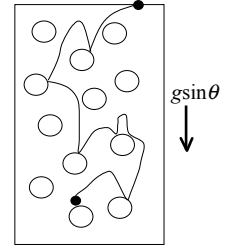


## Many-particle Systems, 8

The classical picture of how electrons migrate through a resistor driven by an applied potential difference draws an analogy with a kind of pinball machine. In a pinball machine, gravity accelerates the pinball down the table, but the ball's progress is impeded by collisions with bumpers, as depicted to the right. ( $\theta$  is the angle the table makes with the horizontal direction.) When averaged over many collisions the ball's average equation of motion is  $ma = mg \sin \theta - mv/\tau$ , where  $a$  and  $v$  are directed down the table, and  $\tau$  is the average time between collisions. If the table is sufficiently long, the average acceleration is approximately zero and the ball achieves a "terminal" or "drift" speed:



$$v_d = \tau g \sin \theta.$$

In the pinball model of electrical conduction, the electron is the ball, the bumpers are atoms, and the accelerating force is due to an electric field. The pinball model average equation of motion is  $ma = e\mathcal{E} - mv/\tau$  and the drift speed is  $v_d = e\mathcal{E}\tau/m$ . It is usual to replace the mean time between collisions by  $\lambda/\langle v \rangle$ , where  $\lambda$  is the average distance between collisions—also called the *mean free path*—and  $\langle v \rangle$  is the average speed *between* collisions, and typically  $\gg v_d$ . In other words,

$$v_d = e\mathcal{E}\lambda/m\langle v \rangle. \quad (1)$$

Equation (1) is equivalent to Ohm's Law for electrical conduction:  $V = IR$ . To see this, consider a resistor of length  $L$  and cross-sectional area  $A$ . The potential difference across the resistor is due to an electric field inside it:  $V = \mathcal{E}L$ . The current through the resistor is  $I = JA$ , where  $J$  is the *current density* (amp-m<sup>-2</sup>). Thus, Ohm's Law can also be expressed  $\mathcal{E} = \rho_E J$ , where  $\rho_E = RA/L$  is the electric resistivity (ohm-m). Furthermore, the current density is the electron charge,  $e$ , times the drift speed,  $v_d$ , times the number of conduction (i.e., able to move when the field is turned on) electrons per unit volume,  $n_e$ :  $J = ev_d n_e$ . Combining these results with (1) we find

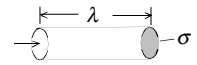
$$\rho_E = \frac{m\langle v \rangle}{n_e e^2 \lambda}. \quad (2)$$

If the electron were a classical particle, one might expect that  $\langle v \rangle$  would be related to the average thermal kinetic energy at the resistor's temperature,  $T$ . In which case,  $\langle v \rangle = \sqrt{3k_B T/m}$  (Mn6, p4). In addition, in the pinball model, the mean free path is probably close to the interatomic spacing  $a$  (especially since atoms in a solid are so closely spaced). The classical pinball picture says nothing about what  $n_e$  might be; that has to be guessed. The net result is that the pinball model predicts that  $\rho_E$  should increase with increasing temperature proportional to  $\sqrt{T}$  and, because  $n_e$  is so similar for all solids, should have about the same magnitude for all solids. In fact, it is observed that for metals  $\rho_E$  increases with increasing temperature roughly proportional to  $T$  (except at very low temperatures); for semiconductors

$\rho_E$  decreases with increasing temperature; and differences in  $\rho_E$  between materials can be at least 20 orders of magnitude! The classical pinball model is really profoundly wrong.

Nonetheless, Equation (2) can correctly describe observed resistivities provided quantum mechanics is used to evaluate  $\langle v \rangle$ ,  $\lambda$ , and  $n_e$ . The wavefunctions of electrons in a solid are localized, atomic-like functions for the lowest atomic energy states, but for higher energy states they are diffusely spread throughout the solid. As noted in Mn7, these latter wavefunctions are approximately harmonic standing waves—sums or differences of traveling plane waves: e.g.,  $e^{ipx/\hbar} \pm e^{-ipx/\hbar}$ , in the  $x$ -direction. The first term corresponds to momentum  $+p$  in the  $x$ -direction, the second to momentum  $-p$  in the  $x$ -direction. If an electric field is turned on in the  $+x$ -direction, say, the magnitude of the momentum (and hence energy) in the first term will tend to decrease while that in the second will tend to increase (because electrons are negatively charged, they accelerate opposite to the applied electric field). This will happen if there are unfilled electron states at nearby lower or higher energies, respectively. If there are no available unfilled nearby states the electron cannot lose or gain energy (i.e., the Fermi repulsion electrons exert on one another balances the electric force). Electrons near the Fermi energy in an unfilled (conduction) band can gain energy (moving opposite the applied field direction) because there are many nearby unfilled higher energy states; they are less likely to lose energy, however, because lower energy states are all (or mostly) filled. The net effect is that the applied field produces electron motion opposite the field (i.e., conventional current in the field direction). In other words, in Equation (2)  $\langle v \rangle \doteq v_F$ , where  $\varepsilon_F = mv_F^2/2$ ; unlike the classical pinball model,  $\langle v \rangle$  is roughly independent of temperature. The quantity  $n_e$  in (2) is the number density of electrons in the conduction band. This is large and more-or-less independent of temperature for metals, close to zero for insulators (and due mostly to impurities), and small but a rapidly increasing function of temperature for semiconductors.

Finally, the mean free path,  $\lambda$ , is determined by imperfections in the periodic arrangement of the solid's underlying atoms. This can occur dynamically through thermally excited vibrations of the atoms (*phonons*, collective modes of vibration associated with sound-waves) or statically because impurity atoms alter the perfect periodicity of the solid lattice. The excited conduction electron wavefunctions are not eigenstates of the solid potential energy with imperfections and therefore undergo de-excitation transitions. This process is called *scattering* of the conduction electrons. The mean free path has the geometric interpretation depicted in the figure to the right. An electron entering the cylinder at the left will (almost) certainly scatter before leaving on the right. The volume of the cylinder is  $\lambda \sigma_{scatt}$ , where  $\sigma_{scatt}$  is the *cross-sectional area* of a scatterer, which, depending on the nature of interaction, might be much smaller to or much larger than the geometric cross-section of the scatterer. There is supposedly one scatterer in this volume so  $n_{scatt} \lambda \sigma_{scatt} = 1$ , where  $n_{scatt}$  is the number density of scatterers. Thus,  $\lambda = 1/n_{scatt} \sigma_{scatt}$ .



The mean free path for phonon scattering is  $\lambda_{phonons} = 1/n_{phonons} \sigma_{phonons}$ , where  $\sigma_{phonons}$  is the “electron-phonon interaction cross-sectional area.” The latter is probably independent of temperature and proportional to the cross-sectional area of an atom. On the other hand,  $n_{phonons}$  is temperature-dependent. Phonons and photons share many characteristics. A “box” full of phonons is similar to blackbody radiation. The wavefunction for phonons spreads over the entire solid and can therefore have long wavelengths. As a result, the allowed energies for

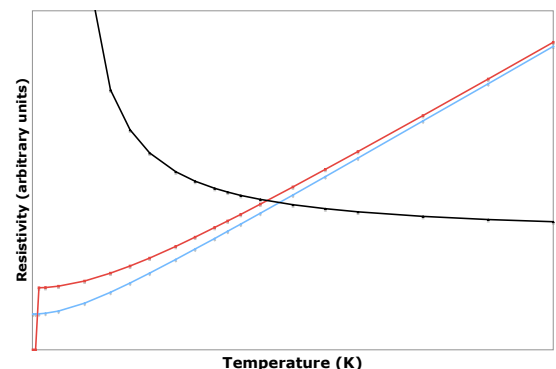
phonons are very closely spaced. Near room temperature, the density of phonons is proportional to  $T$ . Thus, scattering by phonons produces a mean free path that is proportional to  $1/T$  (the higher  $T$  the shorter the mean free path, while the lower  $T$  the longer the mean free path) and a corresponding resistivity that is proportional to  $T$ . For impurity scattering,  $\lambda_{imp} = 1/n_{imp} \sigma_{imp}$ , which is independent of temperature (as is the corresponding resistivity).

**Example:** Consider solid copper. Cu has a mass density of  $8.96 \text{ gm/cm}^3$  and an atomic mass of 63.5;  $8.96 \text{ gm}$  of copper is  $8.96/63.5 \times 6.02 \times 10^{23} \text{ atoms} = 8.5 \times 10^{22} \text{ atoms}$ . A copper atom has one 4s electron so the 4s band in solid copper has one conduction electron per atom. Thus  $n_e = 8.5 \times 10^{28} \text{ m}^{-3} = n_{atoms}$ . The Fermi energy for the conduction electrons is 6.9 eV (taking the energy at the bottom of the conduction band to be zero). Consequently,  $v_F = 1.58 \times 10^6 \text{ m/s}$  (about 0.5%  $c$ , pretty fast and about 10x greater than the thermal speed of a classical electron at 300 K). At 300 K, the electrical resistivity of reasonably pure copper is  $1.68 \times 10^{-8} \Omega\text{-m}$ . Inserting these values into (2) allows us to calculate  $\lambda$  for copper at 300 K: 39.3 nm or 173 atomic spacings! Note this mean free path includes contributions from both phonon scattering and whatever impurities are present.

**Example:** Suppose a solid copper sample is 99.9% pure (1 impurity atom per  $10^3$  copper atoms). The number density of impurity atoms is  $8.5 \times 10^{25} \text{ m}^{-3}$ . Assuming that  $\sigma_{imp}$  is just the geometric cross-section of a typical atom, about  $5 \times 10^{-20} \text{ m}^2$ , the mean free path for impurity scattering would be 235 nm. Thus, at room temperature scattering by phonons is much more important than by impurity atoms (in reasonably pure copper).

**Example:** Because  $\lambda_{phonons}$  is proportional to  $1/T$  we can write  $\lambda_{phonons}(T) = \lambda_{phonons}(300\text{K}) \frac{300\text{K}}{T}$ . The mean free paths due to phonon and impurity scattering are equal (for the values above) at a temperature of about  $T = 50 \text{ K}$ . At temperatures lower than this, impurity resistance is larger than phonon resistance, and roughly constant. In other words, for metals, as temperature decreases resistivity decreases linearly because of phonon scattering, then flattens out when the mean free path for phonon scattering becomes larger than the mean free path for impurity scattering.

The discussion above is for metals; what about semiconductors? For a pure semiconductor,  $n_e = 0$  at  $T = 0 \text{ K}$ , and therefore its resistivity is infinite (it's an insulator at  $T = 0 \text{ K}$ ) as long as the applied fields aren't large. As the temperature is raised electrons are promoted into the conduction band. The temperature dependence of this "ionization" process is exponential. The phonon scattering process only increases linearly with increasing temperature, so the resistivity *decreases* as  $T$  increases—a completely different trend from a metal, and certainly not predicted by a classical picture of electrical resistance. The resistivity for a pure semiconductor is always *much* higher than for a metal, also not predicted by a classical picture. The figure to the right depicts qualitatively how resistivity varies with temperature for Cu (bottom curve, a good conductor), Sn (middle, a not so good conductor), and Si (top, a semiconductor). Note that at



very low temperatures the resistivity of Sn suddenly plummets. That is called “superconductivity.” We’ll get to that presently.

To summarize: The resistivity of a solid can only be understood in terms of the wave nature of electrons and the Pauli Exclusion Principle.

Table of comparison of classical and quantum versions of electrical conduction:

Quantity	Classical pinball	Quantum theory
Average speed $\langle v \rangle$	Proportional to $\sqrt{T}$	Fermi speed, $T$ -independent
Conduction electron density	No clue (?)	Density of electrons in conduction band; constant for metals and insulators, rapidly increasing with $T$ for semiconductors
Mean free path	Constant, average atomic spacing (?), about the same for all solids	For phonon scattering, varies like $1/T$ ; for impurity scattering, independent of $T$ , varying inversely with impurity concentration
Resistivity	Proportional to $\sqrt{T}$ , about the same for all materials	Proportional to $T$ for metals at highish temperatures; decreasing with increasing $T$ for semiconductors; wide range of values for different materials

Electrical resistivity is a *macroscopic* variable that is easy to measure as a function of temperature in the laboratory. Nonetheless, none of its characteristics can be understood without reference to *microscopic* quantum mechanics.

Very cool.