Foundations, 3

Photon probability waves

Our goal here is to try to reconcile classical EM with the existence of photons. The electric and magnetic fields associated with EM radiation, propagating in a direction we designate as x, obey the *Maxwell wave equation*:

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = c^2 \frac{\partial^2 \mathcal{E}}{\partial x^2}. \tag{1}$$

The "partial derivatives" that appear in (1) are defined, for example, as

$$\frac{\partial \mathcal{E}}{\partial t} \equiv \lim_{dt \to 0} \frac{\mathcal{E}(x,t+dt) - \mathcal{E}(x,t)}{dt} \tag{2}$$

(i.e., "hold x fixed, vary t") and so on. One possible harmonic traveling wave solution is of the form $\mathcal{E}(x,t) = \mathcal{E}_{\max}\cos(k_x x - \omega t + \phi)$, where when $k_x = +k$ the wave is propagating in the $+x$-direction, and when $k_x = -k$, it's propagating in the $-x$.

<u>Example</u>: For the $\mathcal{E}(x,t)$ above with $k_x = +k$, $\frac{\partial \mathcal{E}}{\partial x} = -k\mathcal{E}_{\max}\sin(kx - \omega t + \phi)$, $\frac{\partial \mathcal{E}}{\partial t} = +\omega\mathcal{E}_{\max}\sin(kx - \omega t + \phi)$, $\frac{\partial^2 \mathcal{E}}{\partial x^2} = -k^2\mathcal{E}_{\max}\cos(kx - \omega t + \phi) = -k^2\mathcal{E}$, and $\frac{\partial^2 \mathcal{E}}{\partial t^2} = -\omega^2\mathcal{E}_{\max}\cos(kx - \omega t + \phi) = -\omega^2\mathcal{E}$. Clearly, for this plane wave to be a solution to (1), $\omega^2 = c^2k^2$.

Recall that photon energy is $E = hf$ and photon momentum magnitude is $p = h/\lambda$. We can write E as $E = h\omega/2\pi$ and p as $p = hk/2\pi$. Because $h/2\pi$ appears so often in quantum mechanics, it is given its own symbol: $\hbar$ (pronounced "aitch bar"). Then an electric field plane wave can be expressed as $\mathcal{E}(x,t) = \mathcal{E}_{\max}\cos[(p_x x - Et)/\hbar + \phi]$, where $p_x = \pm p$; in this form the electric field contains photon energy and momentum information. OK, but where is the randomness of photon-arrival hiding?

To see where, requires the EM energy density, u. The bright light double slit interference pattern, for example, is the time-average of the variation of u over the surface of the detector. But this pattern is identical to the variation of the probability that a photon will hit the detector when the apparatus is illuminated with very dim light. A way of connecting u and the photon-arrival probability is to declare that in a *single-photon* experiment (see the Appendix in Fn2) the *probability density* (probability per unit volume) of detecting the photon at a given point in space at a given time is $\rho(x,y,z,t) = \frac{u(x,y,z,t)}{\int u\,dV}$, where the integral is over all space. The probability that the photon will be detected in a small volume dV surrounding the point x,y,z is ρdV. Note that $\int \rho\,dV = 1$, which implies that the photon is *somewhere* at time t.

To complete the story of how field and probability are connected we note that for an EM traveling wave in vacuum, $\mathcal{E} = cB$, so that $u = \varepsilon_0 \mathcal{E}^2$. This enables us to define a *one-photon wavefunction* as, $\Psi = N\mathcal{E}$. The "normalization factor" N is *tentatively* chosen so that $\int \Psi^2 dV = 1$ which allows Ψ^2 to be *tentatively* interpreted as the probability density ρ above. We'll see why this is only tentative below.

Substituting $\mathcal{E} = \Psi/N$ into Equation (1) shows that the photon wavefunction is also a solution of the Maxwell wave equation:

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2}.$$

Since Ψ depends on x and t through, for example, $(p_x x - Et)/\hbar$, the wave equation becomes

$$-\frac{E^2 \Psi}{\hbar^2} = -\frac{c^2 p_x^2 \Psi}{\hbar^2}.$$

This suggests that **time derivatives "measure" energy while spatial derivatives "measure" momentum**. In fact, for photons we can re-express the Maxwell wave equation as an energy-momentum **operator equation**:

$$E_{op}^2 \Psi = c^2 p_{x,op}^2 \Psi \tag{4}$$

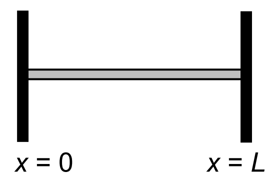
where $E_{op}^2 = -\hbar^2 \frac{\partial^2}{\partial t^2}$ and $p_{x,op}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$. Note that $p_{x,op}^2 \Psi = p_x^2 \Psi$ and $E_{op}^2 \Psi = E^2 \Psi$. The quantities p_x^2 and E^2 on the right-hand sides of these equations are *not* operators; they are just numbers. These equations are of the form

operator[*function*]=*number*•(*same function*).

Such equations are called *eigenvalue* equations. The *number* in the equation is the *eigenvalue* and the *function* is the *eigenfunction* for the given *operator*. (In German, *eigen* means "proper;" the function is a "proper function" and the associated value, a "proper value.") The wavefunction Ψ is simultaneously an eigenfunction of both the $p_{x,op}^2$ (with eigenvalue p_x^2) and E_{op}^2 (with eigenvalue E^2) operators.

Finally, we expect that a photon with constant wavelength will have a unique momentum and a unique energy. That is, we expect that $p_{x,op}\Psi = p_x \Psi$ and $E_{op}\Psi = E\Psi$. But, now something interesting happens. Traveling harmonic waves, that is, $\sin(p_x x/\hbar - Et/\hbar + \phi)$ and $\cos(p_x x/\hbar - Et/\hbar + \phi)$ are solutions to the Maxwell wave equation with unique values of p_x and E. But they are *not* eigenfunctions of $p_{x,op}$ and E_{op}. The operator $p_{x,op}^2 = p_{x,op} p_{x,op}$, so we expect that $p_{x,op}$ is proportional to just $\partial/\partial x$ (and similarly for E_{op}). But, for example, $\partial[\sin(p_x x/\hbar - Et/\hbar + \phi)]/\partial x = (p_x/\hbar)\cos(p_x x/\hbar - Et/\hbar + \phi)$. That is, $p_{x,op}$ and E_{op} convert a sine into a cosine and vice versa, so the functions are not the same on both sides of the equations. The day is saved, however, by **Euler's formula**: $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$, where i is the *unit imaginary number*, $i = \sqrt{-1}$. If sin and cos are solutions to the Maxwell wave equation so is exp. Thus, if we set $\Psi(x,t) = N\exp[i(p_x x/\hbar - Et/\hbar + \phi)]$, we find $\partial\Psi/\partial x = ip_x\Psi/\hbar$, a valid eigenvalue equation with $p_{x,op} = -i\hbar\partial/\partial x$. Similarly, we find $E_{op} = i\hbar\partial/\partial t$. (Can you see how $p_{x,op}^2 = -\hbar^2\partial^2/\partial x^2$ and $E_{op}^2 = -\hbar^2\partial^2/\partial t^2$?) This Ψ is complex and so is Ψ^2. In order to obtain a positive real number for a probability density we have to use $|\Psi|^2 = \Psi^*\Psi$ where Ψ^* is the complex conjugate of Ψ (in it, i is replaced by $-i$ wherever it appears in Ψ).

Example: In the 1-photon double slit experiment you can treat the slits as being "sources of probability wavefunctions." Assuming that these wavefunctions are complex exponential plane waves and using the same geometry as in Fn1, we have $\Psi_1 = \Psi_{\max}\exp\{i[(p_x x_1 - Et)/\hbar + \phi_1]\}$ and $\Psi_2 = \Psi_{\max}\exp\{i[(p_x x_1 + pd\sin\theta - Et)/\hbar + \phi_2]\}$. Adding and factoring common terms produce $\Psi = \Psi_1 + \Psi_2 = \Psi_{\max}\exp[i(p_x x_1 - Et)/\hbar][\exp(i\phi_1) + \exp(ip_x d\sin\theta/\hbar + i\phi_2)]$. Calculating $|\Psi|^2 = \Psi^*\Psi$ yields $|\Psi|^2 = \Psi_{\max}^2\{2 + \exp[i(p_x d\sin\theta/\hbar + \phi_2 - \phi_1)] + \exp[-i(p_x d\sin\theta/\hbar + \phi_2 - \phi_1)]\}$. Euler's formula can be used to show that $\cos\theta = [\exp(i\theta) + \exp(-i\theta)]/2$. Thus, $|\Psi|^2 = 2\Psi_{\max}^2[1 + \cos(p_x d\sin\theta/\hbar + \phi_2 - \phi_1)]$, which has the same form as the bright light interference pattern (Equation (2) in Fn1). Note that this pattern arises from the waviness of the probability, but because p_x appears explicitly it also contains particle (i.e., momentum) information. That is, the wavefunction idea manifests the wave-particle duality of photons.



Example: Consider an apparatus consisting of a very thin fiber optic cable the ends of which are terminated by perfect mirrors at $x = 0$ and $x = L$, as depicted to the right. Somehow, a single photon is injected into the cable. Because the mirrors are perfectly reflecting the electric field is always zero at the cable ends. A single plane wave of the form $\Psi = \Psi_{\max}\exp[i(px - Et)/\hbar]$ cannot satisfy this condition, but two waves with equal and opposite momenta can. For example, the two-plane wave wavefunction $\Psi = \Psi_{\max}\{\exp[i(px - Et)/\hbar] - \exp[i(-px - Et)/\hbar]\}$ automatically vanishes at $x = 0$. (In it, $p > 0$ is the *magnitude* of the momentum, p_x; half the time the photon travels to the right, half the time to the left) This wavefunction can also be written as $\Psi = 2i\Psi_{\max}\sin(px/\hbar)\exp(-iEt/\hbar)$ (because Euler's formula tells us $\sin\theta = [\exp(i\theta) - \exp(-i\theta)]/2i$). This wavefunction will vanish at $x = L$ provided $pL/\hbar = n\pi$, where $n = 1,2,3,...$. In other words, the reflecting ends of the cable force the magnitude of the momentum of the photon to be limited to only certain discrete values, $p_n = n\pi\hbar/L$, and its energy to be $E_n = p_n c = n\pi\hbar c/L$. A free photon can have any momentum and energy, but a trapped one only has values that are consistent with the cable's end conditions. This ***"quantization" of momentum and energy occurs whenever a particle (a photon, an electron, …) is confined to a finite region of space***. (Note: the wavefunction described here is only part of the whole wavefunction, because the electric and magnetic fields inside the wire are no longer in phase, so $\mathcal{E} \neq cB$; the momentum and energy quantization conditions are still the same, though.)

There are no particles, only fields

Though we have just been discussing photons, most of the rest of this course will be massive particles. Importantly, the version of quantum mechanics discussed in these notes is called the “Schrödinger Picture.” It works brilliantly (that is, you can make money using it, $QM = \$$) for describing the behavior of electrons in atoms, molecules, and solids. It’s all about the waviness of “particles.” It describes wave-particle duality, but doesn’t explain it (i.e., it is simply an observed property of matter). Schrödinger quantum mechanics is a *nonrelativistic* limit, in which rest energy is much larger than kinetic or potential energy, of a more general description of matter called “quantum field theory (QFT).” QFT postulates that the “world” consists only of fundamental fields such as photon field, electron field, etc. Quantum fields carry intrinsic properties such as mass and charge, and extrinsic properties such as energy and momentum. Quantum fields interact when their “fundamental excitations,” or “quanta,” change each other’s state. In QFT interactions are *local*: they occur at a point in space and a moment in time. When a quantum of a quantum field interacts with the quanta of the fields of a “particle detector,” the detector records the properties of the field excitation and the excitation changes state. (In the case of photons, the field excitation actually disappears–“the photon state is annihilated”–when this occurs.) Real detectors, however small, are made of lots of atoms. The smallest pixel in a CCD camera, for example, contains over 10^{12} atoms. The quantum field, photon or electron, is spread over all the detectors in an array, but an excitation of the quantum field only interacts at a point with one detector at a time (provided the detectors are “reliable”), making it appear as if the field consists of particles. In other words, in the QFT model of the world, there are no particles, only fields and their interactions. A “particle” is what is measured in a “particle” detector as a result of an interaction between fields; in this version of physical reality it doesn’t make any sense to ask whether the particle existed before the detection event. The QFT view removes the head-splitting dichotomy of wave-particle duality by asking that belief in little, hard, Newtonian nuggets be suspended–as difficult as that task might be for most of us who learned about physics starting with Newton-first.