Many-particle Systems, 5

Photons as $T \rightarrow 0$ **K**

Photons are *massless* bosons. Because they are massless, any number of them can be added or subtracted without changing the photon system energy. (For example, a 1 eV photon can be exchanged for one hundred 0.01 eV photons, without changing system energy.) As a result, the chemical potential for photons doesn't control the number of particles in the system and consequently is taken to be zero. Thus, the average occupation number for photons is

$$\bar{N}_{\sigma \ ph} = \frac{1}{\exp(\varepsilon_{\sigma}/k_{B}T) - 1}.$$
(1)

At T = 0 the exponential in (1) blows up for all $\varepsilon_{\sigma} > 0$. (If a photon exists it has positive kinetic energy; by definition, the zero-energy state of this system has no photons.) Here, there's no chemical potential, as was the case for massive particles, to "save the day" from the total loss of particles. As a result, at T = 0 all photons vanish. Photons do not undergo Bose-Einstein Condensation at low temperature, they simply disappear.

Photon gas at finite temperature: blackbody radiation

One of the most important examples of a statistical mechanical system is a "gas" of photons in thermal equilibrium with the walls of a box that contains them. The goal here is to relate the average energy in the photon gas to the wall temperature. It is necessary to evaluate $\overline{E} = \sum_{\sigma} \overline{N}_{\sigma} \varepsilon_{\sigma}$. For simplicity, assume the box is a cube with side length L and that the photon wavefunction vanishes at the walls. The single-particle eigenstates (σ) in the box are standing waves described by 3 quantum numbers n_x, n_y, n_z , each of which can be 1, 2, 3, ..., and one spin quantum number that can have one of two values, ± 1 (see Sc5, pg4). The wavelengths of these standing waves (as in the infinite 3D well) are $2L/n_x, 2L/n_y, 2L/n_z$. To each wavelength there is a corresponding momentum magnitude: $p_x = h/(2L/n_x) = (h/2L)(n_x^2 + n_y^2 + n_z^2)^{1/2}$. For photons, energy = pc, so $\varepsilon_{\sigma} = (hc/2L)(n_x^2 + n_y^2 + n_z^2)^{1/2}$.

The wavelengths that contribute most to the system energy are typically much smaller than *L* so the relevant *n* values are typically >> 1. To calculate \overline{E} , the required sum is $\sum_{\sigma} \dots = 2 \sum_{n_x n_y n_z} \dots \Delta n_x \Delta n_y \Delta n_z$, where the 2 is for the *two* spin states and the Δn s are each equal to 1

(every time a new *n* is entered in the sum it increases from the previous value by 1). The Δn s are not usually included in expressions for sums, but here the important *n*-values are large so the following approximation is excellent

$$\sum_{n_x n_y n_z} \dots \Delta n_x \Delta n_y \Delta n_z \simeq \iint_{n_x n_y} \iint_{n_z} \dots dn_x dn_y dn_z$$

The integrand in the integral depends only on $n = (n_x^2 + n_y^2 + n_z^2)^{1/2}$, which is spherically symmetric. The coordinates in the triple integral should therefore be changed from Cartesian (n_x, n_y, n_z) to spherical. The values of the *n*'s are positive, so the integration occurs only over one octant of a sphere:

$$\iint_{n_x n_y n_z} \dots dn_x dn_y dn_z = \frac{1}{8} 4\pi \int_{n} \dots n^2 dn$$

As a result,

$$\overline{E} = 2\frac{4\pi}{8}\frac{hc}{2L}\int_{n=0}^{\infty}\frac{n^3}{\exp(hcn/2Lk_BT)-1}dn.$$

Of course, the limits are not really 0 and ∞ , but close enough. If we let $x = hcn/2Lk_{p}T$ we obtain

$$\overline{E} = 8\pi L^3 \frac{(k_B T)^4}{(hc)^3} \int_{n=0}^{\infty} \frac{x^3}{\exp(x) - 1} dx$$

The integral is tabulated and has the value $\pi^4/15$. Consequently, the energy density (energy/volume) in the photon gas is

$$u = \frac{\overline{E}}{L^3} = \frac{8\pi^5}{15} \frac{(k_B T)^4}{(hc)^3}.$$
 (2)

Remarkably, this result is independent of what the walls of the box are made of; it depends on the single variable T-raised to the power 4.

<u>Example</u>: The quantity $8\pi^5 k_B^4 / 15(hc)^3 = 4.7 \times 10^{-24} \text{ eV/nm}^3 \text{-K}^4$. For T = 300 K, $u = 3.8 \times 10^{-14} \text{ eV/nm}^3$; that's about $5 \times 10^{-12} \text{ eV}$ in the volume of an atom. At room temperature a gas of photons produces no atomic electronic excitations. In macroscopic units $u = 6 \times 10^{-6} \text{ J/m}^3$, still not much.

If we write $u = \int_{0}^{\infty} \phi(\varepsilon) d\varepsilon$, $\phi(\varepsilon)$ is the photon *energy density spectrum*;

$$\phi(\varepsilon) = \frac{8\pi}{(hc)^3} \frac{\varepsilon^3}{\exp(\varepsilon/k_B T) - 1}.$$
(3)

Equation (3) is also called the **blackbody spectrum**. The figure to the right shows spectra for T = 1500 K and 2000 K. Clearly, the maximum of the spectrum shifts to higher energies as Tincreases and also the magnitude of the maximum increases with increasing T. By differentiating (3) with respect to ε and setting the result equal to zero we find that the spectrum maximum occurs at $\varepsilon_{max} = 2.82k_BT$. (For the spectra in the figure $\varepsilon_{max} = 0.36$ eV and 0.49 eV.) Plugging this into (3) shows that $\phi(\varepsilon_{max}) \propto T^3$. If (3) is divided by the photon energy ε we obtain the photon *number density* spectrum



$$n_{ph}(\varepsilon) = \frac{8\pi}{(hc)^3} \frac{\varepsilon^2}{\exp(\varepsilon/k_B T) - 1}$$

The maximum in the photon number density occurs at $\varepsilon_{max} = 1.59k_BT$. Despite having a pronounced peak at $\varepsilon_{max} = 2.82k_BT$, the blackbody energy spectrum still has 5% of the maximum value at an energy $\varepsilon = 9.3k_BT$, so the "tail" of the distribution is fairly long and photons (albeit few) exist with energies well in excess of k_BT .

<u>Example</u>: The universe is filled uniformly (as far as we know) with a **cosmic blackbody spectrum**, shown to the right. The shape implies the temperature of the spectrum is T = 2.725 K. At such a low temperature, these photons cannot interact with atomic matter. It is generally agreed that over 10 billion years ago the cosmic blackbody was 1000 times hotter. That was the last time these photons interacted with atomic matter.



<u>Example</u>: What is the wavelength of a photon with $\varepsilon_{max} = 2.82k_BT$ at room temperature?

<u>Solution</u>: At room temperature $k_B T = 1/40 \text{ eV}$, so the photon energy is 0.07 eV. This is also hc/λ so the wavelength is about 18 µm, about the size of a biological cell (about 100 x smaller than for the CMB).

<u>Example</u>: What is the wavelength of a photon with $\varepsilon_{max} = 2.82k_BT$ at the temperature of the Cosmic Microwave Background (CMB)? <u>Solution</u>: Set $\lambda = hc/2.82k_BT$ to find that at T = 2.725 K (about 1/100 room temperature) the

wavelength is about 2 mm (a microwave photon). (Incidentally, there are about 1.6×10^9 CMB photons for every proton in the universe, but because these photons carry so little energy, the total mc^2 for protons is about 2000 times greater than the energy of the CMB.)