceleration. It is not at all obvious that *one* kind of mass is responsible for these three effects, but all attempts to experimentally distinguish between different kinds of mass have failed. Such experiments include the continuous measurement of the accelerations of Moon and Earth toward Sun since 1969 (when mirrors were placed on Moon by Apollo astronauts), as well as a series of Earth-based laboratory experiments. To date, **these measurements allow us to conclude that any departure from equality of the different kinds of mass can be no more than a few parts in 10^{13} . Thus, we speak with great confidence of only mass.**

For gravitational interactions, Newton's Second Law says $m\vec{a} = -G \frac{Mm}{r^2} \hat{r}$. Because the inertial mass, m , on the left equals the gravitational mass, m , on the right, $\vec{a} = -G \frac{M}{r^2} \hat{r}$ irrespective of the mass of the body interacting with M . For situations near the surface of Earth, $M = M_E$, $r \approx R_E$, and $-\hat{r}$ points vertically downward toward Earth's center (ignoring local variations due to irregular mass distributions). Thus, the *gravitational* acceleration for all masses near the surface of Earth is $\vec{a} = \vec{g}_E$, where $\vec{g}_E = -\hat{r} G M_E / R_E^2$. Note that the units of the right hand side are N/kg, while on the left hand side they are m/s^2 . Of course, because a N is a $\text{kg}\cdot\text{m/s}^2$, the two are equivalent.

Static electric fields can be derived from scalar electric potential fields by taking gradients, i.e., $\vec{\mathcal{E}} = -\text{grad}\phi_E$. Electric field has the units of force/charge so electric potential has units [force \times distance]/charge = energy/charge. You get *electric potential energy* from electric

potential by multiplying by charge. The Newtonian gravitational field of a point mass falls off as $1/r^2$, just like the Coulomb electric field of a point charge. Because of this strong mathematical similarity, \vec{g} can also be derived from a *scalar (gravitational) potential* field, ϕ_G , by taking a gradient: $\vec{g} = -\text{grad}\phi_G$. Since g is measured in N/kg—force per unit mass, ϕ_G is measured in N·m/kg = J/kg—energy per unit mass. In other words, you get *gravitational potential energy* by multiplying gravitational potential by mass.

The gravitational field associated with a point mass or a spherically symmetric mass distribution depends only on distance, so the gradient operator can be written $\text{grad} = \hat{r} \frac{\partial}{\partial r}$.

Consequently, outside the source mass $\frac{\partial\phi_G}{\partial r} = \frac{GM}{r^2}$, or $\phi_G = \phi_0 - GM/r$, where ϕ_0 is an arbitrary “integration” constant that allows one to set ϕ_G to zero at any reference radius. Gravitational potential is related to the gravitational potential energy, U , a mass m experiences by $U = m\phi_G$. Often the reference point for zero potential is taken at $r = \infty$, in which case, $\phi_G = -GM/r$. For “near-Earth” situations, however, the potential is more often taken to be zero at Earth’s surface. Then, one has $\phi_G = GM_E/R_E - GM_E/r$. Again, writing $r = R_E + h$ and assuming $r \ll R_E$ allows us to approximate the potential as

$$\phi_G = GM_E/R_E - GM_E/(R_E + h) \approx GM_E/R_E - GM_E(R_E^{-1} - hR_E^{-2}) = GM_E h / R_E^2 = g_E h.$$

This, of course, produces the familiar gravitational potential energy, $U = mg_E h$.

Lastly, it should be noted that *inside a uniform sphere of mass* of radius R the gravitational field a distance $r < R$ from the center depends only on the mass within a concentric sphere of radius r . The gravitational fields due to all mass points at distances greater than r exactly cancel. As the mass inside r is related to the whole mass M by $M(r) = Mr^3 / R^3$,

$$g_{\text{inside}}(r) = GM(r)/r^2 = GMr / R^3 = g_s r / R,$$

where g_s is the gravitational field at the surface (GM/R^2). In other words, within the uniform spherical mass, g increases linearly from 0 at the center to a maximum, g_s , at the surface, then falls off as $1/r^2$ outside (i.e., $g_{\text{outside}}(r) = GM/r^2 = g_s R^2 / r^2$). The figure to the right depicts this behavior. The difference between inside and outside will arise again when we consider Einstein’s theory of gravity.

